30. *On Steiner's Curvature-centroid, II.*

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1. Making use of convex polygons, Steiner⁽¹⁾ deduced some properties of the oval concerning the curvature-centroid. One of his interesting results is: When an oval rolls on a fixed straight line until it returns to the original position of contact, then the area $A(P)$, described by the radius vector joining the point of contact and the point P , fixed in the plane of the oval, becomes minimum when and only when P is the curvature-centroid of the oval; further the locus of P for which $A(P)$ is constant is a circle with the curvature-centroid as its centre.

On the other hand, I have found, in my former paper⁽²⁾ under the same title, that somewhat similar properties exist when the oval touches two fixed straight lines and rolls about them. It, seems therefore, to me that in whatever manner the oval rolls or revolves on a fixed straight line kept in contact with the latter until it returns to the original point of contact, the same fact would be true. From this standpoint, I will first prove the following theorems:

(1) When the oval revolves in such a manner that it is always kept in contact with a fixed straight line L at a fixed point Q , then the area $F(P)$ described by the radius vector joining Q and a point P fixed in the plane of the oval becomes minimum when and only when P is Steiner's curvature-centroid.

(2) The locus of P for which $F(P)$ is constant is a circle with the curvature-centroid as its centre.

These theorems can be proved by using Fourier's series as in my former paper. But here we will establish them by the elegant method due to Steiner. In other words, we have to show the facts above quoted for the convex polygon in the place of the oval, since the latter may be approximated by the former.

In fact, consider a convex polygon $ABCD$ which revolves in such a manner that it always touches the straight line L at a point Q .

(¹) J. Steiner, Werke II, 99—159.

(²) B. Su, this Journal, 4 (1927), 195—201.

Let P be a fixed point in the plane of the polygon; $PA=a$, $PB=b$, $PC=c$, $PD=d$; and let the adjacent angles of $ABCD$ be $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ respectively.

First we consider the position where D coincides with Q and the side DA lies on L . Translate DA from right to left till A coincides with Q , then P moves to P_1 and PP_1 is parallel to L , while $ABCD$ comes to the position $A_1(=Q)$, B_1 , C_1 , D_1 . The area APP_1 is easily seen to be equal to the area PAD . Now turn the polygon about Q till A_1B_1 coincides with L so that P_1 moves on a circular arc with radius a and centre Q and the central angle $\hat{A}$, and $A_1B_1C_1D_1$ comes to the position $A_2(=Q)$, B_2 , C_2 , D_2 . If we joint P_2 and B_2 and operate in quite a similar manner for the polygon $A_2B_2C_2D_2$, etc. till the polygon returns to its original position, we finally get

$$(1) \quad F(P) = V + \frac{1}{2} \sum a^2 \hat{A},$$

where V is the area of the polygon.

Let S denote the point of mean distance of A, B, C, D , with coefficients $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ and let

$$PS=s, \quad SA=a_1, \quad SB=b_1, \quad SC=c_1, \quad SD=d_1.$$

Then the area described by S is

$$F(S) = V + \frac{1}{2} \sum a_1^2 \hat{A}.$$

But after Steiner⁽³⁾ we have

$$(2) \quad \sum a^2 \hat{A} = \sum a_1^2 \hat{A} + s^2 (\sum \hat{A}).$$

(³) Steiner, loc. cit, p. 110.

From (1) and (2), we get

$$(3) \quad F(P) = F(S) + \pi s^2.$$

Therefore

$$F(P) \geq F(S),$$

where the equality sign holds good when and only when P coincides with S .

If the number of sides of the polygon increases indefinitely, while their length tends to zero, then the polygon tends to a convex oval, whence follow our theorems.

From (3), we further obtain:

The area $F(P)$ described by the radius vector QP is greater than the minimum area $F(S)$ by the area of a circle with centre S and radius PS .

The area $F(P)$ is equal to that described by the radius vector joining the point of contact and P when the oval rolls on the line until it returns to its original point of contact.

N.B. Using Fourier's series we can show that the area of the closed curve described by point P becomes minimum when and only when P is the curvature-centroid of the oval. The locus of P for which the area is constant is a circle with the curvature-centroid as its centre.

2. For the sake of brevity, we call the motion of the oval in the last paragraph "slip at P ". We will now prove the theorem:

Let the oval roll on a fixed line L and slip at any point or points of L , the combination of rolling and slipping being arbitrary, until it turns to its original point of contact. Then (1) the area described by the radius vector drawn from the point of contact to any fixed point P in the plane of the oval becomes minimum when and only when P is the curvature-centroid of the oval; (2) the locus of P for which $F(P)$ is constant is a circle with the curvature-centroid as its centre.

We will demonstrate these theorems for the case when the oval subjects to only one time of slipping and only one time of rolling. For the other cases, they can be proved *mutatis mutandis*.

Consider, as former, a n -sided convex polygon $A_1A_2\dots A_kA_{k+1}\dots A_nA_{n+1}(=A_1)$ which is supposed to slip from A_1 to A_k and to roll from A_k to A_{n+1} . Then from Steiner's and above results, the area described is

$$F(P) = PA_1A_k + \frac{1}{2} \sum_{i=1}^k a_i^2 \hat{A}_i + PK_kA_{n+1} + \frac{1}{2} \sum_{i=k+1}^n a_i^2 \hat{A}_i,$$

or

$$(4) \quad F(P) = F(S) + \pi s^2.$$

Making n tend to ∞ , we get the results immediately.

3. An oval E rolls kept in contact with another fixed oval E_1 or a convex curve in the outside until E returns to the original point of contact. If the corresponding radius of curvature at the point of contact be $1/\rho$, $1/\rho_1$ respectively, then the generalized curvature-centroid is defined by Steiner as the centroid of the masses distributed at the point of contact by the quantity $1/\rho + 1/\rho_1$.

Now if an oval E rolls and slips on the outside of a fixed oval E_1 , we modify the generalized curvature-centroid i.e. the mass of E is $1/\rho$ or $1/\rho + 1/\rho_1$ according as the oval in motion slips or rolls. The curvature-centroid thus obtained is defined as "mixed".

By a similar method as in last paragraph, we can prove:

(1) When the oval E rolls on a fixed oval E_1 or a part of it and slips at any point or points, the combination of rolling and slipping being arbitrary, until it returns to its original point of contact, then the area described by the radius vector drawn from the point of contact and a fixed point P in the plane of the oval E becomes minimum when and only when P is the mixed curvature-centroid of E .

(2) The locus of P for which $F(P)$ is constant is a circle with the mixed curvature-centroid as its centre.

4. Since every motion of an oval can be considered as combinations of slipping and rolling, provided the oval is kept in contact with a fixed convex curve, we get:

(1) In whatever manner the oval moves kept in contact with a given line (or a convex curve) until it returns to the original point of contact, the area $A(P)$ described by the radius vector joining the point of contact and a fixed point P in the plane of the oval becomes minimum when and only when P is the curvature-centroid (or the mixed curvature-centroid) of the oval.

(2) The locus of P for which $A(P)$ is constant is a circle with the curvature-centroid (or the mixed curvature-centroid) as its centre.

Especially, we get the following case.

If an oval moves in a plane kept in contact with two fixed circles

of equal radius, until it returns to its original position, then the area $A(P)$ described by the radius vectors, which are drawn from a fixed point P in the plane of the oval to the points of contact with the circles, becomes minimum when and only when P is the generalized curvature-centroid of the oval. The locus of P for which $A(P)$ is constant is a circle with the generalized curvature-centroid of the oval as its centre.