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Tangent Coalgebras and Hyperalgebras I

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Contents

Introduction

This paper presents some applications of the theory of coalgebras and Hopf algebras to the theory of locally algebraic schemes and groups over a

field k. Our starting point is to define *the tangent coalgebra* $\mathbf{T}_x(\mathfrak{X})$ to a locally algebraic k-scheme $\mathfrak{X}$ at a point x of $\mathfrak{X}$. The functor $(\mathfrak{X}, x) \mapsto \mathbf{T}_x(\mathfrak{X})$ permits us to compare the two languages of schemes and coalgebras. A property $\mathscr{P}$ of morphisms of locally algebraic k-schemes is said to be translatable into the language of coalgebras if there exists a property $\bar{\mathscr{P}}$ of morphisms of coalgebras such that a morphism $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ of locally algebraic k-schemes satisfies the property $\mathscr{P}$ at a closed point x of $\mathfrak{X}$ iff the induced map $\mathbf{T}_x(\mathfrak{f}): \mathbf{T}_x(\mathfrak{X}) \to \mathbf{T}_y(\mathfrak{Y})$, where $y = \mathfrak{f}(x)$, satisfies the property $\bar{\mathscr{P}}$. The flatness [7, I, § 2, 2.4] is an example of translatable properties. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two properties of morphisms of locally algebraic k-schemes translatable into the coalgebra language. Then the equivalence of their translations $\bar{\mathscr{P}}_1$ and $\bar{\mathscr{P}}_2$ implies of course the equiqalence of $\mathscr{P}_1$ and $\mathscr{P}_2$. In this way, among other things, we can give another proof of Theorem of smoothness [7, I, § 4, 4.2].

The tangent coalgebra $\mathbf{T}_e(\mathfrak{G})$ to a locally algebraic k-group $\mathfrak{G}$ at unit $e \in \mathfrak{G}(k)$ has a natural structure of *hyperalgebra* and is called *the hyperalgebra of* $\mathfrak{G}$ and denoted by $\mathbf{hy}(\mathfrak{G})$, where the word 'hyperalgebra' is a synonym of 'connected cocommutative Hopf algebra'. The Lie algebra $\mathrm{Lie}(\mathfrak{G})$ of $\mathfrak{G}$ [7, II, § 4, 4.8] is nothing else than the primitive elements $\mathrm{P}(\mathbf{hy}(\mathfrak{G}))$ of $\mathbf{hy}(\mathfrak{G})$. Our main task in this connection is to establish the hyperalgebra theory of locally algebraic k-groups which is analogous to the Lie algebra theory of algebraic groups over a field of characteristic 0.

The purpose of this paper is to develop some of the basic theory of tangent coalgebras and hyperalgebras. The reader is expected to be familiar with the languages of coalgebras and schemes. We refer to [11] for the former and to [7] for the latter.

This paper divides into three Chapters. Chapter 1 is a collection of algebraic preliminaries and complements to the theory of coalgebras and Hopf algebras and is free from the scheme theory. We introduce some new concepts as follows: the categories $\boldsymbol{W}_R$, where $R \in \boldsymbol{M}_k$, and $\boldsymbol{W}$, set- or group-functors on a category $\boldsymbol{A}$, the largest sub-Hopf algebra of a cocommutative bialgebra, the Krull dimension of a coalgebra, the multiplicative Birkhoff-Witt hyperalgebras, the smoothness of a map of coalgebras, the reduced part of a hyperalgebra, actions of hyperalgebras on hyperalgebras and so on. We set up a coalgebra theoretical version of the theorem of smoothness and characterize the Birkhoff-Witt coalgebras by the extension property of sequences of divided powers. We show that a finitely generated commutative A-algebra B, where A is assumed to be a finitely generated commutative k-algebra, is flat (over A) iff B^0 is an injective A^0-comodule.

Chapter 2 deals with the theory of *tangent coalgebras* and *underlying coalgebras*.

The tangent coalgebra to a k-scheme $\mathfrak{X}$ at a point x is by definition $(\mathscr{O}_x)^0$,

the dual k-coalgebra of the fibre $\mathcal{O}_x$ over x of the structure sheaf $\mathcal{O}_{\mathfrak{X}}$. The underlying coalgebra of $\mathfrak{X}$, written $\mathbf{T}(\mathfrak{X})$, is $\bigoplus_{x \in \mathfrak{X}} \mathbf{T}_x(\mathfrak{X})$ the direct sum of all tangent coalgebras. For instance if $\mathfrak{X} = \mathfrak{Sp}\, A$ the affine scheme of A ($\in M_k$), then $\mathbf{T}(\mathfrak{X}) = A^0$. We show that the restriction of the functor $\mathbf{T}(-) : \mathfrak{X} \mapsto \mathbf{T}(\mathfrak{X})$ to the full subcategory of locally algebraic k-schemes satisfies the following properties: Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a map of locally algebraic k-schemes. Then $\mathfrak{f}$ is a monomorphism iff the induced map $\mathbf{T}(\mathfrak{f}) : \mathbf{T}(\mathfrak{X}) \to \mathbf{T}(\mathfrak{Y})$ is injective (as a map of sets). If $\mathfrak{f}$ is faithfully flat then $\mathbf{T}(\mathfrak{f})$ is surjective. If $\mathfrak{f}$ is quasi-compact and $\mathbf{T}(\mathfrak{f})$ is bijective, then $\mathfrak{f}$ is an isomorphism. Let $\mathfrak{f}, \mathfrak{g} : \mathfrak{X} \rightrightarrows \mathfrak{Y}$ be two maps of locally algebraic k-schemes. If $\mathbf{T}(\mathfrak{f}) = \mathbf{T}(\mathfrak{g})$ then $\mathfrak{f} = \mathfrak{g}$. $\mathbf{T}(-)$ converts the Frobenius map $\mathfrak{F} : \mathfrak{X} \to \mathfrak{X}^{(p)}$ [7, II, § 7, 1.1] into the $\mathscr{V}$-map of $\mathbf{T}(\mathfrak{X})$ [9, § 4.1].

Further we translate some concepts such as the local dimension $\dim_x \mathfrak{X}$, flatness or *smoothness* into the language of coalgebras. When a k-group $\mathfrak{G}$ acts on a locally algebraic k-scheme $\mathfrak{X}$ and stabilities a rational point $e \in \mathfrak{X}(k)$, we show that there is a natural linear representation of $\mathfrak{G}$ on $\mathbf{T}_e(\mathfrak{X})$ such that $\mathbf{T}_e(\mathfrak{X}^{\mathfrak{G}})$ is the largest subcoalgebra of $\mathbf{T}_e(\mathfrak{X})$ contained in $\mathbf{T}_e(\mathfrak{X})^{\mathfrak{G}}$. This is an example of situations in which the above "geometric" definition of tangent and underlying coalgebras is inadequate. We give a more "categorical" definition and prove the last result among other things.

Chapter 3 deals with the theory of hyperalgebras and set up an analogy of the Lie algebra theory of algebraic groups over a field of characteristic 0.

After defining the hyperalgebra $\mathbf{hy}(\mathfrak{G})$ of a k-group-functor $\mathfrak{G}$ in § 3.1, we compute in § 3.2 the hyperalgebras of V_a, $\mathfrak{GL}(V)$ and $\mathfrak{Aut}(A)$, where V is a vector space over k and A is a k-algebra [7, II, § 1, 2.1; 2.4; 2.6] and show that given a linear representation of a k-group $\mathfrak{G}$ on V we have a natural left $\mathbf{hy}(\mathfrak{G})$-module structure on V.

In § 3.3 we show that the map $\mathfrak{H} \mapsto \mathbf{hy}(\mathfrak{H})$ from the set of *connected* subgroups of a locally algebraic k-group $\mathfrak{G}$ to the set of sub-hyperalgebras of $\mathbf{hy}(\mathfrak{G})$ is injective. We compute $\mathbf{hy}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H}))$ and $\mathbf{hy}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H}))$ in § 3.4 and $\mathbf{hy}(\mathscr{D}(\mathfrak{G}))$ in § 3.5. In § 3.6 we develop the theory of algebraic sub-hyperalgebras which is analogus to [7, II, § 6, 2.4 ~ 2.11].

Notations and Conventions

We shall adopt the terminology, definitions, notations and main results of [11].

Throughout this paper, k will denote a fixed ground field. The characteristic exponent of k ($= \mathrm{Max}\,(1, \mathrm{char}\,(k))$) is denoted by p. Vector spaces, algebras, coalgebras etc. will mean k-vector spaces, k-algebras, k-coalgebras

etc.

All rings or monoids are assumed to be associative and unitary and their homomorphisms preserve units. For a monoid M, $\mathrm{U}(M)$ denotes the group of invertible elements in M. For a ring A, the multiplicative monoid of A is denoted by $A^{\times}$. We put $\mathrm{U}(A)=\mathrm{U}(A^{\times})$. *The Krull dimension* of a commutative ring R is denoted by $K \dim R$.

For a category $\boldsymbol{A}$, $\boldsymbol{A}(X, Y)$ denotes the set of $\boldsymbol{A}$-maps from X to Y, where X and Y are objects of $\boldsymbol{A}$.

Categories:

$\boldsymbol{E}$	sets
$\boldsymbol{Mon}$	monoids
$\boldsymbol{Gr}$	groups
$\boldsymbol{Lie}_k$	Lie algebras
$\boldsymbol{Hopf}_k$	Hopf algebras
$\boldsymbol{Mod}_A$	left A-modules, where A is a ring
$\boldsymbol{Comod}_C$	right C-comodules, where C is a coalgebra
$\boldsymbol{Alg}_R$	R-algebras, where R is a commutative ring
$\boldsymbol{Coalg}_R$	R-coalgebras, where R is a commutative ring
$\boldsymbol{M}_R$	(small) commutative R-algebras, where R is a (small) commutative ring

Let V and W be vector spaces. We put

$$V \otimes W = V \otimes_k W$$
$$V^* = \boldsymbol{Mod}_k\,(V, k)$$
$$\mathrm{End}_k\,(V) = \boldsymbol{Mod}_k\,(V, V).$$

For a linear map $f: V \to W$, its transpose is denoted by

$${}^t f : W^* \to V^*.$$

For $x \in V$ and $X \in V^*$, we put

$$\langle X, x \rangle = \langle x, X \rangle = X(x).$$

For subsets T of V and P of V^*,

$$T^{\perp} = \{X \in V^* \mid \langle X, T \rangle = 0\}$$
$$P^{\perp} = \{x \in V \mid \langle P, x \rangle = 0\}.$$

A subspace V' of V is said to be *cofinite* if V/V' is finite dimensional. A subspace V'' of V^* is said to be *dense* if $V''^{\perp}=0$. Finally $\mathbf{S}^n V$ and $\mathbf{E}^n V$ denote the n-fold symmetric power and exterior power of V respectively.

Let A and C be an algebra and a coalgebra respectively. The structure maps will be denoted by

$$\mu_A : A\otimes A \to A, \eta_A : k \to A$$
$$\Delta_C\ C \to C\otimes C, \varepsilon_C : C \to k.$$

The index A or C is often omitted. The vector space $\boldsymbol{Mod}_k\,(C, A)$ is always viewed as an algebra, whose multiplication is defined by

$$f*g=\mu\circ(f\otimes g)\circ\Delta.$$

We adopt the following "sigma notation":

$$\Delta(c)=\sum c_{(1)}\otimes c_{(2)} \qquad \text{for } c\in C.$$

The coradical filtration [11, § 9.1] of C is denoted by $\{C_i\}_{i\geqq 0}$. $\mathrm{G}(C)$ denotes the set of *group-like* elements of C

$$\{g\in C \mid \Delta(g)=g\otimes g \text{ and } \varepsilon(g)=1\}.$$

$\mathrm{G}(C)$ can be naturally identified with the set of coalgebra maps from k to C. For $g\in \mathrm{G}(C)$, C^g denotes *the irreducible component* [11, § 8.0] of C containing g and $\mathrm{P}_g(C)$ the set of *primitive* elements of C with respect to g

$$\{x\in C \mid \Delta(x)=x\otimes g+g\otimes x\}.$$

C is said to be *connected* if C_0 is one-dimensional. A connected coalgebra C has a unique group-like element, which we shall denote by g_C and put

$$\mathrm{P}(C)=\mathrm{P}_{g_C}(C).$$

A Hopf algebra means a bialgebra with *an antipode.* The antipode of a Hopf algebra H is denoted by S_H (or S).

The symbols $\coprod_{\alpha\in A} X_\alpha$ and $\bigoplus_{\alpha\in A} V_\alpha$ will mean the disjoint union of sets X_α ($\alpha\in A$) and the direct sum of vector spaces V_α ($\alpha\in A$) respectively.

