17. On a Power Series which has only Algebraic Singularities on its Convergence Circle, III.

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1. Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be regular for \( |x| < 1 \) and all its singular points on \( |x| = 1 \) be algebraic, then I have proved\(^{(1)}\) that

\[
\lim_{n \to \infty} \left[ |f_n(x_1)| + |f_n(x_2)| + \ldots + |f_n(x_k)| \right] = \infty
\]

where

\[
f_n(x) = a_0 + a_1 x + \ldots + a_n x^n,
\]

and \( k \) is the number of singular points on \( |x| = 1 \) and \( x_1, x_2, \ldots, x_k \), are any \( k \) points outside \( |x| = 1 \).

In this paper I will prove the following

Theorem. Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be regular for \( |x| < 1 \), and all its singular points on \( |x| = 1 \) be algebraic, then

\[
\lim_{n \to \infty} f_n(x) = \infty \quad \text{almost everywhere}^{(2)} \quad |x| > 1.
\]

Firstly I could prove the theorem only for functions with simple poles on \( |x| = 1 \); Prof. Kakeya being interested in my result remarked to me that the theorem still holds for functions with only poles on \( |x| = 1 \).

Modifying his method and using his lemma, I could reach to the above result. But for his suggestions I could not have reached to my result. Here I will express my cordial thanks to him.

2. Let the singular points on \( |x| = 1 \) be

\[
\frac{1}{\omega_1} = e^{i\theta_1}, \quad \frac{1}{\omega_2} = e^{i\theta_2}, \quad \ldots, \quad \frac{1}{\omega_k} = e^{i\theta_k} (\theta_1 < \theta_2 < \ldots < \theta_k).
\]

Then by Caucy’s theorem,

\[
a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} \, dz,
\]

where \( C \) is any closed curve about the origin and lying in the domain of regularity of \( f(x) \).

\(^{(1)}\) M. Tsuji, this journal 3 (1926) 69-85.

\(^{(2)}\) I.e. with exception at most a point-set of measure zero in Lebesgue’s sense.
When $x$ lies outside $C$

\[(1) \quad f(x) = a_0 + \cdots + a_n x^n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z} \left[ 1 + \frac{x}{z} + \cdots + \left( \frac{x}{z} \right)^n \right] dz\]

\[= \frac{z^{n+1}}{2\pi i} \int_C \frac{f(z)}{z^{n+1}(x-z)} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{x-z} dz = \frac{z^{n+1}}{2\pi i} \int_C \frac{f(z)}{z^{n+1}(x-z)} dz.\]

From the hypothesis, $f(x)$ can be expanded about $\frac{1}{\omega_1}$ in the form,

\[(2) \quad f(z) = b^{(1)}_0 (1 - \omega_1 z)^{p_1} + b^{(1)}_1 (1 - \omega_2 z)^{p_1} + \cdots + b^{(1)}_q (1 - \omega_{q+1} z)^{p_1} + \sum_{n=0}^{\infty} \alpha_n z^n,\]

where $\sum_{n=0}^{\infty} \alpha_n z^n$ has the convergence radius $>1$ and $m_1$, $p_1$ are integers, $p_1 > 0$ and $\frac{m_1}{p_1}$ is neither zero nor a positive integer.

Since

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n x^n = x^{n+1} O(\varepsilon^n)$$

where $0 < \varepsilon < 1$ and as will be shown later, it can be neglected without any essential influence on the final result, we will assume $\sum_{n=0}^{\infty} \alpha_n z^n = 0$ in the following discussion.

We will take the path of integration $C$ in the following way:

$$C = (L_1' + l_1 + L_1) + C_1 + (L_2' + l_2 + L_2) + C_2 + \cdots + (L_k' + l_k + L_k) + C_k,$$

where

- $L_r'$; $z = e^{i\varphi_r}$, $t$ varying from $1 + \frac{1}{\sqrt{n}}$ to $1 + \frac{1}{n}$;
- $l_r$; $z - \frac{1}{\omega_r} = \frac{1}{n} e^{i\theta}$, $\theta$ varying from $0$ to $-2\pi$;
- $L_r$; $z = e^{i\varphi_r}$, $t$ varying from $1 + \frac{1}{n}$ to $1 + \frac{1}{\sqrt{n}}$;
- $C_r$; $z = \left(1 + \frac{1}{\sqrt{n}}\right)e^{i\theta}$, $\theta$ varying from $\varphi_r$ to $\varphi_{r+1}$. 
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We put

\[
\varphi_1(z) = b_0^{(1)} (1 - \omega_1 z)^{p_1} + \ldots + b_q^{(1)} (1 - \omega_1 z)^{p_1 + q}
\]

and similar expressions for \( \varphi_i(z) \), and \( F_i(z) \); \( q \) will be determined later.

