22. *On Steiner's Curvature-centroid.*

By Buchin Su, Sendai.

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1. When a closed convex curve is defined by the equation :

$$x=x(\theta), \quad y=y(\theta),$$

where θ stands for the angle between the x -axis and the tangent of the curve, the point with coordinates

$$x_1 = \frac{1}{2\pi} \int_0^{2\pi} x(\theta) d\theta, \quad y_1 = \frac{1}{2\pi} \int_0^{2\pi} y(\theta) d\theta$$

is called by Steiner⁽¹⁾ "Krümmungsschwerpunkt" of the curve, which Prof. T. Hayashi⁽²⁾ rendered into "the curvature-centroid" of the curve.

An interesting property of the curvature-centroid has recently been proved by Prof. M. Fujiwara: When a curve E of constant breadth revolves in a square kept in contact with all sides, the area of the curve described by a fixed point P in E becomes minimum when P is the curvature-centroid of E . He further proposed the question: Does the same fact exist in the case of an in-revolvable curve E of a polygon? In the following lines I will answer this question by establishing a general theorem, which contains the results due to Prof. Fujiwara as a special case.

2. Given two straight lines A, B with an angle α in a plane and an oval E revolves touching the straight lines. If we consider a point P in the domain of the curve, then P describes certain curve (closed) when the oval revolves touching A, B and finally returns to its original position. Our theorem runs as follows:

The area $A(P)$ of the curve described by P becomes minimum when and only when P is Steiner's curvature-centroid of the oval.

More generally, we have :

(¹) J. Steiner, Von dem Krümmungsschwerpunkt ebener Kurven, Crelle's Journ. **21** (1838).

(²) T. Hayashi, On Steiner's curvature-centroid, Science Reports, Tôhoku Imp. Univ. (I) **13** (1924—25).

The locus of P for which $A(P)$ is constant is a circle with steiner's curvature-centroid as its centre.

Proof. Take one of the two straight lines and its orthogonal one as y -axis and x -axis respectively. Let the oval take its original position E and let P be a fixed point in E . Rotate E to the position E' so that the tangent g of E coincides with y -axis and P becomes P' as shown in

the figure. If the polar tangential equation of E with regard to P as the origin be $p=p(\theta)$, then from the relative position of E' and P' , it follows :

$$P'M'=p(\theta), \quad P'Q=p(\theta+\alpha'), \quad (\alpha'=\pi-\alpha),$$

where $P'M'$, $P'Q$ are two perpendiculars drawn from P' to the given lines respectively. If we take the parallel straight lines of the axes passing through P as new axes, it may be easily seen :

$$(1) \quad \begin{cases} x=P'M=p(\theta)-r\sin\beta, \\ y=P'N=p(\theta+\alpha')/\sin\alpha+p(\theta)/\tan\alpha-r\cos\beta, \end{cases}$$

where $OP=r$, and β stands for the angle between OP and y -axis.

From (1) the area $A(P)$ is given by

$$\begin{aligned} A(P) &= \frac{1}{2} \oint y dx - x dy \\ &= \frac{1}{2} \int_0^{2\pi} (p(\theta+\alpha')/\sin\alpha + p(\theta)/\tan\alpha - r\cos\beta) \cdot p'(\theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2\pi} (p'(\theta)/\tan\alpha + p'(\theta+\alpha')/\sin\alpha) (p(\theta) - r\sin\beta) d\theta, \quad \text{or} \\ (2) \quad 2\sin\alpha \cdot A(P) &= - \int_0^{2\pi} p'(\theta+\alpha') \cdot p(\theta) d\theta + \int_0^{2\pi} p(\theta+\alpha') \cdot p'(\theta) d\theta, \end{aligned}$$

since

$$\int_0^{2\pi} p'(\theta) d\theta = 0, \quad \int_0^{2\pi} p'(\theta + \alpha') d\theta = 0, \quad \int_0^{2\pi} p(\theta) \cdot p'(\theta) d\theta = 0.$$

Expanding $p(\theta)$ in the Fourier's series :

$$(3) \quad p(\theta) \sim a_0/2 + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta),$$

we obtain

$$(4) \quad p(\theta + \alpha') \sim a_0/2 + \sum_{k=1}^{\infty} (\alpha_k \cos k\theta + \beta_k \sin k\theta),$$

where

$$(5) \quad \begin{aligned} \alpha_k &= a_k \cos k\alpha' + b_k \sin k\alpha', \\ \beta_k &= b_k \cos k\alpha' - a_k \sin k\alpha'. \end{aligned}$$