1. Preliminaries

1.1 The structure of A^0

1.1.1 Let C be a k-coalgebra. If W_1 and W_2 are subspaces of C we denote by $W_1\wedge W_2$ the kernel of the map

$$C\xrightarrow{\Delta} C\otimes C \longrightarrow C/W_1\otimes C/W_2$$

[11, § 9.0]. Let A be a subalgebra of C^*. For a subset S of A we put

$$S^{\perp}=\{x\in C\,|\,\langle x, S\rangle=0\}.$$

LEMMA. *If V_1 and V_2 are subspaces of A, then $V_1^{\perp}\wedge V_2^{\perp}=(V_1V_2)^{\perp}$.*

PROOF. Since $V_i^{\perp}$ is the kernel of

$$C\longrightarrow A^*\xrightarrow{\text{restr.}}V_i^*,$$

we have $C/V_i^{\perp}\hookrightarrow V_i^*$. The commutative diagram

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d] & C\otimes C \arrow[r] & C/V_1^{\perp}\otimes C/V_2^{\perp} \arrow[r, hook] & V_1^*\otimes V_2^* \arrow[d, hook] \\
A^* \arrow[r, "\text{restr.}"] & (V_1V_2)^* \arrow[rr, hook, "{}^t\mu"] & & (V_1\otimes V_2)^*,
\end{tikzcd}$$

where Δ (resp. μ) is the comultiplication (resp. multiplication) of C (resp. A), implies that $(V_1V_2)^{\perp}=V_1^{\perp}\wedge V_2^{\perp}$.

1.1.2 We are concerned with the structure of A^0, where A is a commutative k-algebra. But we see later (2.1.7) that

$$A^0=\bigoplus_{P\in\operatorname{Spec}A}(A_P)^0$$

where A_P is the localization of A at P. Hence enough to consider the case where A is local.

PROPOSITION. *Let A be a commutative local k-algebra with maximal ideal m.*

(i) *If m is not cofinite in A, then $A^0=0$.*

(ii) *If m is cofinite, then A^0 is irreducible with coradical filtration $\{A^0\cap(A/m^{i+1})^*\}_{i\geqq 0}$. In particular we have*

$$A^0\subset\varinjlim_i (A/m^{i+1})^*.$$

(iii) *If m is cofinite and A is Noetherian, then we have*

$$A^0=\varinjlim_i (A/m^{i+1})^*$$

and A^0 is dense in A^*.

PROOF. We have $A^0=\varinjlim_{A\supset I \text{ cofinite ideals}}(A/I)^*$ by [11, § 6.0]. Because m is nilpotent mod I for any cofinite ideal I of A, we have

$$A^0\subset\varinjlim_i (A/m^{i+1})^*.$$

If m is not cofinite then A^0 is 0, because A has no cofinite ideals. If m is cofinite then $(A/m)^*$ is a simple subcoalgebra of A^0. Since

$$\wedge^{i+1}(A/m)^* = A^0 \cap (A/m^{i+1})^*$$

by 1.1.1, we have $A^0 = \bigcup_i \wedge^{i+1}(A/m)^*$. It follows from [11, Lemma 11.0.8] that A^0 is irreducible with coradical $(A/m)^*$. This proves (ii). If furthermore A is Noetherian then m^{i+1}'s are cofinite and $\cap m^{i+1} = 0$ by the theorem of Krull. This means (iii).

1.1.3 Let A be a commutative k-algebra. For an ideal I of A such that I^{i+1} is cofinite for all i we put

$$(A^0)_I = \varinjlim_i (A/I^{i+1})^*.$$

LEMMA. *Let $\{M_1, \cdots, M_t\}$ be the set of maximal ideals of A containing I. Then we have*

$$(A^0)_I = (A^0)_{M_1} \oplus \cdots \oplus (A^0)_{M_t} \quad \text{and} \quad (A^0)_{M_i} = (A_{M_i})^0.$$

PPOOF. $M_1 \cap \cdots \cap M_t = M_1 \cdots M_t$ is nilpotent modulo I, because A/I is Artinian. I is nilpotent mod $M_1 \cdots M_t$. Hence we have

$$(A^0)_I = (A^0)_{M_1 \cdots M_t}.$$

Because $A \div (M_1 \cdots M_t)^{i+1} = A \div M_1^{i+1} \times \cdots \times A \div M_t^{i+1}$, we have

$$(A^0)_{M_1 \cdots M_t} = (A^0)_{M_1} \oplus \cdots \oplus (A^0)_{M_t}.$$

$(A^0)_{M_i} = (A_{M_i})^0$ follows from 1.1.2.

1.1.4 Let A be a commutative local k-algebra with maximal ideal m. Suppose that m^{i+1}'s are cofinite in A. Let $K|k$ be a field extension. Then the K-coalgebra $K \otimes A^0$ is clearly equal to $((K \otimes A)^0)_{K \otimes m}$, where $(K \otimes A)^0$ denotes the dual K-coalgebra of $K \otimes A$. Hence by 1.1.3, we have

COROLLARY. $K \otimes A^0 = ((K \otimes A)_{M_1})^0 \oplus \cdots \oplus ((K \otimes A)_{M_t})^0$, *where* $\{M_1, \cdots, M_t\}$ *is the set of maximal ideals of $K \otimes A$ containing $K \otimes m$, and $(-)^0$ in the right hand side denotes the K-coalgebra dual.*

1.2 The categories W_R, W_k, W_k^{cn} and W

1.2.1 For each $R \in M_k$, we define a category W_R as follows: An object of W_R is a cocommutative k-coalgebra; A morphism from C to D in W_R is an R-coalgebra map $f: R \otimes C \to R \otimes D$ such that $f(R \otimes C_0) \subset R \otimes D_0$, where C_0 (resp. D_0) is the coradical of C (resp. D). In view of [11, Theorem 8.0.8 d)], W_k is the same as the category of cocommutative coalgebras. For an M_k-map $\phi: R \to S$ we define a functor $W_\phi: W_R \to W_S$ as follows: W_ϕ is the identity

on the class of objects; $W_\phi(f)=S\otimes_R f$ for any W_R-morphism f. Finally let W_R^{f} be the full subcategory of W_R whose class of objects consists of all finite *dimensional* cocommutative k-coalgebras.

1.2.2 It is known that the category W_k has finite products [11, Theorem 6.4.5]. More precisely for any $C, D \in W_k$, the diagram:

$$C \xleftarrow{1\otimes\varepsilon} C\otimes D \xrightarrow{\varepsilon\otimes 1} D$$

is a direct product diagram in W_k and k the final object.

Given two W_k-maps $f, g: C \rightrightarrows D$, the largest subcoalgebra contained in Ker $(f-g)$ is clearly the equalizer (kernel) of the pair (f, g) in W_k. Thus W_k has finite products and equalizers. Therefore:

PROPOSITION. *The category W_k has finite limits.*

1.2.3 PROPOSITION. *A map $f: C \to D$ of cocommutative coalgebras is a monomorphism in W_k iff it is injective* (*as a map of sets*).

PROOF. Enough to prove the "only if" part. Assume that f is a monomorphism in W_k. Then for any subcoalgebra C' of C, the induced map

$$f: C' \to f(C')$$

is also a monomorphism. Since C is the directed union[2)] of its finite dimensional subcoalgebras [11, Theorem 2.2.1], we can assume without loss of generality that C is finite dimensional and that $f(C)=D$. Then the induced algebra map ${}^tf: D^* \to C^*$ is an epimorphism in M_k. This follows if one notes that for any $\phi, \psi \in M_k(C^*, A)$ with $A \in M_k$, $\phi(C^*)\cdot\psi(C^*)$ is a finite dimensional subalgebra of A. Hence the assertion follows from the following:

LEMMA. *Let $\phi: A \to B$ be an M_k-map. If B is a finitely generated A-module then ϕ is an epimorphism in M_k iff it is surjective.*

PROOF. By [2, II, § 3, n°3, Proposition 11], ϕ is surjective iff the induced map: $A/m \to B/Bm$ is surjective for any maximal ideal m of A. Suppose that ϕ is an epimorphism in M_k. Then the induced map: $A/m \to B/Bm$ is also an epimorphism. In particular we have $x\otimes 1 = 1\otimes x$ in $(B/Bm)\otimes_{A/m}(B/Bm)$ for any $x \in B/Bm$. But since A/m is a field, this means the surjectivity of the map: $A/m \to B/Bm$. Hence ϕ is surjective.

1.2.4 Let $R \in M_k$. Then k is the final object of W_R also. Let $C, C' \in W_R$. Then the diagram:

$$C \xleftarrow{W_\eta(1\otimes\varepsilon)} C\otimes C' \xrightarrow{W_\eta(\varepsilon\otimes 1)} C'$$

is a direct product diagram in the category of cocommutative R-coalgebras [6, page 28, Proposition 4.1] (but not necessarily so in W_R), where $\eta: k \to R$ is the structure map. Let $D \in W_R$. The above diagram induces an injection:

$$W_R(D, C \otimes C') \hookrightarrow W_R(D, C) \times W_R(D, C').$$

Note that $(C \otimes C')_0 \subset C_0 \otimes C'_0$ in general [8, 2.3.13]. If $(C \otimes C')_0 = C_0 \otimes C'_0$, then the above map is clearly isomorphic and hence $C \otimes C'$ is the direct product of C and C' in W_R. This occurs for example if C or C' is pointed [11, § 8.0] or if k is perfect.

1.2.5 LEMMA. *Let $C \in W_k$ and $A \subset C$ be a subspace. Let B be the largest subcoalgebra of C which is contained in A. Then for any $R \in M_k$, $D \in W_R$ and $\sigma \in W_R(D, C)$, we have*

$$\sigma(R \otimes D) \subset R \otimes A \Leftrightarrow \sigma(R \otimes D) \subset R \otimes B.$$

PROOF. Suppose that $\sigma(R \otimes D) \subset R \otimes A$. We want to show that $\sigma(R \otimes D) \subset R \otimes B$. Since D is the directed union of its finite dimensional subcoalgebras, we can assume that D is finite dimensional. Then there exists a finite dimensional subspace A' of A such that $\sigma(R \otimes D) \subset R \otimes A'$. Let $\omega: C^* \to R \otimes D^*$ be the composite

$$C^* \xrightarrow{\text{cano.}} \boldsymbol{Mod}_R(R \otimes C, R) \xrightarrow{\boldsymbol{Mod}_R(\sigma, R)} R \otimes D^*.$$

This is an algebra map, since σ is an R-coalgebra map. Since $\sigma(R \otimes D) \subset R \otimes A'$, ω factors as

$$C^* \xrightarrow{\text{cano.}} A'^* \xrightarrow{\text{cano.}} R \otimes A'^* \xrightarrow{\boldsymbol{Mod}_R(\sigma, R)} R \otimes D^*.$$

But since A' is finite dimensional, there exist a unique subspace B' of A' and an isomorphism $\theta: B'^* \simeq \omega(C^*)$ such that ω factors as

$$\omega: C^* \xrightarrow{\text{cano.}} A'^* \xrightarrow{\text{cano.}} B'^* \overset{\theta}{\simeq} \omega(C^*) \hookrightarrow R \otimes D^*.$$

Since $B'^{\perp} = \operatorname{Ker}(\omega)$ is an ideal of C^*, B' is a subcoalgebra of C [11, Proposition 1.4.3]. Hence $B' \subset B$. The fact that ω factors through the canonical projection: $C^* \to B'^*$ means clearly that $\sigma(R \otimes D) \subset R \otimes B'$. Thus we have $\sigma(R \otimes D) \subset R \otimes B$.

1.2.6 Let $C, D \in W_k$ and $f, g: C \rightrightarrows D$ be two coalgebra maps. Then $\operatorname{Ker}(f, g)$ the equalizer of (f, g) in W_k is the largest subcoalgebra contained in $\operatorname{Ker}(f-g)$. Since $\operatorname{Ker}(R \otimes f - R \otimes g) = R \otimes \operatorname{Ker}(f-g)$, it follows from above that $\operatorname{Ker}(f, g)$ is the equalizer of $(W_\eta(f), W_\eta(g)): C \rightrightarrows D$ in W_R for any

$R \in \boldsymbol{M}_k$. In other words the functor

$$W_\eta : W_k \to W_R$$

preserves equalizers (but not finite products).