Then from (2) we have

\[
f(z) = \varphi_1(z) + F_1(z) = \ldots = \varphi_k(z) + F_k(z).
\]

Further we put

\[
f(z) - [\varphi_1(z) + \ldots + \varphi_k(z)] = G(z)
\]

then

\[
G(z) = F_i(z) + \text{a regular function about } \frac{1}{\alpha_i} (i = 1, \ldots, k).
\]

From (1) we have

\[
f_n(x) = z^{n+1} \left[ \frac{1}{2\pi i} \int_C \frac{\varphi_1(z) + \ldots + \varphi_k(z)}{z^{n+1}(x - z)} \, dz + \frac{1}{2\pi i} \int_C \frac{G(z)}{z^{n+1}(x - z)} \, dz \right].
\]

Now

\[
\frac{1}{2\pi i} \int_C \frac{G(z)}{z^{n+1}(x - z)} \, dz = \frac{1}{2\pi i} \int_{C_1 + \ldots + C_k} \frac{G(z)}{z^{n+1}(x - z)} \, dz
\]

\[
+ \frac{1}{2\pi i} \sum_{r=1}^k \int_{L_r + L_r + (x - z)} \frac{G(z)}{z^{n+1}(x - z)} \, dz.
\]

Since on \( C_1 + \ldots + C_k \),
\[ |G(z)| < K \left[ \left| 1 - \omega_n z \right|^{\frac{m_1 + q + 1}{p_1}} + \ldots + \left| 1 - \omega_k z \right|^{\frac{m_k + q + 1}{p_k}} \right], \]

putting \( \lambda = \text{Max} \left( \left| \frac{m_1 + q + 1}{p_1} \right|, \ldots, \left| \frac{m_k + q + 1}{p_k} \right| \right) \), we have on \( C_1 + \ldots + C_k \),

\[(9) \quad |G(z)| < Kn^{\lambda} \]

[We will use the same letter \( K \) to denote a positive constant independent of \( x \) and \( n \).]

Hence we have for \( 1 < r_0 \leq |x| \) uniformly,

\[(10) \quad \frac{1}{2\pi i} \int_{C_1 + \ldots + C_k} \frac{G(z)}{z^{n+1}(x-z)} \, dz \leq K \frac{n^{\lambda}}{\left(1 + \frac{1}{\sqrt{n}} \right)^n} \leq Kn^{\lambda} e^{-\sqrt{n}}. \]

Since on \( L_1 + L_1 + L_1 \), \( |F_i(z)| < K \left| 1 - \omega_i z \right|^{-\frac{m_i + q + 1}{p_i}} \), we have from \( (7) \),

\[(11) \quad \left| \frac{1}{2\pi i} \int_{L_1 + L_1 + L_1} \frac{G(z)}{z^{n+1}(x-z)} \, dz \right| = \left| \frac{1}{2\pi i} \int_{L_1 + L_1 + L_1} \frac{F(z)}{z^{n+1}(x-z)} \, dz \right| < K \int_{L_1} \left| 1 - \omega_i z \right|^{-\frac{m_i + q + 1}{p_i}} \left| \frac{1}{z^{n+1}|x-z|} \right| \, dz \]

\[ \leq Kn \frac{n^{\lambda}}{\left(1 - \frac{1}{n} \right)^n} \leq Kn \frac{n^{\lambda}}{\left(1 - \frac{1}{n} \right)^n}. \]

\[(12) \quad \left| \frac{1}{2\pi i} \int_{L_1 + L_1 + L_1} \frac{G(z)}{z^{n+1}(x-z)} \, dz \right| = \left| \frac{1}{2\pi i} \int_{L_1 + L_1 + L_1} \frac{F(z)}{z^{n+1}(x-z)} \, dz \right| \]

\[ \leq K \int_{L_1 + L_1 + L_1} \left| 1 - \omega_i z \right|^{-\frac{m_i + q + 1}{p_i}} \left| \frac{1}{z^{n+1}|x-z|} \right| \, dz \leq Kn \frac{n^{\lambda}}{\left(1 - \frac{1}{n} \right)^n}. \]

where we take \( q \) so large that \( \frac{m_i + q + 1}{p_i} \geq 0 \) \((i = 1, \ldots, k)\).