Hence it follows :

$$(6) \quad p'(\theta) \sim \sum_{k=1}^{\infty} k(b_k \cos k\theta - a_k \sin k\theta),$$

$$(7) \quad p'(\theta + \alpha') \sim \sum_{k=1}^{\infty} k(\beta_k \cos k\theta - \alpha_k \sin k\theta).$$

Making use of a theorem of Hurwitz's,⁽³⁾ we have

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} p(\theta + \alpha') \cdot p'(\theta) d\theta &= \sum_{k=1}^{\infty} k(b_k \alpha_k - a_k \beta_k), \\ \frac{1}{\pi} \int_0^{2\pi} p'(\theta + \alpha') \cdot p(\theta) d\theta &= \sum_{k=1}^{\infty} k(a_k \beta_k - b_k \alpha_k), \end{aligned}$$

whence

$$(8) \quad \frac{\sin \alpha}{\pi} \cdot A(P) = \sum_{k=1}^{\infty} k(\alpha_k b_k - \beta_k a_k) = \alpha_1 b_1 - \beta_1 a_1 + \sum_{k=2}^{\infty} k(\alpha_k b_k - \beta_k a_k).$$

From (5), we get

$$\alpha_1 b_1 - \beta_1 a_1 = \sin \alpha (a_1^2 + b_1^2),$$

while the quantity

$$C = \sum_{k=2}^{\infty} k(\alpha_k b_k - \beta_k a_k),$$

does not depend upon the position of P , since a_k, b_k ($k=2, 3, \dots$) are the same wherever P be taken. Thus we get finally

⁽³⁾ A. Hurwitz, Sur quelques applications géométriques des séries de Fourier, *Annales de l'École Normale Supérieure*, Ser. 3, **19** (1902).

$$(9) \quad A(P) = \pi(a_1^2 + b_1^2) + C\pi/\sin\alpha.$$

Therefore $A(P)$ is minimum when and only when

$$(10) \quad a_1 = 0, \quad b_1 = 0,$$

which shows nothing but the fact that the point P is Steiner's curvature-centroid of the oval⁽⁴⁾. Thus the first part of our theorem is completely demonstrated.

From (9) it is evident that if $A(P)$ is constant, $a_1^2 + b_1^2 = \text{const.}$, which proves the second part of our theorem.

As a special case we obtain:

When an in-revolvable curve E revolves in certain polygon kept in contact with all the sides of the polygon⁽⁵⁾, the area of the curve described by a point P fixed in the domain of E becomes minimum when and only when P is Steiner's curvature-centroid of E .

3. The above method suggests us an application for some integrals with respect to the oval. For example, taking an inner point P of the oval E , we may show that the integral

$$I(P) = \int_0^{2\pi} p^{(\mu)}(\theta + \alpha) \cdot p^{(\nu)}(\theta + \beta) d\theta, \quad (\mu, \nu = 0, 1, 2, \dots)$$

when $p = p(\theta)$ is the polar tangential equation of E with regard to P , is minimum when and only when P is Steiner's curvature-centroid of E .

Especially, when $\mu = 0, \nu = 0, \alpha = 0, \beta = 0$, we have that the area of the pedal curve of E with respect to the point P becomes minimum when P is Steiner's curvature-centroid⁽⁶⁾.

Further if $\mu = \nu = 1, \alpha = \beta = 0$, then

$$I(P) = \int_0^{2\pi} p'^2(\theta) d\theta.$$

Thus we have: The area of the pedal curve of the involute of E becomes minimum when P is Steiner's curvature-centroid⁽⁷⁾.

4. If the polar tangential equation of the oval referring to the point 0 (fig.) be

$$(11) \quad p = p(\theta) \sim a_0/2 + \sum_{k=3}^{\infty} (a_k \cos k\theta + b_k \sin k\theta),$$

(4) T. Kubota, Tôhoku Math. Journ. **14** (1918).

(5) M. Fujiwara, Science Reports, Tôhoku Imp. Univ. **4** (1914).

(6) J. Steiner, loc. cit.