1.2.7 A coalgebra C is said to be connected if the coradical C_0 is 1-dimensional [9, Definition 3.1]. Such a coalgebra has a unique group-like element g_C and $C_0 = kg_C$. We denote by W_k^{cn} the category of cocommutative connected coalgebras. If $C, D \in W_k^{\mathrm{cn}}$, then $C \otimes D \in W_k^{\mathrm{cn}}$, since $(C \otimes D)_0$ is contained in $C_0 \otimes D_0$ (cf. [9, Proposition 3.2.4]). Let $f, f' : C \rightrightarrows D$ be the maps in W_k^{cn}. Since $f(g_C) = g_D = f'(g_C)$, the kernel Ker (f, g) is a non-zero, hence connected, subcoalgebra of C. Thus we have proven:

PROPOSITON. *The full subcategory W_k^{cn} of W_k is closed under finite limits.*

1.2.8 Let B_1 be the coalgebra on basis b_0 and b_1 with

$$\Delta(b_0) = b_0 \otimes b_0, \varepsilon(b_0) = 1,$$
$$\Delta(b_1) = b_0 \otimes b_1 + b_1 \otimes b_0, \varepsilon(b_1) = 0.$$

Clearly $B_1 \in W_k^{\mathrm{cn}}$ and we have

$$W_k^{\mathrm{cn}}(B_1, C) \simeq \mathrm{P}(C)$$

for any $C \in W_k^{\mathrm{cn}}$, where $\mathrm{P}(C)$ is the set of primitive elements with respect to the unique group-like element g_C, that is

$$\mathrm{P}(C) = \{x \in C \mid \Delta(x) = x \otimes g_C + g_C \otimes x\}.$$

In particular the functor

$$\mathrm{P} : W_k^{\mathrm{cn}} \to \boldsymbol{Mod}_k, C \mapsto \mathrm{P}(C)$$

commutes with finite limits.

Let $f : C \to D$ be a W_k^{cn}-map. We denote by $\mathrm{Ker}_0(f)$ the equalizer Ker $(f, g_D \circ \varepsilon)$ in W_k^{cn}, where g_D is identified with the unique coalgebra map: $k \to D$. It is also defined by the following pullback diagram in W_k:

$$\begin{tikzcd} C \arrow[r, "f"] & D \\ \mathrm{Ker}_0(f) \arrow[u, hook] \arrow[r] & kg_D \arrow[u, hook] \end{tikzcd}$$

Since $\mathrm{P}(-)$ commutes with finite limits, we have

$$\mathrm{P}(\mathrm{Ker}_0(f)) = \mathrm{Ker}\,(\mathrm{P}(f): \mathrm{P}(C) \to \mathrm{P}(D)).$$

LEMMA. *A W_k^{cn}-map $f: C \to D$ is injective (as a map of sets) iff* $\mathrm{Ker}_0(f) = kg_C$.

PROOF. It is known that f is injective iff $\mathrm{P}(f): \mathrm{P}(C) \to \mathrm{P}(D)$ is injective [11, Lemma 11.0.1]. In particular the structure map $\varepsilon: C \to k$ is injective iff $\mathrm{P}(C) = 0$. Hence we have

$$\begin{aligned} f \text{ is injective } &\Leftrightarrow \mathrm{P}\,(\mathrm{Ker}_0(f)) = 0 \\ &\Leftrightarrow \mathrm{Ker}_0(f) = kg_C. \end{aligned}$$

1.2.9 We define a category W as follows: The class of objects in W consists of all pairs (R, C), where $R \in M_k$ and $C \in W_k$; The W-maps from (S, D) to (R, C), where (S, D) and (R, C) are objects in W, consist of all pairs (ϕ, σ) with $\phi \in M_k(R, S)$ and $\sigma \in W_S(D, C)$. The composite of W-maps

$$(T, E) \xrightarrow{(\psi, \tau)} (S, D) \xrightarrow{(\phi, \sigma)} (R, C)$$

is of course $(\psi \circ \phi, W_\psi(\sigma) \circ \tau)$.

W^{f} will denote the full subcategory of W consisting of all (R, C) such that C is finite dimensional.

1.3 Representable functors and hyperalgebras

1.3.1 Let A be a category. A *contravariant* functor from A to E (resp. Mon, resp. Gr) is called a *set-* (resp. *monoid-*, resp. *group-*) *functor* on A.

Suppose that A has finite products and especially the final object, written e. A monoid (resp. group) object in A is by definition a triple (resp. 4–tuple) (G, μ, η) (resp. (G, μ, η, S)) with $G \in A$, $\mu \in A(G \times G, G)$ and $\eta \in A(e, G)$ (resp. in addition $S \in A(G, G)$) such that

$$\mu \circ (\mu \times 1_G) = \mu \circ (1_G \times \mu): G \times G \times G \to G$$

and

$$\mu \circ (\eta \times 1_G) = 1_G = \mu \circ (1_G \times \eta): G \to G$$

(resp. in addition

$$\mu \circ (S \times 1_G) \circ \Delta = \eta \circ \varepsilon = \mu \circ (1_G \times S) \circ \Delta: G \to G),$$

where $\Delta: G \to G \times G$ is the diagonal map and $\varepsilon: G \to e$ is the unique map.

Let G be an object (resp. a monoid object, resp. a group object) in A. Then the functor

$$A(-,G): C\mid\to A(C,G)$$

is a set- (resp. monoid-, resp. group-) functor on A, where the multiplication on $A(C,G)$ is defined by

$$A(C,G)\times A(C,G)\simeq A(C,G\times G)\xrightarrow{A(C,\mu)}A(C,G),$$

(in case G is a monoid or group object).

A set- (resp. monoid-, resp. group-) functor T on A is said to be *representable* if T is isomorphic to $A(-,G)$ for some object (resp. monoid object, resp. group object) G in A. Such an object (resp. a monoid object, resp. a group object) G, which is clearly unique up to isomorphisms, is said to *represent* T.

A group- or monoid-functor on A is representable iff the underlying set-functor is representable as is easily verified.

1.3.2 The category W_k is seen to have finite products (1.2.2). More precisely if C and D are two objects in W_k, the following is a direct product diagram in W_k:

$$C\xleftarrow{1\otimes\varepsilon}C\otimes D\xrightarrow{\varepsilon\otimes 1}D.$$

The final object in W_k is k. It follows that a cocommutative bialgebra (resp. cocommutative Hopf algebra) is nothing other than a monoid (resp. group) object in W_k (cf. [6, page 33 and 11, page 71]).

Let H be a cocommutative bialgebra (resp. cocommutative Hopf algebra). Then the functor

$$W_k(-,H): C\mid\to W_k(C,H)$$

is a monoid- (resp. group-) functor on W_k, where the monoid (resp. group) structure on $W_k(C,H)$ is defined as follows:

$$\begin{array}{ll}\text{(multiplication)} & f*g=\mu\circ(f\otimes g)\circ\Delta \\ \text{(unity)} & \eta\circ\varepsilon\end{array}$$

(resp. in addition

$$\text{(inverse)}\qquad f^{-1}=S\circ f).$$

Recall that W_k^f is the category of finite dimensional cocommutative coalgebras. A set- (resp. monoid-, resp. group-) functor T on W_k^f is said to be *represented* by a cocommutative coalgebra (resp. bialgebra, resp. Hopf algebra) H if T is isomorphic to the restriction of $W_k(-,H)$ to W_k^f. From Lemma below it follows that such an H is unique up to isomorphisms.

LEMMA. *Let T and U be two set- (resp. monoid-, resp. group-) functors on W_k. We assume that T and U commute with limits. (This means that if $C=\varinjlim_j C_j$ in W_k then $T(C)=\varprojlim_j T(C_j)$ and $U(C)=\varprojlim_j U(C_j)$ in* **E** *(resp.* ***Mon****, resp.* ***Gr****)). Then any natural transformation*

$$\alpha: T\,|\,W_k^{\mathrm{f}} \to U\,|\,W_k^{\mathrm{f}}$$

can be uniquely extended to a natural transformation

$$\tilde{\alpha}: T \to U.$$

If in particular $T\,|\,W_k^{\mathrm{f}}$ and $U\,|\,W_k^{\mathrm{f}}$ are isomorphic to each other then so are T and U.

PROOF. Since every coalgebra is the directed union of its finite dimensional subcoalgebras, we have naturally

$$T(C) \simeq \varprojlim_{C\supset D\ \mathrm{f.d.}} T(D) \quad \text{and} \quad U(C) \simeq \varprojlim_{C\supset D\ \mathrm{f.d.}} U(D)$$

for any $C \in W_k$. Let $\tilde{\alpha}(C): T(C) \to U(C)$ be the composite

$$T(C) \simeq \varprojlim_{C\supset D\ \mathrm{f.d.}} T(D) \xrightarrow{\varprojlim_{C\supset D\ \mathrm{f.d.}} \alpha(D)} \varprojlim_{C\supset D\ \mathrm{f.d.}} U(D) \simeq U(C).$$

Then $\tilde{\alpha}: T \to U$ is a unique natural transformation such that

$$\tilde{\alpha}\,|\,W_k^{\mathrm{f}} = \alpha.$$

1.3.3 Let $\alpha: T \to U$ be a natural transformation of monoid- (resp. group-) functors on a categary $\boldsymbol{A}$. The kernel of α, written $\mathrm{Ker}\,(\alpha)$, is defined by

$$\mathrm{Ker}\,(\alpha)(A) = \mathrm{Ker}\,(\alpha(A): T(A) \to U(A))$$

for all $A \in \boldsymbol{A}$. This is clearly a monoid- (resp. group-) functor on $\boldsymbol{A}$.

Let $f: H \to H'$ be a map of cocommutative bialgebras. *The bialgebra kernel* of f, written $\mathrm{Ker}_0\,(f)$, is the equalizer of $(f, \eta\circ\varepsilon)$ in the category W_k[3].

LEMMA. $\mathrm{Ker}_0\,(f)$ *is a subbialgebra of H and represents the kernel of the map*

$$W_k(-, f): W_k(-, H) \to W_k(-, H')$$

of monoid-functors on W_k. If H and H' have the antipode, then $\mathrm{Ker}_0\,(f)$ *is a sub-Hopf algebra of H.*

PROOF. $\mathrm{Ker}\,(f - \eta\circ\varepsilon)$ is a subalgebra of H. In general if A is a subal-

gebra of H then the largest subcoalgebra of H contained in A is also a subalgebra of H as is easily seen. Hence $\mathrm{Ker}_0(f)$, which is the largest subcoalgebra contained in $\mathrm{Ker}(f-\eta\circ\varepsilon)$, is a subbialgebra. The other statements are clear.

1.3.4 Let $C, D \in W_k$, $g \in \mathrm{G}(D)$ and $\sigma \in W_k(C, D)$. We denote as usual by C_0 the coradical of C and by D^g the irreducible component of D which contains g. We claim that

$$\sigma(C) \subset D^g \Leftrightarrow \sigma(C_0) \subset kg.$$

Indeed if $\sigma(C) \subset D^g$, then $\sigma(C_0) \subset (D^g)_0 = kg$. Conversely if $\sigma(C_0) \subset kg$, then $\sigma(\wedge^n C_0) \subset \wedge^n kg$ by [11, Lemma 9.1.3]. Since $D^g = \cup \wedge^n kg$ [11, Lemma 11.0.8], we have $\sigma(C) \subset D^g$.

Let H be a cocommutative bialgebra. Then H^1 the irreducible component of H containing 1 is a subbialgebra [11, Proposition 8.1.1]. It follows from above that the sequence

$$1 \longrightarrow W_k(C, H^1) \longrightarrow W_k(C, H) \xrightarrow{\text{cano.}} W_k(C_0, H)$$

is exact for any $C \in W_k$. This gives a characterization of the monoid-functor $W_k(-, H^1)$.

1.3.5 A cocommutative irreducible bialgebra is called *a hyperalgebra*. A hyperalgebra is connected and has an antipode [11, Propposition 9.2.5]. A subbialgebra or a quotient bialgebra of a hyperalgebra is also a hyperalgebra. The tensor product of two hyperalgebras is a hyperalgebra.

A hyperalgebra is nothing other than a monoid object, in fact a group object, in W_k^{cn}. Let $R \in M_k$ with $\eta: k \to R$ the structure map. It follows from (1.2.4) that the functor

$$W_\eta : W_k^{\mathrm{cn}} \to W_R$$

preserves finite products and so group objects. Hence a hyperalgebra H can be considered as a group object in W_R and so the functor

$$W_R(-, H) : C \mapsto W_R(C, H)$$

a group-functor on W_R.

LEMMA. *Let $f: H \to H'$ be a map of hyperalgebras. Then* $\mathrm{Ker}_0(f)$ *the Hopf kernel of f* (1.3.3) *is a subhyperalgebra of H and represents the kernel of the map*

$$W_R(-, W_\eta(f)) : W_R(-, H) \to W_R(-, H')$$

of group-functors on W_R.