From \( (11) \) and \( (12) \), we get,

\[(13) \quad \frac{1}{2\pi i} \int_{G} \frac{G(z)}{z^{n+1}(x-z)} \, dz = O \left( n^{-\frac{1}{2}} \left( \frac{m_1 + p + 1}{p_1} + 1 \right) \right) + \ldots + O \left( n^{-\frac{1}{2}} \left( \frac{m_k + p + 1}{p_k} + 1 \right) \right) \]

\[ = O \left( n^{-\frac{1}{2}} \alpha \right), \]

where we put

\[(14) \quad \alpha = \text{Min} \left( \frac{m_1 + q + 1}{p_1} + 1, \ldots, \frac{m_k + q + 1}{p_k} + 1 \right). \]
From (8) and (13) we have,

$$f_{n}(x) = x^{n+1} \left[ \frac{1}{2\pi i} \int_{c} \frac{\varphi_{1}(z) + \ldots + \varphi_{k}(z)}{z^{n+1}(x-z)} \, dz + O\left( n^{-\frac{1}{2}} \right) \right]$$

uniformly for $1 < r_{0} \leq |x|$.

3. Now

$$\frac{1}{2\pi i} \int_{c} \frac{\varphi_{1}(z) + \ldots + \varphi_{k}(z)}{z^{n+1}(x-z)} \, dz = \frac{1}{2\pi i} \int_{c} \frac{\varphi_{1}(z)}{z^{n+1}(x-z)} \, dz + \ldots + \frac{1}{2\pi i} \int_{c} \frac{\varphi_{k}(z)}{z^{n+1}(x-z)} \, dz.$$

And

$$\frac{1}{2\pi i} \int_{c} \frac{\varphi_{1}(z)}{z^{n+1}(x-z)} \, dz = \frac{1}{2\pi i} \int_{c} \frac{\omega_{1}\varphi_{1}(z)}{z^{n+1}(\omega_{x}x - 1 + 1 - \omega_{x})} \, dz =$$

$$\frac{1}{2\pi i} \int_{c} \frac{\omega_{1}\varphi_{1}(z)}{z^{n+1}(\omega_{x}x - 1)} \left[ 1 + \frac{\omega_{x} - 1}{\omega_{x}x - 1} + \ldots + \frac{(\omega_{x} - 1)^{r}}{(\omega_{x} - 1)^{r+1}} \right] \, dz.$$

Hence

$$\frac{1}{2\pi i} \int_{c} \frac{\varphi_{1}(z)}{z^{n+1}(x-z)} \, dz = \frac{1}{2\pi i} \int_{c} \frac{\varphi_{1}(z)}{z^{n+1}(x-z)} \left[ 1 + \frac{\omega_{x} - 1}{\omega_{x}x - 1} + \ldots + \frac{(\omega_{x} - 1)^{r}}{(\omega_{x} - 1)^{r+1}} \right] \, dz.$$

Since $\frac{\varphi_{1}(z)}{z^{n+1}(x-z)} \left( \frac{\omega_{x} - 1}{\omega_{x}x - 1} \right)^{r+1}$ is regular about $\frac{1}{\omega_{2}}, \ldots, \frac{1}{\omega_{k}}$, we can change the path of integration $C$ into $C' = L_{1} + L_{1} + C_{1} + \ldots + C_{k}$.

On $C_{1} + \ldots + C_{k}$, $|\varphi_{1}(z)| < K |1 - \omega_{x}| \left| \frac{m_{1}}{p_{1}} \right|$, therefore

$$\left| \frac{1}{2\pi i} \int_{C_{1} + \ldots + C_{k}} \frac{\varphi_{1}(z)}{z^{n+1}(x-z)} \left( \frac{\omega_{x} - 1}{\omega_{x}x - 1} \right)^{r+1} \, dz \right| \leq K \frac{n^{\frac{1}{2}}}{p_{1}} \left| \frac{m_{1}}{p_{1}} \right| \left( \frac{1}{\sqrt{n}} \right)^{n} \leq K \frac{n^{\frac{1}{2}}}{p_{1}} e^{-\sqrt{n}}.$$

And on $L_{1} + L_{1} + L_{1}$,

$$|\varphi_{1}(z)| < K |1 - \omega_{x}| \left| \frac{m_{1}}{p_{1}} \right|.$$
where we take \( r \) so large that \( \frac{m_i}{p_i} + r + 2 \geq 0 \) \((i=1, \ldots, k)\).