(7) T. Hayashi, loc. cit.

then the equation referring to the point P becomes

$$(12) \quad \bar{p} = p(\theta) - r \sin(\theta - \beta),$$

and the coordinates of point P' (x, y) are

$$(13) \quad \begin{cases} x = p(\theta) - r \sin(\theta - \beta), \\ y = (p(\theta + \pi - \alpha) - r \sin(\alpha + \beta - \theta)) / \sin \alpha + (p(\theta) - r \sin(\theta - \beta)) / \tan \alpha, \end{cases}$$

which may be written in the form

$$(14) \quad \begin{cases} x \sim a_0/2 + (a_1 + r \sin \beta) \cos \theta + (b_1 - r \cos \beta) \sin \theta \\ \quad + \sum_{k=2}^{\infty} (a_k \cos k\theta + b_k \sin k\theta), \\ y \sim (a_0/2) \cot \frac{\alpha}{2} + (b_1 - r \cos \beta) \cos \theta + (-a_1 - r \sin \beta) \sin \theta \\ \quad + \sum_{k=2}^{\infty} \{ (\alpha_k / \sin \alpha + a_k / \tan \alpha) \cos k\theta + (\beta_k / \sin \alpha + b_k / \tan \alpha) \sin k\theta \}, \end{cases}$$

where

$$(15) \quad \begin{cases} \alpha_k = a_k \cos k(\pi - \alpha) + b_k \sin k(\pi - \alpha), \\ \beta_k = b_k \cos k(\pi - \alpha) - a_k \sin k(\pi - \alpha). \end{cases}$$

If we determine r and β so as to satisfy the equations

$$(16) \quad \begin{cases} b_1 - r \cos \beta = (a_1 + r \sin \beta) \cot(\alpha/2), \\ -a_1 - r \sin \beta = (b_1 - r \cos \beta) \cot(\alpha/2), \end{cases}$$

then we have

$$(17) \quad y - x \cot(\alpha/2) \sim \sum_2^{\infty} (\alpha'_k \cos k\theta + \beta'_k \sin k\theta),$$

where α'_k, β'_k are expressible by a_k, b_k and α .

Applying a theorem due to Hurwitz⁽⁸⁾, we get that the equation

$$y - x \cot(\alpha/2) = 0$$

has at least four roots. But relation (16) gives

$$\tan \beta = -a_1/b_1, \quad r = \sqrt{a_1^2 + b_1^2}.$$

Therefore we have the following result:

The curve described by Steiner's curvature-centroid has at least four points of intersection with the bisector of the given straight lines A, B .

(8) A. Hurwitz, Über die Fourierschen Konstanten integrierbarer Funktionen, Math. Ann. 57, (1903), 425-446, especially p. 444.

Combining this result with the former theorem, we see that if the curve described by a fixed point in the oval has only two points of intersection or none with the bisector of the given straight lines, then it cannot assume a minimum area, provided the curve does not reduce to a point. Especially we get:

If the curve described by a fixed point is an oval⁽⁹⁾, then it cannot assume a minimum area, provided the curve is not a point.

If the curve described be a single point, then from (14), we have

$$r\sin\beta = -a_1, \quad r\cos\beta = b_1; \quad a_k = 0, \quad b_k = 0 \quad (k=2, 3, \dots)$$

i.e. the oval must be a circle, while the point is its centre as a matter of course.

5. For Steiner's curvature-centroid the relation (14) may be written in the form

$$x - a_0/2 \sim \sum_{k=2}^{\infty} (a_k \cos k\theta + b_k \sin k\theta),$$

$$y - a_0/2 \cot(\alpha/2) \sim \sum_{k=2}^{\infty} \{ (a_k/\sin\alpha + a_k/\tan\alpha) \cos k\theta + (\beta_k/\sin\alpha + b_k/\tan\alpha) \sin k\theta \};$$

whence follows that the equation

$$x - a_0/2 - m(y - (a_0/2) \cot \frac{\alpha}{2}) = 0$$

has at least four roots, m being taken arbitrary. But

$$a_0 = L/\pi,$$

where L is the curve length of the oval. Hence we arrive at the result:

If P_0 be a point on the bisector of A, B in the domain, where the oval lies and with the distance $L/(2\pi \sin \frac{\alpha}{2})$ from origin, then any straight line passing through P_0 has at least four points of intersection with the curve described by Steiner's curvature-centroid, provided the latter is not a single point.

This result obviously contains the former one as a special case.

6. Let us apply the result in the last paragraph to the case of in-revolvable curve of a certain polygon. It was shown by Prof. Fujiwara⁽⁵⁾ that for the existence of such curves every side of the polygon

(⁹) Here we exclude the case for which the locus of curvature-centroids is a doubly superposing curve as in the case of central ovals.

necessarily touches a circle i.e. the in-circle of the polygon ; and further the length of inrevolvable curve is always equal to $L=2\pi R$, where R is the radius of the in-circle.

Therefore we see that the point P_0 in § 6 becomes the centre of the in-circle i.e. the in-centre of the polygon. Thus we get:

If Σ be the polygon for which the in-revolvable curve exists, then any straight line passing through the in-centre of Σ has at least four points of intersection with the curve described by Steiner's curvature-centroid of the in-revolvable curve of Σ .