PROOF. This follows immediately from (1.2.6).

1.3.6 As a corollary to 1.2.8 we obtain:

PROPOSITION. *Let $f: H \to H'$ be a map of hyperalgebras. Then we have*

$$\mathrm{P}(\mathrm{Ker}_0(f)) = \mathrm{Ker}\,(\mathrm{P}(f): \mathrm{P}(H) \to \mathrm{P}(H'))$$

and f is injective iff $\mathrm{Ker}_0(f) = k$.

1.3.7 Let $R \in \boldsymbol{M}_k$. A set-functor T on $\boldsymbol{W}_R^{\mathrm{f}}$ is said to be represented by an object C of $\boldsymbol{W}_R$ if

$$T \simeq \boldsymbol{W}_R(-, C) \mid \boldsymbol{W}_R^{\mathrm{f}}.$$

Such an object C is unique up to $\boldsymbol{W}_R$-isomorphisms by the following lemma which can be proved by the same method as in 1.3.2:

LEMMA. *Let $R \in \boldsymbol{M}_k$ and $C, D \in \boldsymbol{W}_R$. Let*

$$\alpha: \boldsymbol{W}_R(-, C) \mid \boldsymbol{W}_R^{\mathrm{f}} \to \boldsymbol{W}_R(-, D) \mid \boldsymbol{W}_R^{\mathrm{f}}$$

be a map of set-functors on $\boldsymbol{W}_R^{\mathrm{f}}$. Then there exists a unique $\boldsymbol{W}_R$-map $\sigma: C \to D$ such that

$$\alpha = \boldsymbol{W}_R(-, \sigma) \mid \boldsymbol{W}_R^{\mathrm{f}}.$$

1.3.8 Let H be an object in $\boldsymbol{W}_k$. Then the correspondence

$$(R, C) \mid\!\to \boldsymbol{W}_R(C, H),\ \boldsymbol{W} \to \boldsymbol{E}$$

turns into a contravariant functor if we associate the composite

$$\boldsymbol{W}_R(C, H) \xrightarrow{\boldsymbol{W}_\phi} \boldsymbol{W}_S(C, H) \xrightarrow{\boldsymbol{W}_S(\sigma, H)} \boldsymbol{W}_S(D, H)$$

with each $\boldsymbol{W}$-map $(\phi, \sigma): (S, D) \to (R, C)$. This set-functor on $\boldsymbol{W}$ is said to be *represented by H* and denoted by Θ_H. (If we embed $\boldsymbol{W}_k$ into $\boldsymbol{W}$ via

$$\boldsymbol{W}_k \to \boldsymbol{W},\ C \mid\!\to (k, C)$$

then Θ_H is nothing other than the representable set-functor $\boldsymbol{W}(-, (k, H))$). If H has a hyperalgebra structure, then Θ_H becomes naturally a group-functor on $\boldsymbol{W}$ by (1.3.5).

LEMMA. *Let H and H' be two cocommutative coalgebras (resp. hyperalgebras). Let*

$$\alpha: \Theta_H \mid \boldsymbol{W}^{\mathrm{f}} \to \Theta_{H'} \mid \boldsymbol{W}^{\mathrm{f}}$$

be a map of set- (resp. group-) functors on W^{f}. Then there exists a unique coalgebra map (resp. hyperalgebra map)

$$\sigma: H \rightarrow H'$$

such that

$$\alpha(R, C) = W_R(C, W_\eta(\sigma))$$

for each $(R, C) \in W^{\mathrm{f}}$. In particular α can be uniquely extended to a map

$$\tilde{\alpha}: \Theta_H \rightarrow \Theta_{H'}$$

of set- (resp. group-) functors on W.

PROOF. $R \in M_k$ being fixed, α induces a map of set- (resp. group-) functors on W_R^{f}

$$\alpha_R: W_R(-, H) \mid W_R^{\mathrm{f}} \rightarrow W_R(-, H') \mid W_R^{\mathrm{f}}.$$

This is of the form $W_R(-, \sigma(R)) \mid W_R^{\mathrm{f}}$ by 1.3.7, where $\sigma(R)$ is a uniquely determined W_R-map: $H \rightarrow H'$. (In case of hyperalgebras, $\sigma(R)$ is also an R-algebra map from $R \otimes H$ to $R \otimes H'$). Since α_R, hence $\sigma(R)$ also, is "natural with respect to R", it follows that

$$\sigma(R) = W_\eta(\sigma(k)).$$

This proves Lemma.

We say that a set- (resp. group-) functor T on W^{f} is said to be represented by a cocommutative coalgebra (resp. hyperalgebra) H if $T \simeq \Theta_H \mid W^{\mathrm{f}}$. Such an H is unique up to a unique isomorphism.

1.3.9 Consider the correspondence

$$(R, C) \mapsto R \otimes C^*, \; W^{\mathrm{f}} \rightarrow M_k.$$

This turns into a contravariant functor if we associate the composite

$$R \otimes C^* \xrightarrow{\phi \otimes 1} S \otimes C^* \xrightarrow{Mod_S(\sigma, S)} S \otimes D^*$$

with each W^{f}-map $(\phi, \sigma): (S, D) \rightarrow (R, C)$. We always in this manner consider $R \otimes C^*$ as a functor of $(R, C) \in W^{\mathrm{f}}$.

1.3a The maximum sub-Hopf algebra of a cocommutative bialgebra

1.3a.1 For a monoid M, we denote by $\mathrm{U}(M)$ the group of invertible elements in M.

Let H be a cocommutative bialgebra. Consider the group-functor on W_k:

$$C \mapsto \mathrm{U}(W_k(C, H)).$$

The purpose of this section is to prove the following:

PROPOSITION. *The above functor is representable. More precisely there exists a (unique) subbialgebra H' of H such that*

(i) *H' has an antipode and*

(ii) *For any $C \in W_k$ and $\sigma \in W_k(C, H)$ we have*

$$\sigma(C) \subset H' \Leftrightarrow \sigma \text{ is invertible in the algebra } \boldsymbol{Mod}_k(C, H).$$

Then H' represents the above group-functor.

1.3a.2 Let A be an associative (not necessarily commutative) algebra and R a commutative *finite dimensional* algebra. Let S be a subalgebra of R. Then $S \otimes A$ is a subalgebra of $R \otimes A$.

LEMMA. *If an element x of $S \otimes A$ is invertible in the algebra $R \otimes A$, then it is invertible in $S \otimes A$.*

PROOF. If S is a field, then the assertion is clear since $R \otimes A = R \otimes_S (S \otimes A)$.

In general let $\{M_1, \cdots, M_n\}$ be the set of all maximal ideals of R. Since R is finite dimensional, $m_i = M_i \cap S$ is a maximal ideal of S. Applying the above remark to the field extension $(R/M_i) | (S/m_i)$, it follows that x is invertible modulo $m_i \otimes A$. Thus we have

$$S \otimes A = x(S \otimes A) + m_i \otimes A.$$

Notice that the set $\mathscr{X}$ of ideals a of S such that

$$S \otimes A = x(S \otimes A) + a \otimes A$$

is closed under the formation of finite products. But since

$$M = M_1 \cap \cdots \cap M_n$$

is nilpotent, $m = m_1 \cap \cdots \cap m_n$ is a nilpotent ideal of S and belongs to the set $\mathscr{X}$ because it contains $m_1 \cdots m_n$. This implies that $0 \in \mathscr{X}$, that is

$$S \otimes A = x(S \otimes A).$$

Similarly we have $S \otimes A = (S \otimes A)x$. This proves Lemma.

1.3a.3 Let A be an algebra and $\tau : C \to D$ a surjective map of cocommutative coalgebras.

COROLLARY. *Let $\sigma \in \boldsymbol{Mod}_k(D, A)$. If $\sigma\circ\tau$ is invertible in the algebra $\boldsymbol{Mod}_k(C, A)$, then σ is invertible in $\boldsymbol{Mod}_k(D, A)$.*

PROOF. Let $\{C_\alpha\}$ be the set of all finite dimensional subcoalgebras of C. Put

$$D_\alpha=\tau(C_\alpha),\ \tau_\alpha=\tau\,|\,C_\alpha : C_\alpha\to D_\alpha \quad \text{and}$$
$$\sigma_\alpha=\sigma\,|\,D_\alpha : D_\alpha\to A.$$

Since $\sigma_\alpha\circ\tau_\alpha \in \boldsymbol{Mod}_k(C_\alpha, A)\simeq(C_\alpha)^*\otimes A$ is invertible, it follows from Lemma above that $\sigma_\alpha \in \boldsymbol{Mod}_k(D_\alpha, A)\simeq(D_\alpha)^*\otimes A$ is invertible. But since $\{D_\alpha\}$ forms a directed set of subcoalgebras such that $D=\cup D_\alpha$, this means that σ is invertible.

1.3a.4 Let H be a cocommutative bialgebra. Then $W_k(C, H)$ is a multiplicative submonoid of the algebra $\boldsymbol{Mod}_k(C, H)$ for any $C\in W_k$. We claim that if $\sigma\in W_k(C, H)$ is invertible in $\boldsymbol{Mod}_k(C, H)$ then $\sigma^{-1}\in W_k(C, H)$.

Indeed consider the commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\sigma} & H \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_H} \\ C\otimes C & \xrightarrow{\sigma\otimes\sigma} & H\otimes H. \end{array}$$

Since $\Delta_H\in \boldsymbol{Alg}_k(H, H\otimes H)$ and $\Delta_C\in W_k(C, C\otimes C)$, it follows that

$$\Delta_H\circ\sigma^{-1}=(\Delta_H\circ\sigma)^{-1}=((\sigma\otimes\sigma)\circ\Delta_C)^{-1}=(\sigma^{-1}\otimes\sigma^{-1})\circ\Delta_C.$$

Similarly we have $\varepsilon_H\circ\sigma^{-1}=\varepsilon_C$.

Let $\mathscr{X}$ be the set of subcoalgebras C of H such that the inclusion $i_C: C\hookrightarrow H$ is invertible in the algebra $\boldsymbol{Mod}_k(C, H)$ (or equivalently in the monoid $W_k(C, H)$).

LEMMA. (i) $k\in\mathscr{X}$.

(ii) *If C and D are in $\mathscr{X}$ then $C+D$ and $C\cdot D$ are in $\mathscr{X}$ also.*

(iii) *For any $\sigma\in W_k(E, H)$, where $E\in W_k$, σ is invertible in the algebra $\boldsymbol{Mod}_k(E, H)$ iff $\sigma(E)\in\mathscr{X}$.*

PROOF. (i) is clear and (iii) follows immediately from Corollary 1.3a.3.

(ii) Suppose that $C, D\in\mathscr{X}$. Then $C\cap D$ is a subcoalgebra of H and

$$(i_C)^{-1}\,|\,C\cap D=(i_D)^{-1}\,|\,C\cap D$$

clearly. Hence there exists a unique linear map

$$j: C+D\to H$$

such that $j|C=(i_C)^{-1}$ and $j|D=(i_D)^{-1}$. Since the map j is easily seen to be the inverse of $i_{(C+D)}$, it follows that $C+D \in \mathscr{X}$.

On the other hand the coalgebra map:

$$C\otimes D \to H,\ c\otimes d \mapsto c\cdot d$$

has the map:

$$C\otimes D \to H,\ c\otimes d \mapsto (i_D)^{-1}(d)\cdot(i_C)^{-1}(c)$$

as its inverse. By (iii) this means that $C\cdot D \in \mathscr{X}$.

1.3a.5 Let H and $\mathscr{X}$ be as above. Put

$$H'=\sum_{C\in\mathscr{X}} C.$$

Then it follows from Lemma above that H' is a subbialgebra of H and the largest element of $\mathscr{X}$. Since the image of the map

$$(i_{H'})^{-1}: H' \to H$$

belongs to $\mathscr{X}$, We have $(i_{H'})^{-1}(H') \subset H'$. Hence H' has $(i_{H'})^{-1}$ as its antipode. In view of the condition (iii) of Lemma above, we see that the subbialgebra H' satisfies the conditions (i) and (ii) of Proposition 1.3a.1. Let $C \in W_k$ and $\sigma \in W_k(C, H)$. Since then

$$\begin{aligned}\sigma(C)\subset H' &\Leftrightarrow \sigma \text{ is invertible in } \boldsymbol{Mod}_k(C,H)\\ &\Leftrightarrow \sigma \in \mathrm{U}(W_k(C,H)),\end{aligned}$$

it follows that the Hopf algebra H' represents the group-functor on W_k:

$$C \mapsto \mathrm{U}(W_k(C,H)).$$

This proves Proposition 1.3a.1.