Consequently

\[
\frac{1}{2\pi i} \int_{\gamma_i} \frac{\varphi_i(z)}{z^{n+1}(x-z)} \left( \frac{\omega_i z - 1}{\omega_i x - 1} \right)^{r+1} \frac{dz}{x-z} = O \left( n^{-\frac{1}{2}} \left( \frac{m_i + r + 2}{p_i} \right) \right) \]

uniformly for \( 1 < r_0 \leq |x| \).

From (16) and (17) we get

\[
\frac{1}{2\pi i} \int_{\gamma_i} \frac{\varphi_i(z) + \cdots + \varphi_k(z)}{z^{n+1}(x-z)} \left( \frac{\omega_i z - 1}{\omega_i x - 1} \right)^{r+1} \frac{dz}{x-z} = O \left( n^{-\frac{1}{2}} \left( \frac{m_i + r + 2}{p_i} \right) \right)
\]

where we put

\[
\beta = \min \left( \frac{m_1}{p_1} + r + 2, \ldots, \frac{m_k}{p_k} + r + 2 \right).
\]

or

\[
\frac{1}{2\pi i} \int_{\gamma_i} \frac{\varphi_i(z) + \cdots + \varphi_k(z)}{z^{n+1}(x-z)} \frac{dz}{x-z} = \sum_{i=1}^{k} \frac{1}{\omega_i x - 1} \int_{\gamma_i} \frac{\omega_i \varphi_i(z)}{z^{n+1}} dz
\]

\[
+ \frac{1}{(\omega_i x - 1)^2} \int_{\gamma_i} \frac{\omega_i \varphi_i(z)(\omega_i z - 1)^{\beta}}{z^{n+1}} dz + \cdots + \frac{1}{(\omega_i x - 1)^{r+1}} \int_{\gamma_i} \frac{\omega_i \varphi_i(z)(\omega_i z - 1)^{r}}{z^{n+1}} dz
\]

\[
+ O \left( n^{-\frac{1}{2}} \beta \right).
\]

But \( \int_{\gamma_i} \frac{\omega_i \varphi_i(z)(\omega_i z - 1)^{\lambda}}{z^{n+1}} dz \) is the coefficient of \( z^n \) in the expansion of \( \omega_i \varphi_i(z)(\omega_i z - 1)^{\lambda} \) in the power series of \( z \) and since \( (1-z)^{\lambda} = \sum_{n=0}^{\infty} \binom{-\lambda + n - 1}{n} z^n \),

we have from (3) and (20).
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\[ (21) \quad \frac{1}{2\pi i} \int_{C} \frac{p_{1}(z) + \ldots + p_{k}(z)}{z^{n+1}(x-z)} \, dz \]

\[ = \sum_{i=1}^{k} \left( \frac{b_{0}(i) (-m_{1} - n + 1) + \ldots + b_{q}(i) (-m_{q} + q + n - 1)}{n \omega_{i}x - 1} + \ldots \right) \]

\[ + (-1)^{r} \frac{b_{0}(i) (-m_{1} + n - n - 1) + \ldots + b_{q}(i) (-m_{q} + q + n - n - 1)}{(n \omega_{i}x - 1)^{r+1}} \omega_{i}^{n+1} + O(n^{-\frac{1}{2} \beta}) \]

But since \(-\lambda + n - n - 1 \sim A_{i} n^{-\lambda - 1}\) we have

\[ (22) \quad \frac{1}{2\pi i} \int_{C} \frac{p_{1}(z) + \ldots + p_{k}(z)}{z^{n+1}(x-z)} \, dz = \sum_{i=1}^{k} \left( \frac{b_{0}(i) + \bar{E}_{1}(i)}{n \omega_{i}x - 1} + \frac{\bar{E}_{2}(i)}{(n \omega_{i}x - 1)^{2}} + \ldots + \frac{\bar{E}_{r+1}(i)}{(n \omega_{i}x - 1)^{r+1}} \right) \omega_{i}^{n+1} + O(n^{-\frac{1}{2} \beta}) \]

where \(\bar{E}_{j}(i) \to 0\) for \(n \to \infty\) \((i=1, 2, \ldots, k, j=1, 2, \ldots, r+1)\)