Now let H'' be a sub-Hopf algebra of H, that is H'' is a subbialgebra of H and has an antipode. Since then the inclusion $i_{H''}: H'' \hookrightarrow H$ is invertible, it follows that $H'' \subset H'$. In this sense the subbialgebra H' may be referred to as *the largest* sub-Hopf algebra of H. In particular H^1 the irreducible component of H containing 1 is contained in H' since it is a sub-Hopf algebra of H. Thus we have $H^1 = H'^1$.

1.3b The Lie algebra of primitive elements of a hyperalgebra

1.3b.1 Let H be a hyperalgebra. The set

$$\mathrm{P}(H)=\{x\in H \mid \Delta(x)=x\otimes 1+1\otimes x\}$$

is a vector subspace of H and closed under the maps:

$$(g, h) \mapsto [g, h] = gh - hg \quad \text{and} \quad g \mapsto g^p$$

where p is the characteristic exponent of k. Thus $\mathrm{P}(H)$ has a natural structure of (restricted) Lie algebra.

Let B_1 be the coalgebra on basis b_0, b_1 with

$$\Delta(b_0) = b_0 \otimes b_0, \quad \varepsilon(b_0) = 1,$$
$$\Delta(b_1) = b_0 \otimes b_1 + b_1 \otimes b_0, \quad \varepsilon(b_1) = 0$$

(1.2.8). The map

$$\theta : W_k(B_1, H) \to \mathrm{P}(H), \sigma \mapsto \sigma(b_1)$$

is clearly a bijection. By transporting the structure along the bijection θ^{-1}, $W_k(B_1, H)$ becomes a (restricted) Lie algebra. In the following we shall describe that structure.

1.3b.2 For an element $\lambda \in k$, we define a coalgebra map

$$u_\lambda : B_1 \to B_1$$

by

$$u_\lambda(b_0) = b_0 \quad \text{and} \quad u_\lambda(b_1) = \lambda b_1.$$

Let k act on $W_k(B_1, H)$ as follows:

$$k \times W_k(B_1, H) \to W_k(B_1, H), (\lambda, \sigma) \mapsto \sigma \circ u_\lambda.$$

PROPOSITION. *The above action makes the group $W_k(B_1, H)$ into a vector space and then the bijection*

$$\theta : W_k(B_1, H) \xrightarrow{\simeq} \mathrm{P}(H)$$

becomes a morphism of vector spaces.

PROOF. Let $\lambda \in k$ and $\sigma, \tau \in W_k(B_1, H)$. Then

$$(\sigma * \tau)(b_1) = \sigma(b_1)\tau(b_0) + \sigma(b_0)\tau(b_1) = \sigma(b_1) + \tau(b_1) \quad \text{and}$$
$$(\sigma \circ u_\lambda)(b_1) = \sigma(\lambda b_1) = \lambda\sigma(b_1).$$

This proves Proposition.

1.3b.3 Define a coalgebra map

$$\beta : H \otimes H \to H$$

by

$$\beta(x\otimes y)=\sum x_{(1)}y_{(1)}S(x_{(2)})S(y_{(2)})$$

(cf. (1.10.5)).

LEMMA. *Let $C, D \in W_k$. If we put*

$$pr_1=1\otimes\varepsilon: C\otimes D\to C \quad \textit{and}$$
$$pr_2=\varepsilon\otimes 1: C\otimes D\to D$$

then the following two maps coincide with each other:

$$W_k(C,H)\times W_k(D,H)\rightrightarrows W_k(C\otimes D,H)$$
$$(\sigma,\tau)\mapsto\beta\circ(\sigma\otimes\tau) \quad \text{and}$$
$$\mapsto(\sigma\circ pr_1)*(\tau\circ pr_2)*(\sigma\circ pr_1)^{-1}*(\tau\circ pr_2)^{-1}.$$

PROOF. For any $C\in W_k$, the composite:

$$W_k(C,H)^2\simeq W_k(C,H\otimes H)\xrightarrow{W_k(C,\beta)}W_k(C,H)$$

is easily seen to be the map:

$$(\sigma,\rho)\mapsto\sigma*\rho*\sigma^{-1}*\rho^{-1}.$$

The assertion follows immediately, since the element $\sigma\otimes\tau\in W_k(C\otimes D,H\otimes H)$ corresponds to $(\sigma\circ pr_1,\tau\circ pr_2)\in W_k(C\otimes D,H)^2$ canonically.

1.3b.4 We define a linear map

$$\nu: B_1\otimes B_1\to B_1$$

as follows:

$$\nu(b_0\otimes b_0)=b_0,\ \nu(b_0\otimes b_1)=\nu(b_1\otimes b_0)=0,\ \nu(b_1\otimes b_1)=b_1.$$

The map ν is a coalgebra map, since it is the linear dual of the algebra map:

$$k[X]/(X^2)\to k[X]/(X^2)\otimes k[X]/(X^2),\ X\mapsto X\otimes X.$$

PROPOSITION. *There exists a unique map*

$$\alpha: W_k(B_1,H)\times W_k(B_1,H)\to W_k(B_1,H)$$

such that for any $\sigma,\tau\in W_k(B_1,H)$

$$\alpha(\sigma,\tau)\circ\nu=(\sigma\circ pr_1)*(\tau\circ pr_2)*(\sigma\circ pr_1)^{-1}*(\tau\circ pr_2)^{-1}$$

in the group $W_k(B_1\otimes B_1,H)$, where the maps

$$pr_1, pr_2: B_1\otimes B_1\rightrightarrows B_1$$

are the canonical projections. Then the bracket product

$$[\sigma, \tau] = \alpha(\sigma, \tau)$$

makes $W_k(B_1, H)$ *into a Lie algebra over* k *such that the bijection*

$$\theta : W_k(B_1, H) \xrightarrow{\simeq} \mathrm{P}(H)$$

becomes a Lie algebra isomorphism.

PROOF. We have only to show the commutativity of the diagram:

$$\begin{tikzcd}
\mathrm{P}(H) \times \mathrm{P}(H) \arrow[r, "{[\,,\,]}"] \arrow[d, "\theta^{-1}\times\theta^{-1}", "\wr"'] & \mathrm{P}(H) \arrow[dd, "\theta^{-1}", "\wr"'] \\
W_k(B_1, H) \times W_k(B_1, H) \arrow[d, "\bar{\beta}"] & \\
W_k(B_1 \otimes B_1, H) & W_k(B_1, H) \arrow[l, "{W_k(\nu, H)}"']
\end{tikzcd}$$

where $\bar{\beta}$ is the map: $(\sigma, \tau) \mid\rightarrow \beta \circ (\sigma \otimes \tau)$.

Indeed let $x, y \in \mathrm{P}(H)$. Then we have

$$\beta(x \otimes y) = xy - yx, \quad \beta(x \otimes 1) = 0 = \beta(1 \otimes y) \quad \text{and} \quad \beta(1 \otimes 1) = 1.$$

Hence if we put

$$\sigma = \theta^{-1}(x), \quad \tau = \theta^{-1}(y) \quad \text{and} \quad \rho = \theta^{-1}(xy - yx)$$

then it follows that

$$\bar{\beta}(\sigma \otimes \tau) = \rho \circ \nu : B_1 \otimes B_1 \rightarrow H.$$

This proves Proposition.

1.3b.5 Let n be an integer > 0 and C a coalgebra. Let $\mathbf{S}^n C$ be the n-th symetric power of C. Then the maps:

$$\otimes^n C \xrightarrow{\Delta} (\otimes^n C) \otimes (\otimes^n C) \xrightarrow{\textit{canonic}} \mathbf{S}^n C \otimes \mathbf{S}^n C \quad \text{and} \quad \otimes^n C \xrightarrow{\varepsilon} k$$

are easily seen to factor through the canonical projection:

$$\otimes^n C \rightarrow \mathbf{S}^n C.$$

This means that $\mathbf{S}^n C$ is a quotient coalgebra of $\otimes^n C$.

Consider the map:

$$W_k(C, H) \rightarrow W_k(\otimes^n C, H), \sigma \mid\rightarrow (\sigma \circ pr_1) * (\sigma \circ pr_2) * \cdots * (\sigma \circ pr_n)$$

where $pr_i: \otimes^n C \to C$ is the i-th projection, that is

$$pr_i(x_1 \otimes \cdots \otimes x_n) = \varepsilon(x_1) \cdots x_i \cdots \varepsilon(x_n).$$

Just as in Lemma 1.3b.3, one sees that

$$(\sigma \circ pr_1) * \cdots * (\sigma \circ pr_n) = \mu_n \circ (\otimes^n \sigma)$$

where $\mu_n: \otimes^n H \to H$ is the map:

$$x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n.$$

Take $C = B_1$ in particular. Then for any $\sigma \in W_k(B_1, H)$ the image $\sigma(B_1)$ is contained in a commutative subalgebra of H, since $\sigma(b_0) = 1$. This means that the map

$$\mu_n \circ (\otimes^n \sigma): \otimes^n B_1 \to H$$

factors through $\mathbf{S}^n B_1$. Thus we can well-define a map

$$f_n: W_k(B_1, H) \to W_k(\mathbf{S}^n B_1, H), \sigma \mapsto (\sigma \circ pr_1) * \cdots * (\sigma \circ pr_n).$$

Now let p be the characteristic exponent of k. Let

$$g: B_1 \to \mathbf{S}^p B_1$$

be the composite:

$$\begin{aligned} &B_1 \to \otimes^p B_1 \xrightarrow{canonic} \mathbf{S}^p B_1 \\ &b_0 \mapsto \otimes^p b_0 \\ &b_1 \mapsto \otimes^p b_1. \end{aligned}$$

We claim that g is a coalgebra map. Indeed we have

$$\Delta(\otimes^p b_1) = \sum_{i \in \{0,1\}^p} (b_{i_1} \otimes \cdots \otimes b_{i_p}) \otimes (b_{1-i_1} \otimes \cdots \otimes b_{1-i_p})$$

in $\otimes^p B_1$. Consider the natural action of the cyclic group of order p on the set $\{0, 1\}^p$. Since each orbit has length 1 or p, it follows that the image $\overline{\otimes^p b_1}$ of $\otimes^p b_1$ in $\mathbf{S}^p B_1$ is a primitive element. This means that g is a coalgebra map.

Let $P: W_k(B_1, H) \to W_k(B_1, H)$ be the composite:

$$W_k(B_1, H) \xrightarrow{f_p} W_k(\mathbf{S}^p B_1, H) \xrightarrow{W_k(g, H)} W_k(B_1, H).$$

Then the diagram:

$$\begin{tikzcd} W_k(B_1,H) \arrow[r, "P"] \arrow[d, "\theta", "\wr"'] & W_k(B_1,H) \arrow[d, "\theta", "\wr"'] \\ \mathrm{P}(H) \arrow[r, "(-)^p"] & \mathrm{P}(H) \end{tikzcd}$$

commutes. Indeed let $\sigma \in W_k(B_1, H)$. Then the map

$$f_p(\sigma): \mathbf{S}^p B_1 \to H$$

is induced by the composite:

$$\otimes^p B_1 \xrightarrow{\otimes^p \sigma} \otimes^p H \xrightarrow{\mu_p} H.$$

Hence we have

$$P(\sigma)(b_1) = f_p(\sigma)(g(b_1)) = \mu_p((\otimes^p \sigma)(\otimes^p b_1)) = \sigma(b_1)^p.$$

Thus we have proven:

PROPOSITION. *The map $P: W_k(B_1, H) \to W_k(B_1, H)$ makes the Lie algebra $W_k(B_1, H)$ into a restricted Lie algebra such that the canonical bijection*

$$\theta: W_k(B_1, H) \xrightarrow{\simeq} \mathrm{P}(H)$$

is an isomorphism of restricted Lie algebras.

1.3b.6 Let $R \in \boldsymbol{M}_k$ with $\eta: k \to R$ the structure map. Let $\sigma, \rho \in W_R(C, H)$ and $\tau \in W_R(D, H)$, where $C, D \in W_R$. The map

$$\sigma \otimes_R \tau: R \otimes C \otimes D \to R \otimes H \otimes H$$

is a W_R-map from $C \otimes D$ to $H \otimes H$, since

$$(C \otimes D)_0 \subset C_0 \otimes D_0 \quad \text{and} \quad (H \otimes H)_0 = H_0 \otimes H_0 = k.$$

Now $W_R(C, H)$ is a group by (1.3.5). It is easy to see that the multiplication satisfies

$$\sigma * \rho = W_\eta(\mu) \circ (\sigma \otimes_R \rho) \circ W_\eta(\Delta).$$

In particular if we define maps

$$\begin{aligned} \beta &: H \otimes H \to H \\ \mu_n &: \otimes^n H \to H \\ pr_1 &: C \otimes D \to C \\ pr_2 &: C \otimes D \to D \\ pr_i &: \otimes^n C \to C \end{aligned}$$

just as in the previous paragraphs, it follows that

$$W_\eta(\beta)\circ(\sigma\otimes_R\tau)=(\sigma\circ W_\eta(pr_1))*(\tau\circ W_\eta(pr_2))*(\sigma\circ W_\eta(pr_1))^{-1}*(\tau\circ W_\eta(pr_2))^{-1}$$

in $W_R(C\otimes D, H)$ and

$$W_\eta(\mu_n)\circ(\otimes_R^n\sigma)=(\sigma\circ W_\eta(pr_1))*\cdots*(\sigma\circ W_\eta(pr_n))$$

in $W_R(\otimes^n C, H)$. For $\lambda\in R$, we define a W_R-map $u_\lambda\colon B_1\to B_1$ by

$$u_\lambda(1\otimes b_0)=1\otimes b_0 \quad\text{and}\quad u_\lambda(1\otimes b_1)=\lambda\otimes b_1.$$

Finally let

$$\nu\colon B_1\otimes B_1\to B_1 \quad\text{and}\quad g\colon B_1\to \mathbf{S}^p B_1$$

be the coalgebra maps defined in (1.3b.4 and 5).

PROPOSITION. (i) *The action:*

$$R\times W_R(B_1,H)\to W_R(B_1,H),\ (\lambda,\sigma)\mapsto\sigma\circ u_\lambda$$

makes the group $W_R(B_1,H)$ into an R-module.

(ii) *There exists a unique map*

$$\alpha\colon W_R(B_1,H)\times W_R(B_1,H)\to W_R(B_1,H)$$

such that

$$\alpha(\sigma,\tau)\circ W_\eta(\nu)=(\sigma\circ W_\eta(pr_1))*(\tau\circ W_\eta(pr_2))*(\sigma\circ W_\eta(pr_1))^{-1}*(\tau\circ W_\eta(pr_2))^{-1}$$

in the group $W_R(B_1\otimes B_1, H)$ for any $\sigma,\tau\in W_R(B_1,H)$. Then the bracket product

$$[\sigma,\tau]=\alpha(\sigma,\tau)$$

turns the R-module $W_R(B_1,H)$ into a Lie algebra over R.

(iii) *Let p be the characteristic exponent of k. The map:*

$$\begin{aligned} W_R(B_1,H)&\to W_R(\otimes^p B_1,H)\\ \sigma&\mapsto(\sigma\circ W_\eta(pr_1))*\cdots*(\sigma\circ W_\eta(pr_p))\end{aligned}$$

induces a map

$$f_p\colon W_R(B_1,H)\to W_R(\mathbf{S}^p B_1,H).$$

Let $P\colon W_R(B_1,H)\to W_R(B_1,H)$ be the composite:

$$W_R(B_1,H)\xrightarrow{f_p}W_R(\mathbf{S}^p B_1,H)\xrightarrow{W_R(W_\eta(g),H)}W_R(B_1,H).$$

Then the map P makes the R-Lie algebra $W_R(B_1, H)$ into a restrictive Lie algebra over R.

(iv) *The map*: $\sigma \mapsto \sigma(1 \otimes b_1)$, $W_R(B_1, H) \to R \otimes H$ *induces an isomorphism of restricted Lie algebras over R*:

$$W_R(B_1, H) \xrightarrow[\simeq]{} R \otimes \mathrm{P}(H).$$

PROOF. The set of primitive elements of the R-coalgebra $R \otimes H$ with respect to $1 \otimes 1$ is equal to $R \otimes \mathrm{P}(H)$ as is easily seen. Since for any $\sigma \in W_R(B_1, H)$ we have $\sigma(1 \otimes b_0) = 1 \otimes 1$, it follows that the map:

$$W_R(B_1, H) \to R \otimes \mathrm{P}(H), \sigma \mapsto \sigma(1 \otimes b_1)$$

is bijective. This implies Proposition by the same method just as in case of $R = k$ (where one should notice that $R \otimes \mathbf{S}^p B_1$ is the p-th symmetric power of the R-module $R \otimes B_1$).

1.4 The Krull dimension of coalgebras

1.4.1 PROPOSITION. *Let C be a cocommutative coalgebra with coradical filtration $\{C_i\}_{i \geqq 0}$. Suppose C_1 is finite dimensional. Then C_n is finite dimensional for any $n \geqq 0$, C^* is Noetherian and every ideal of C^* is of the form $D^\perp$, where D is a uniquely determined subcoalgebra of C.*

This is contained in [8, 5.2.1, 2.4.3 and 1.3.1]. We give here another:

PROOF. Since $C_n = C_0 \wedge C_{n-1}$, we have

$$\Delta: C/C_n \hookrightarrow C/C_0 \otimes C/C_{n-1}.$$

But because $\Delta(C_{n+1}) \subset \sum_{i+j=n+1} C_i \otimes C_j$ [11, Corollary 9.1.7], we have

$$\Delta: C_{n+1}/C_n \hookrightarrow C_1/C_0 \otimes C_n/C_{n-1}.$$

Therefore C_n is finite dimensional by induction. We have for $i \leqq j$ $C_i = \bigwedge^{i+1} C_0 = ((C_0^\perp)^{i+1})^\perp$ in C_j by 1.1.1 or [11, Proposition 9.0.0 b)]. Since C_j is finite dimensional this means that

$$\mathrm{Ker}\,(C_i^* \leftarrow C_j^*) = \mathrm{Ker}\,(C_0^* \leftarrow C_j^*)^{i+1}.$$

If we apply [3, III, § 2, n°11, Proposition 14] and its two Corollaries to

$$C^* = \varprojlim_n C_n^*,$$

it follows that C^* is Noetherian and the topology on C^*, which is the projective limit of discrete rings C_n^*, is the same as the $C_0^\perp$-adic topology.

More precisely we have

$$C_n^{\perp}=\mathrm{Ker}\,(C_n^*\leftarrow C^*)=\mathrm{Ker}\,(C_0^*\leftarrow C^*)^{n+1}=(C_0^{\perp})^{n+1},$$

i.e. C^* is the $C_0^{\perp}$-adic completion of C^*. Hence by [3, III, § 3, n°4, Proposition 8 (i)] C^* is a Zariski ring. It follows from [3, III, § 3, n°3, Proposition 6 c)] that every ideal I of C^* is closed. The closure of I in C^* is easily seen to be $I^{\perp\perp}$. If we notice that $I^{\perp}$ is a subcoalgebra of C [11, Proposition 1.4.3 b)] the proof is complete.

1.4.2 Let $C\in W_k$. A subcoalgebra D of C is said to be a *coprime* subcoalgebra, if $D\neq 0$ and for any subcoalgebras A and B of C, $D\subset A\wedge B$ implies $D\subset A$ or $D\subset B$.

LEMMA. *D is a coprime subcoalgebra of C iff $D^{\perp}$ is a prime ideal of C^*.*

PROOF. $D\neq 0$ is equivalent to $D^{\perp}\neq C^*$. For any ideals I and J of C^*, $I^{\perp}$ and $J^{\perp}$ are subcoalgebras of C [11, Proposition 1.4.3 b)], and $IJ\subset D^{\perp}$ is equivalent to $D\subset(IJ)^{\perp}=I^{\perp}\wedge J^{\perp}$ (1.1.1). Hence if D is coprime then $D^{\perp}$ is prime. Conversely if A and B are subcoalgebras of C, then $A^{\perp}$ and $B^{\perp}$ are ideals of C^* [11, Proposition 1.4.3 a)]. Suppose $D^{\perp}$ is prime. Then $D\subset A\wedge B=(A^{\perp}B^{\perp})^{\perp}$ implies $A^{\perp}B^{\perp}\subset D^{\perp}$ and so $A^{\perp}\subset D^{\perp}$ or $B^{\perp}\subset D^{\perp}$. This means $D\subset A$ or $D\subset B$. Hence D is coprime.

1.4.3 DEFINITION. *Let $C\in W_k$. The Krull dimension of C, which we denote by $K\dim C$, is the supremum of the length n of chains*

$$D_0\subsetneqq D_1\subsetneqq\cdots\subsetneqq D_n$$

of coprime subcoalgebras of C.

1.4.4 COROLLARY. *If C_1 is finite dimensional then $K\dim C$ is equal to the Krull dimension $K\dim C^*$ of C^*.*[4)]

1.4.5 REMARK. A subcoalgebra E of a cocommutative coalgebra C is coprime iff E^* is an integral domain. Hence if D is a subcoalgebra of C it is clear that

$$K\dim D\leqq K\dim C.$$

If $K\dim C<\infty$ and C^* is an integral domain, the equality holds iff $D=C$.

1.4.6 Following [8, § 2.2] a coalgebra C is said to be *of finite type* if C_1 is finite dimensional.

Let C be a cocommutative coalgebra of finite type. By 1.4.1 every

ideal of C^* is of the form $D^\perp$. $D^\perp$ is cofinite in C^* iff D is finite dimensional. Hence we have

$$C=\varinjlim D \xrightarrow{\simeq} \varinjlim (C^*/D^\perp)^* = (C^*)^0,$$

where D is taken over all finite dimensional subcoalgebras of C.

In this case, since we have

$$\boldsymbol{M}_k(C^*, C'^*) \simeq \boldsymbol{W}_k(C', (C^*)^0)$$

for any $C' \in \boldsymbol{W}_k$, the canonical map

$$\boldsymbol{W}_k(C', C) \to \boldsymbol{M}_k(C^*, C'^*),\ f \mapsto {}^t f$$

is a bijection. See [8] for detailed or generalized results on coalgebras of finite type.

1.5 Some Hopf algebras

In this section we consider some representable group-functors on $\boldsymbol{W}_k$, $\boldsymbol{W}_k^{\mathrm{cn}}$ or $\boldsymbol{W}$ and determine the representing Hopf algebras or hyperalgebras.

1.5.1 Let V be a vector space. Then there exists a pair $(\mathrm{C}(V), \pi_V)$ with $\mathrm{C}(V) \in \boldsymbol{W}_k$ and $\pi_V \in \boldsymbol{Mod}_k(\mathrm{C}(V), V)$ such that for any $C \in \boldsymbol{W}_k$, the map

$$\boldsymbol{W}_k(C, \mathrm{C}(V)) \to \boldsymbol{Mod}_k(C, V),\ \sigma \mapsto \pi_V \circ \sigma$$

is bijective [11, Theorem 6.4.3]. Since such a pair $(\mathrm{C}(V), \pi_V)$ is uniquely determined up to isomorphisms, we say that $\mathrm{C}(V)$ is *the* cofree cocommutative coalgebra on V and π_V the canonical projection.

If $V, W \in \boldsymbol{Mod}_k$ and $f: V \to W$ a linear map, then there exists a unique coalgebra map

$$\mathrm{C}(f): \mathrm{C}(V) \to \mathrm{C}(W)$$

such that $\pi_W \circ \mathrm{C}(f) = f \circ \pi_V$ [11, Proposition 6.4.6]. Thus the correspondence

$$\mathrm{C}: \boldsymbol{Mod}_k \to \boldsymbol{W}_k,\ V \mapsto \mathrm{C}(V)$$

becomes a covariant functor (which is the right adjoint of the forgetfull functor: $\boldsymbol{W}_k \to \boldsymbol{Mod}_k$).