Let

\[ (23) \quad \mu = \text{Min}(\frac{m_{1}}{p_{1}}, \ldots, \frac{m_{k}}{p_{k}}) \]

Then we have

\[ (24) \quad \frac{1}{2\pi i} \int_{C} \frac{p_{1}(z) + \ldots + p_{k}(z)}{z^{n+1}(x-z)} \, dz = \left( -\mu + n - 1 \right) \left( \frac{c_{1} \omega_{1}^{n+1}}{n \omega_{1}x - 1} + \ldots + \frac{c_{k} \omega_{k}^{n+1}}{n \omega_{k}x - 1} \right) \]

\[ + \sum_{i=1}^{k} \left( \frac{\bar{E}_{1}(i)}{n \omega_{i}x - 1} + \ldots + \frac{\bar{E}_{r+1}(i)}{(n \omega_{i}x - 1)^{r+1}} \right) \right) + O \left( n^{-\frac{1}{2} \beta} \right) \left( \bar{E}_{j}(i) \to 0 \text{ for } n \to \infty \right) \]

Some of \(c_{1}, \ldots, c_{k}\) may tend to zero for \(n \to \infty\).

Now we take \(q\) and \(r\) so large that

\[ (25) \quad \frac{1}{2} \alpha \geq k + \mu + 1, \quad \frac{1}{2} \beta \geq k + \mu + 1. \]

Then we have from (15), (24), and (25),

\[ (26) \quad f_{n}(x) = x^{n+1} \left\{ \left( -\mu + n - 1 \right) \left[ \frac{c_{1} \omega_{1}^{n+1}}{n \omega_{1}x - 1} + \ldots + \frac{c_{k} \omega_{k}^{n+1}}{n \omega_{k}x - 1} \right] \right. \]

\[ + \sum_{i=1}^{k} \left( \frac{\bar{E}_{1}(i)}{n \omega_{i}x - 1} + \ldots + \frac{\bar{E}_{r+1}(i)}{(n \omega_{i}x - 1)^{r+1}} \right) \right) + O \left( n^{-\left( k + \mu + 1 \right)} \right) \text{ uniformly for } 1 < r_{0} \leq |x|, \]

where \(\bar{E}_{j}(i) \to 0\) for \(n \to \infty\) \((i=1, \ldots, k, j=1, \ldots, r+1)\).
4. We will prove the following lemma due to Prof. Kakeya.

**Lemma (Kakeya)** Let \( p(x) = a_0x^n + \cdots + a_n \)

\[
|a_0| + |a_1| + \cdots + |a_n| \geq G > 0
\]

and \( x_1, \ldots, x_n \) be the roots of \( p(x) = 0 \) which are so arranged that

\[
|x_1| \leq |x_2| \leq \cdots \leq |x_s| < 2R \leq |x_{s+1}| \leq \cdots \leq |x_n|.
\]

Then for \( |x| \leq R \), we have

\[
|p(x)| \geq E |(x-x_1) \cdots (x-x_s)|
\]

where \( E \) is a positive constant depending only on \( G \) and \( R \).

**Proof.**

Let \( p(x) = a_0(x-x_1) \cdots (x-x_n) \).

Now suppose \( |a_0(x-x_{s+1}) \cdots (x-x_n)| < \varepsilon \), then a fortiori

\[
|a_0| (|x_{s+1}|-R) \cdots (|x_n|-R) < \varepsilon.
\]

But

\[
|x_{s+1}|-R \geq \frac{|x_{s+1}|}{3} + \frac{R}{3}.
\]

Hence

\[
|a_0| (R+|x_{s+1}| \cdots (R+|x_n|) < 3^{n-1} \varepsilon \leq 3^n \varepsilon.
\]

Consequently if by taking \( a_0, \ldots, a_n \) suitably, \( \varepsilon \) can be made very small, then the coefficients of \( a_0(x-x_{s+1}) \cdots (x-x_n) \) become very small, or the coefficients of \( p(x) \) become very small, which contradicts the hypothesis

\[
|a_0| + |a_1| + \cdots + |a_n| \geq G > 0.
\]

Hence there is a positive lower bound \( E \) for \( |a_0(x-x_{s+1}) \cdots (x-x_n)| \)

which proves the lemma.