The functor $\mathrm{C}: \boldsymbol{Mod}_k \to \boldsymbol{W}_k$ clearly commutes with limits (because it has the left adjoint). In particular we have canonically

$$\mathrm{C}(V \oplus W) \xrightarrow{\simeq} \mathrm{C}(V) \otimes \mathrm{C}(W) \quad \text{and} \quad \mathrm{C}(0) \xrightarrow{\simeq} k$$

for any $V, W \in \boldsymbol{Mod}_k$. (Through the first isomorphism the canonical projection $\pi_{(V \oplus W)}: \mathrm{C}(V \oplus W) \to V \oplus W$ corresponds to the map

$$\mathrm{C}(V) \otimes \mathrm{C}(W) \to V \oplus W, \ x \otimes y \mapsto (\pi_V(x)\varepsilon(y), \varepsilon(x)\pi_W(y))).$$

Now Sweedler makes $\mathrm{C}(V)$ a Hopf algebra as follows [11, Theorem 6.4.8]: The linear maps

$$\begin{aligned} &V \oplus V \to V, \qquad (v, w) \mapsto v + w, \\ &0 \to V \\ &V \to V, \ v \mapsto -v \end{aligned}$$

induce coalgebra maps (respectively)

$$\begin{aligned} &\mu: \mathrm{C}(V) \otimes \mathrm{C}(V) \simeq \mathrm{C}(V \oplus V) \to \mathrm{C}(\mathrm{V}) \\ &\eta: k \simeq \mathrm{C}(0) \to \mathrm{C}(V) \\ &S: \mathrm{C}(V) \to \mathrm{C}(V) \end{aligned}$$

which make $\mathrm{C}(V)$ into a cocommutative and commutative Hopf algebra (with antipode S).[5] We leave it to the reader as an exercise to prove:

PROPOSITION. *Let V be a vector space.*

(i) *The cocommutative Hopf algebra* $\mathrm{C}_\mathrm{a}(V)$ *represents the group-functor on* $\boldsymbol{W}_k$:

$$C \mapsto \boldsymbol{Mod}_k(C, V).$$

More precisely the Hopf algebra structure (or group object structure in $\boldsymbol{W}_k$*)* (μ, η, S) *on* $\mathrm{C}_\mathrm{a}(V)$ *is the unique structure which turns the canonical isomorphism*

$$\boldsymbol{W}_k(C, \mathrm{C}(V)) \xrightarrow{\simeq} \boldsymbol{Mod}_k(C, V), \ \sigma \mapsto \pi_V \circ \sigma$$

into a group isomorphism for any $C \in \boldsymbol{W}_k$.

(ii) *The bialgebra structure* (μ, η) *on* $\mathrm{C}_\mathrm{a}(V)$ *is the unique one such that*

$$\pi_V(xy) = \pi_V(x)\varepsilon(y) + \varepsilon(x)\pi_V(y)$$

1.5.2 Let V be a vector space. *The additive B-W hyperalgebra* on V, written $\mathrm{B}_\mathrm{a}(V)$, is $\mathrm{C}_\mathrm{a}(V)^1$ the irreducible component of the Hopf algebra $\mathrm{C}_\mathrm{a}(V)$ containing 1. The underlying coalgebra of $\mathrm{B}_\mathrm{a}(V)$ is called the *B-W* coalgebra on V and denoted by $\mathrm{B}(V)$ (cf. [9, § 4.2] or [11, page 261, Definition]).

The restriction $\pi_V | \mathrm{B}_\mathrm{a}(V)$ of the canonical projection π_V is also denoted by π_V. By (1.3.4) we have

$$
\begin{aligned}
\boldsymbol{W}_k(C, \mathrm{B_a}(V)) &\simeq \mathrm{Ker}\,(\boldsymbol{W}_k(C, \mathrm{C}(V)) \to \boldsymbol{W}_k(C_0, \mathrm{C}(V))) \\
&\simeq \mathrm{Ker}\,(\boldsymbol{Mod}_k(C, V) \to \boldsymbol{Mod}_k(C_0, V)) \\
&\simeq \boldsymbol{Mod}_k(C/C_0, V).
\end{aligned}
$$

In other words, the canonical projection π_V induces an isomorphism of groups:

$$\boldsymbol{W}_k(C, \mathrm{B_a}(V)) \xrightarrow{\simeq} \boldsymbol{Mod}_k(C/C_0, V),\ \sigma \mapsto \pi_V \circ \sigma$$

for any $C \in \boldsymbol{W}_k$.

In particular if $C \in \boldsymbol{W}_k^{\mathrm{cn}}$, then

$$C^+ \xrightarrow{\simeq} C/C_0,$$

where $C^+ = \mathrm{Ker}\,(\varepsilon : C \to k)$. Hence $\mathrm{B_a}(V)$ as a group object in the category $\boldsymbol{W}_k^{\mathrm{cn}}$ represents the following group-functor on $\boldsymbol{W}_k^{\mathrm{cn}}$:

$$C \mapsto \boldsymbol{Mod}_k(C^+, V).$$

In view of [11, Theorem 12.2.5], this means that our B-W coalgebra $\mathrm{B}(V)$ coincides with the coalgebra $\mathrm{B}(V)$ defined by Sweedler.

Now the functor

$$\mathrm{B} : \boldsymbol{Mod}_k \to \boldsymbol{W}_k,\ V \mapsto \mathrm{B}(V)$$

is the right adjoint of the functor:

$$\boldsymbol{W}_k \to \boldsymbol{Mod}_k,\ C \mapsto C/C_0.$$

Hence the functor $\mathrm{B}(-)$ commutes with finite limits. In particular we have canonically

$$\mathrm{B}(V \oplus W) \xrightarrow{\simeq} \mathrm{B}(V) \otimes \mathrm{B}(W) \quad \text{and} \quad \mathrm{B}(0) \xrightarrow{\simeq} k.$$

It is clear that the structure maps μ, η and S on $\mathrm{B_a}(V)$ are the coalgebra maps induced by the following linear maps respectively:

$$
\begin{aligned}
&V \oplus V \to V, \qquad (v, w) \mapsto v + w \\
&0 \to V \\
&V \to V, \qquad v \mapsto -v.
\end{aligned}
$$

There is another characterization of the bialgebra structure (μ, η) on $\mathrm{B_a}(V)$. The map $\eta : k \to \mathrm{B}(V)$ is the unique coalgebra map. The map μ: $\mathrm{B}(V) \otimes \mathrm{B}(V) \to \mathrm{B}(V)$ is the unique coalgebra map such that

$$\pi_V(\mu(x\otimes y))=\pi_V(x)\varepsilon(y)+\varepsilon(x)\pi_V(y)$$

for any $x, y\in \mathrm{B}(V)$.

Sweedler defines a Hopf algebra structure on $\mathrm{B}(V)$ [11, page 261, Definition]. $\mathrm{B}(V)$ being given that structure, it is easy to show that "our" structure map

$$\mu: \mathrm{B}(V)\otimes\mathrm{B}(V)\to\mathrm{B}(V)$$

becomes a map of Hopf algebras, since it is induced by a linear map: $V\oplus V\to V$ (cf. [11, Lemma 12.3.1 d)]. This means that the bialgebra $\mathrm{B}(V)$ defined by Sweedler coincides with our $\mathrm{B}_{\mathrm{a}}(V)$.

1.5.3 Let B be the coalgebra on basis $b_0, b_1, b_2, \cdots$ with $\varepsilon(b_n)=\delta_{0,n}$ and $\Delta(b_n)=\sum b_i\otimes b_{n-i}$. Then $B\in \boldsymbol{W}_k$. Since $\{kb_n\}_{n\geqq 0}$ defines a strict graduation on B [11, page 232], the coradical filtration of B is given by

$$B_n=kb_0+\cdots+kb_n$$

[11, Lemma 11.2.1]. Let

$$\pi: B\to k$$

be the linear map: $b_n\mapsto\delta_{1,n}$.

LEMMA. *Let $R\in \boldsymbol{M}_k$ and $C\in\boldsymbol{W}_R$. Then the map*

$$\boldsymbol{W}_R(C,B)\to\boldsymbol{Mod}_R(R\otimes(C/C_0),R),\ \sigma\mapsto(1_R\otimes\pi)\circ\sigma$$

is bijective.

PROOF. Note that an element σ of $\boldsymbol{W}_R(C,B)$ is an R-coalgebra map: $R\otimes C\to R\otimes B$ such that $\sigma(R\otimes C_0)\subset R\otimes B_0$. Since $1_R\otimes\pi$ is zero on $R\otimes B_0$, the composite $(1_R\otimes\pi)\circ\sigma$ is zero on $R\otimes C_0$. Hence the above map is well-defined. We have only to construct the inverse map.

Recall that $\boldsymbol{Mod}_k(C,R)$ is an algebra with multiplication

$$f*g=\mu\circ(f\otimes g)\circ\Delta$$

and the unit $\eta\circ\varepsilon$. Let $V, W\subset C$ be subspaces and $f, g\in\boldsymbol{Mod}_k(C,R)$. If $f|V=0$ and $g|W=0$, then we have

$$f*g\,|\,V\wedge W=0$$

clearly from the definition of the wedge product.

Now we can naturally identify

$$\boldsymbol{Mod}_R(R\otimes(C/C_0),R)\simeq\boldsymbol{Mod}_k(C/C_0,R).$$

Let $f \in \boldsymbol{Mod}_k(C/C_0, R)$. That is f is an element of $\boldsymbol{Mod}_k(C, R)$ such that $f \mid C_0 = 0$. Then since $C_{n-1} = \bigwedge^n C_0$, we have

$$f^n \mid C_{n-1} = 0$$

for $n > 0$, where f^n is the n-fold power of f in the algebra $\boldsymbol{Mod}_k(C, R)$ and $\{C_n\}$ is the coradical filtration of C. Since $C = \bigcup C_n$, we can well-define

$$\sigma(f) : C \to R \otimes B,\ c \mid\to \textstyle\sum_{n=0}^{\infty} f^n(c) b_n.$$

Let $\overline{\sigma(f)} : R \otimes B$ be the R-linear extension of $\sigma(f)$. Then it is easy to show that $\overline{\sigma(f)}$ is a unique R-coalgebra map: $R \otimes C \to R \otimes B$ such that

$$\overline{\sigma(f)}(R \otimes C_0) \subset R \otimes B_0 \quad \text{and} \quad (1_R \otimes \pi) \circ \overline{\sigma(f)} = \bar{f},$$

where $\bar{f} : R \otimes (C/C_0) \to R$ is the R-linear extension of f. This proves Lemma.

1.5.4 The above Lemma means that the pair (B, π) is "a" B-W coalgebra on k. Hence there exists a unique coalgebra isomorphism $\theta : \mathrm{B}(k) \xrightarrow{\simeq} B$ such that $\pi \circ \theta = \pi_k$, or equivalently there exists a "unique" basis $\{b_0, b_1, b_2, \cdots\}$ for $\mathrm{B}(k)$ such that

$$\Delta(b_n) = \textstyle\sum b_i \otimes b_{n-i}$$
$$\varepsilon(b_n) = \delta_{0,n} \quad \text{and} \quad \pi_k(b_n) = \delta_{1,n}.$$

Such a basis $\{b_n\}$ is called the canonical basis for $\mathrm{B}(k)$.

Let $R \in \boldsymbol{M}_k$, $C \in \boldsymbol{W}_R$ and $V \in \boldsymbol{Mod}_k$. Consider the map

$$\omega_{C,V} : \boldsymbol{W}_R(C, \mathrm{B}_a(V)) \to \boldsymbol{Mod}_R(R \otimes (C/C_0), R \otimes V)$$
$$\sigma \mid\to (1_R \otimes \pi_V) \circ \sigma.$$

This is easily seen to be a well-defined homomorphism of groups, where the left hand side is a group by (1.3.5). The Lemma above means that $\omega_{C,k}$ is an isomorphism. Notice that C being fixed the functors:

$$V \mid\to \boldsymbol{W}_R(C, \mathrm{B}_a(V)) \quad \text{and} \quad V \mid\to \boldsymbol{Mod}_R(R \otimes (C/C_0), R \otimes V)$$

commute with finite products. Hence if V is finite dimensional, then since it is a product of finite copies of k, the map $\omega_{C,V}$ is an isomorphism. Suppose now that C is finite dimensional. Let $\{V_\beta\}$ be the set of all finite dimensional subspaces of V. Since we have

$$\mathrm{B}_a(V) \simeq \varinjlim_{\beta} \mathrm{B}_a(V_\beta)$$

by [11, Lemma 12.3.1], it follows that

$$\boldsymbol{W}_R(C, \mathrm{B}_a(V)) \simeq \varinjlim_{\beta} \boldsymbol{W}_R(C, \mathrm{B}_a(V_\beta)) \quad \text{and}$$

$$\boldsymbol{Mod}_R(R\otimes(C/C_0), R\otimes V) \simeq \varinjlim_{\beta} \boldsymbol{Mod}_R(R\otimes(C/C_0), R\otimes V_\beta).$$

Since each map ω_{C,V_β} is an isomorphism, $\omega_{C,V}$ is isomorphic whenever C is finite dimensional. In general let $\{C_\alpha\}$ be the set of all finite dimensional subcoalgebras of C. Since

$$C = \varinjlim_{\alpha} C_\alpha \quad \text{and} \quad C/C_0 = \varinjlim_{\alpha} C_\alpha/(C_\alpha)_0,$$

it follows that

$$\boldsymbol{W}_R(C, \mathrm{B}_a(V)) \simeq \varprojlim_{\alpha} \boldsymbol{W}_R(C_\alpha, \mathrm{B}_a(V)) \quad \text{and}$$

$$\boldsymbol{Mod}_R(R\otimes(C/C_0), R\otimes V) \simeq \varprojlim_{\alpha} \boldsymbol{Mod}_R(R\otimes(C_\alpha/(C_\alpha)_0), R\otimes V).$$

Since each map $\omega_{C_\alpha,V}$ is an isomorphism, the map $\omega_{C,V}$ is always an isomorphism. Thus we have proven:

PROPOSITION. *Let V be a vector space. Then the map:*

$$\begin{aligned} \boldsymbol{W}_R(C, \mathrm{B}_a(V)) &\to \boldsymbol{Mod}_R(R\otimes(C/C_0), R\otimes V) \\ \sigma &\mapsto (1_R\otimes\pi_V)\circ\sigma \end{aligned}$$

is an isomorphism of groups for any $R \in \boldsymbol{M}_k$ and $C \in \boldsymbol{W}_R$.