6. Proof of the Theorem.

Now we put

\[
\tau_n(x) = \frac{c_1a_1^{n+1}}{\omega_1x-1} + \cdots + \frac{c_ka_k^{n+1}}{\omega_kx-1} + \sum_{i=1}^{k} \left( \frac{E_i^{(i)}}{(\omega_i x-1)^{n+1}} + \frac{E_i^{(n+1)}}{(\omega_i x-1)^{n+2}} \right)
\]

and

\[
\sigma_n(x) = \frac{c_1a_1^{n+1}}{\omega_1x-1} + \cdots + \frac{c_ka_k^{n+1}}{\omega_kx-1} = \frac{A_1(a^{(n)}x^{n-1} + \cdots + A_k(a^{(n)}x^{n-1})}{(\omega_1 x-1) \cdots (\omega_k x-1)}.
\]

Then

\[
\tau_n(x) = \frac{(A_1(a^{(n)}x^{n-1} + \cdots + A_k(a^{(n)}x^{n-1}))}{(\omega_1 x-1) \cdots (\omega_k x-1)} + \sum_{i=1}^{k} B_i^{(n)} x^i
\]

and

\[
\sigma_n(x) = \sum_{i=1}^{k} c_i(a^{(n)}x^i + \cdots + D_i(a^{(n)}x^i) + \sum_{i=1}^{k} B_i^{(n)} x^i
\]

Thus

\[
\frac{\tau_n(x)}{[\prod_i (\omega_i x-1)]^{n+1}} = \frac{\sum c_i(a^{(n)}x^i + \cdots + D_i(a^{(n)}x^i) + \sum_{i=1}^{k} B_i^{(n)} x^i}{[\prod_i (\omega_i x-1)]^{n+1}}
\]
where \(B_i^{(n)} \to 0\) for \(n \to \infty\), and we put
\[
(A_1^{(n)}x^{k-1} + \ldots + A_k^{(n)})[(\omega_1x-1)\ldots(\omega_n-1)]^r = \sum D_i^{(n)}x^i.
\]

First we will prove that
\[
\sum_i |c_i^{(n)}| \geq G > 0,
\]
where \(G\) is independent of \(n\).

For that, it suffices to prove that
\[
\sum_i |D_i^{(n)}| \geq G > 0,
\]
since \(B_i^{(n)} \to 0\) for \(n \to \infty\).

If \(\sum_i |D_i^{(n)}|\) can be made very small by taking \(n\) suitably, then
\[
|\sigma_n(x)| \text{ becomes very small uniformly for } 1 < r_0 \leq |x| \leq R.
\]
But as I have proved before,\(^{(3)}\) for \(k\) distinct points \(x_1, \ldots, x_k\) which lie outside \(|x|=1\),
\[
|\sigma_n(x_1)| + |\sigma_n(x_2)| + \ldots + |\sigma_n(x_k)| \geq \epsilon_0 > 0
\]
where \(\epsilon_0\) is independent of \(n\), which contradicts the hypothesis that \(|\sigma_n(x)|\) becomes very small, hence there exists a positive constant \(G\) such that
\[
\sum_i |c_i^{(n)}| \geq G > 0.
\]

By the lemma above proved,
\[
|\tau_n(x)| \geq E |(x-x_1^{(n)})\ldots(x-x_s^{(n)})| \geq E |x-x_1^{(n)}\ldots(x-x_s^{(n)})| \text{ uniformly for } 1 < r_0 \leq |x| \leq R,
\]
where \(x_1^{(n)}, \ldots, x_s^{(n)}\) are the roots of \(\sum_i c_i^{(n)}x^i = 0\), which lie in the circle \(|x|=2R\). From \(B_i^{(n)} \to 0\) and \(\sum_i |D_i^{(n)}| \geq G\), the roots \(x_1^{(n)}\ldots x_s^{(n)}\) are approximately equal to the roots of \(\sum_i D_i^{(n)}x^i = 0\), which lie in this circle.

And since \(\sum_i D_i^{(n)}x^i = 0\) has \(r\)-ple roots \(1/\omega_i\) \((i=1, \ldots, k)\), for roots of \(\sum_i c_i^{(n)}x^i = 0\) lie in the circle \(|x| = \frac{1+r_0}{2}\) for \(n \geq n_0\), and since for such root \(x^{(n)}\), and \(|x| \geq r_0\), \( |x-x^{(n)}| > \frac{1+r_0}{2} \), we get,

\(^{(3)}\) M. Tsuji, this journal 3 (1926) 80.
(29) \(|\tau_n(x)| \geq E| (x-x_1^{(n)})...(x-x_{o-1}^{(n)})| \) uniformly for \(1 < r_0 \leq |x| \leq R\), where \(E\) is a constant and \(x_1^{(n)}...,x_{o-1}^{(n)}\) are the roots of \(\sum_i c_i^{(n)}x^i = 0\) which lie in the ring domain \(1 < \frac{1+r_0}{2} \leq |x| < 2R\). Since \(\sum_i D_i^{(n)}x^i = 0\) or \(\sigma_n(x) = 0\) has at most \(k-1\) roots in this domain, we have
\[
(30) \quad \sigma \leq k-1 \quad \text{for} \quad n \geq n_0.
\]
Now let the roots \(x_1^{(n)}...,x_{o-1}^{(n)}\) be marked on the complex plane, and draw circles about them with a radius \(\frac{\varepsilon}{\sqrt{n}}\), then the sum of areas of these circles is \(\frac{\varepsilon^2 \pi}{n^2}\).

Hence the total sum of the areas of such circles is \(\frac{\varepsilon^2 \pi}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\) which can be made very small by taking \(\varepsilon\) sufficiently small. For a small \(\varepsilon\), there exist surely points in the region \(1 < r_0 \leq |x| \leq R\) which lie outside of all these circles.

We denote the point-set of such points by \(E_\varepsilon\), then from (29), for \(n \geq n_0\), we have
\[
(31) \quad |\tau_n(x)| \geq E \frac{\varepsilon^o}{\sigma^o n^o} \geq H \frac{1}{n^{k-1}} \quad \text{uniformly in} \quad E_\varepsilon,
\]
where \(E\) and \(H\) are absolute constants.

From (26) (27) (31), we have
\[
(32) \quad |f_n(x)| \geq \left(\frac{-\mu n+1}{n}\right) |\tau_n(x)| - \left(\frac{1}{n^{k+\mu+1}}\right)|x|^{n+1}
\]
Since
\[
(33) \quad |f_n(x)| \geq K \left(\frac{1}{n^{k+\mu+1}}\right)|x|^{n+1} \quad \text{uniformly in} \quad E_\varepsilon.
\]
Consequently
\[
(34) \quad \lim_{n \to \infty} f_n(x) = \infty \quad \text{uniformly in} \quad E_\varepsilon.
\]
By making \(\varepsilon \to 0\), we see that \(\lim_{n \to \infty} f_n(x) = \infty\) almost everywhere \(1 < r_0 \leq |x| \leq R\) and since \(r_0\) and \(R\) are arbitrary, \(\lim_{n \to \infty} f_n(x) = \infty\) almost everywhere \(1 < |x|\) which proves the Theorem.
That \( \lim_{n \to \infty} f_n(x) = \infty \) uniformly in \( E \), reminds us of a theorem of Egoroff \((4)\) that if the measurable functions \( f_n(x) \) converge in a point-set \( E \), then we can find a sub-set \( E_1 \) of \( E \), whose measure differs from that of \( E \) arbitrarily small, and such that in \( E_1 \), \( f_n(x) \) converge uniformly.

**Remark.** (A) By the entirely similar way we can prove the theorem for functions with only algebraico-logarithmic singular points on \( |x| = 1 \).

(B) From (26) we can easily prove that

\[
\lim_{n \to \infty} \left( |f_n(x_1)| + \ldots + |f_n(x_k)| \right) = \infty
\]

where \( x_1, \ldots, x_k \) are any \( k \) points outside \( |x| = 1 \).

\[
\lim_{n \to \infty} \left( |f_n(x)| + |f_{n+1}(x)| + \ldots + |f_{n+k-1}(x)| \right) = \infty
\]

uniformly for \( 1 < r_0 \leq |x| \).

Which are the principal results in my previous paper \((5)\)

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\((4)\) C. R. Vol CLII (1911) 244.

\((5)\) M. Tsuji, this journal 3 (1926) 69-85.