1.5.5 Let A be an associative (not necessarily commutative) k-algebra. Put

$$\pi'_A = \pi_A + \eta\circ\varepsilon : \mathrm{B}(A) \to A,$$

where $\eta: k \to A$ is the structure map. It follows immediately from (1.5.2) that π'_A induces a bijection:

$$\begin{aligned} \boldsymbol{W}_k(C, \mathrm{B}(A)) &\xrightarrow{\simeq} \{f \in \boldsymbol{Mod}_k(C, A) \mid f|C_0 = \eta\circ\varepsilon\} \\ \sigma &\longmapsto \pi'_A\circ\sigma \end{aligned}$$

for any $C \in \boldsymbol{W}_k$. Now the vector space $\boldsymbol{Mod}_k(C, A)$ has a natural structure of k-algebra and the subset

$$\mathrm{T}_A(C) = \{f \in \boldsymbol{Mod}_k(C, A) \mid f|C_0 = \eta\circ\varepsilon\}$$

is clearly a multiplicative submonoid of $\boldsymbol{Mod}_k(C, A)$. The above fact means that the coalgebra $\mathrm{B}(A)$ represents the monoid-functor on $\boldsymbol{W}_k$

$$\mathrm{T}_A : C \to \mathrm{T}_A(C).$$

Therefore there exists a unique hyperalgebra structure (μ', η') on $\mathrm{B}(A)$ such that the bijection

$$W_k(C, \mathrm{B}(A)) \xrightarrow{\simeq} \mathrm{T}_A(C), \sigma \mapsto \pi'_A \circ \sigma$$

turns into a monoid isomorphism. In particular it follows that $\mathrm{T}_A(C)$ is a multiplicative subgroup of $\boldsymbol{Mod}_k(C, A)$.

PROPOSITION. *(μ', η') is the unique bialgebra structure on $\mathrm{B}(A)$ which makes the map*

$$\pi'_A : \mathrm{B}(A) \to A$$

into an algebra map.

DEFINITION. *The hyperalgebra $(\mathrm{B}(A), \mu', \eta')$ is called* the multiplicative *B-W* hyperalgebra *on A and denoted by $\mathrm{B_m}(A)$.*

PROOF. In the group $W_k(\mathrm{B}(A) \otimes \mathrm{B}(A), \mathrm{B_m}(A))$ we have

$$\mu' = (1_{\mathrm{B}(A)} \otimes \varepsilon_{\mathrm{B}(A)}) * (\varepsilon_{\mathrm{B}(A)} \otimes 1_{\mathrm{B}(A)}).$$

Apply the group isomorphism:

$$W_k(\mathrm{B}(A) \otimes \mathrm{B}(A), \mathrm{B_m}(A)) \xrightarrow{\simeq} \mathrm{T}_A(\mathrm{B}(A) \otimes \mathrm{B}(A))$$
$$\sigma \longmapsto \pi'_A \circ \sigma.$$

Then we have

$$\pi'_A \circ \mu' = (\pi'_A \otimes \varepsilon_{\mathrm{B}(A)}) * (\varepsilon_{\mathrm{B}(A)} \otimes \pi'_A) = \mu_A \circ (\pi'_A \otimes \pi'_A).$$

This means that $\mu' : \mathrm{B}(A) \otimes \mathrm{B}(A) \to \mathrm{B}(A)$ is a unique coalgebra map such that $\pi'_A \circ \mu' = \mu_A \circ (\pi'_A \otimes \pi'_A)$. Similarly we have

$$\pi'_A \circ \eta' = \eta_A$$

and such a coalgebra map η' is unique. This proves Proposition.

1.5.6 PROPOSITION. *Let A be an associative k-algebra. Then for any $R \in \boldsymbol{M}_k$ and $C \in \boldsymbol{W}_R$, the composite*

$$W_R(C, \mathrm{B_m}(A)) \to \boldsymbol{Mod}_R(R \otimes C, R \otimes A) \simeq \boldsymbol{Mod}_k(C, R \otimes A)$$
$$\sigma \mapsto (1_R \otimes \pi'_A) \circ \sigma$$

induces an isomorphism of groups:

$$W_R(C, \mathrm{B_m}(A)) \xrightarrow{\simeq} \mathrm{T}_{(R\otimes A)}(C).$$

PROOF. This is a corollary to Propositions 1.5.4 and 1.5.5. Details are left to the reader as an exercise.

1.5.7 PROPOSITION. *Let A be an algebra and H a hyperalgebra. Then the canonical projection* π'_A: $\mathrm{B_m}(A) \to A$ induces a bijection:

$$\boldsymbol{Hopf}_k(H, \mathrm{B_m}(A)) \xrightarrow{\simeq} \boldsymbol{Alg}_k(H, A), \sigma \mapsto \pi'_A \circ \sigma.$$

PROOF. Let $f \in W_k(H, \mathrm{B}(A))$. We show that

$$f \in \boldsymbol{Alg}_k(H, \mathrm{B_m}(A)) \Leftrightarrow \pi'_A \circ f \in \boldsymbol{Alg}_k(H, A).$$

Suppose that $\pi'_A \circ f$ is an algebra map. Since then we have

$$\pi'_A \circ f \circ \mu_H = \mu_A \circ (\pi'_A \circ f \otimes \pi'_A \circ f) = \pi'_A \circ \mu' \circ (f \otimes f) \quad \text{and}$$
$$\pi'_A \circ f \circ \eta_H = \eta_A = \pi'_A \circ \eta',$$

where (μ', η') is the algebra structure on $\mathrm{B_m}(A)$, it follows that

$$f \circ \mu_H = \mu' \circ (f \otimes f) \quad \text{and} \quad f \circ \eta_H = \eta'$$

and so f is an algebra map. Now if $g \in \boldsymbol{Alg}_k(H, A)$, then $g \mid H_0 = \eta \circ \varepsilon$ because H is connected. Hence we have $\boldsymbol{Alg}_k(H, A) \subset \mathrm{T}_A(H)$. This proves Proposition.

1.5.8 Let $\{b_n\}$ be the canonical basis for $\mathrm{B}(k) = B$ (1.5.4).

EXAMPLE. The multiplication μ (resp. μ') on $\mathrm{B_a}(k)$ (resp. $\mathrm{B_m}(k)$) is determined by

$$\mu(b_i \otimes b_j) = \binom{i+j}{i} b_{i+j}$$

$\Bigg($resp. by

$$\mu'(b_i \otimes b_j) = \sum_{r=\mathrm{Max}(i,j)}^{i+j} \frac{r!}{(r-i)!\,(r-j)!\,(i+j-r)!}\, b_r \Bigg).$$

PROOF. Put $\zeta = \pi_k \circ \mu$ and $\zeta' = \pi_k \circ \mu'$. Since

$$\pi_k \circ \mu = \pi_k \otimes \varepsilon + \varepsilon \otimes \pi_k \quad \text{and} \quad \pi'_k \circ \mu' = \mu \circ (\pi'_k \otimes \pi'_k),$$

it follows that

$$\zeta(b_i \otimes b_j) = \delta_{(0,1),(i,j)} + \delta_{(1,0),(i,j)} \quad \text{and}$$
$$\zeta'(b_i \otimes b_j) = \delta_{(0,1),(i,j)} + \delta_{(1,0),(i,j)} + \delta_{(1,1),(i,j)}$$

where δ is the Kronecker symbol. As is seen in Proof of Lemma 1.5.3, we have

$$\mu(x)=\sum \zeta^r(x)b_r \quad \text{and} \quad \mu'(x)=\sum \zeta'^r(x)b_r,$$

for any $x \in B\otimes B$, where ζ^r is the r-fold power of ζ in the algebra $\boldsymbol{Mod}_k(B\otimes B, k)$. Hence we have

$$\mu(b_i\otimes b_j)=\sum_r \sum_{\substack{i_1+\cdots+i_r=i \\ j_1+\cdots+j_r=j}} \zeta(b_{i_1}\otimes b_{j_1})\cdots\zeta(b_{i_r}\otimes b_{j_r})b_r$$

and the analogous formula for μ' and ζ'. Now a simple calculation shows that

$$\sum_{\substack{i_1+\cdots+i_r=i \\ j_1+\cdots+j_r=j}} \zeta(b_{i_1}\otimes b_{j_1})\cdots\zeta(b_{i_r}\otimes b_{j_r})=\binom{i+j}{i}\delta_{i+j,r} \quad \text{and}$$

$$\sum_{\substack{i_1+\cdots+i_r=i \\ j_1+\cdots+j_r=j}} \zeta'(b_{i_1}\otimes b_{j_1})\cdots\zeta'(b_{i_r}\otimes b_{j_r})$$

$$=\begin{cases} \dfrac{r!}{(r-i)!\,(r-j)!\,(i+j-r)!} & \text{if } \operatorname{Max}(i,j)\leqq r\leqq i+j \\ 0 & \text{otherwise.} \end{cases}$$

From this follows the assertion.

1.5.9 Let $V\in \boldsymbol{Mod}_k$ and $C\in \boldsymbol{W}_k$. Then C^* is a commutative k-algebra and $C^*\otimes V$ has a natural structure of C^*-module. Let

$$\mathrm{p}_{C,V}: \mathrm{End}_{C^*}(C^*\otimes V)\hookrightarrow \boldsymbol{Mod}_k(C, \mathrm{End}_k(V))$$

be the injection induced by the canonical injection: $C^*\otimes V\hookrightarrow \boldsymbol{Mod}_k(C, V)$ as follows:

$$\begin{aligned}\mathrm{End}_{C^*}(C^*\otimes V)&\simeq \boldsymbol{Mod}_k(V, C^*\otimes V)\hookrightarrow \boldsymbol{Mod}_k(V, \boldsymbol{Mod}_k(C, V)) \\ &\simeq \boldsymbol{Mod}_k(C, \mathrm{End}_k(V)).\end{aligned}$$

The map $\mathrm{p}_{C,V}$ is easily seen to be a k-algebra map. If C or V is finite dimensional, then $\mathrm{p}_{C,V}$ is an isomorphism.

We denote by $\mathbf{E}^nV$ the n-fold exterior power of V. Then $C^*\otimes\mathbf{E}^nV$ can be naturally identified with the n-fold exterior power of the C^*-module $C^*\otimes V$. For an element $g\in \mathrm{End}_{C^*}(C^*\otimes V)$, we denote by $\mathbf{E}^ng: C^*\otimes\mathbf{E}^nV \to C^*\otimes\mathbf{E}^nV$ the C^*-linear map induced by g, that is

$$\mathbf{E}^ng(1\otimes(v_1\wedge\cdots\wedge v_n))=g(1\otimes v_1)\wedge\cdots\wedge g(1\otimes v_n)$$

for $v_1, \cdots, v_n\in V$.

Let $f \in \boldsymbol{Mod}_k(C, \mathrm{End}_k(V))$. We define a linear map $f_n: C \to \mathrm{End}_k(\mathbf{E}^n V)$ by

$$f_n(c)(v_1 \wedge \cdots \wedge v_n) = \sum f(c_{(1)})(v_1) \wedge \cdots \wedge f(c_{(n)})(v_n)$$

where $c \in C$ and $v_1, \cdots, v_n \in V$. This is well-defined because C is cocommutative. The reader may easily verify the following:

LEMMA. (i) *The map*

$$\boldsymbol{Mod}_k(C, \mathrm{End}_k(V)) \to \boldsymbol{Mod}_k(C, \mathrm{End}_k(\mathbf{E}^n V)),\ f \mapsto f_n$$

is a multiplicative monoid map.

(ii) *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{End}_{C^*}(C^* \otimes V) & \xrightarrow[g \mapsto \mathbf{E}^n g]{} & \mathrm{End}_{C^*}(C^* \otimes \mathbf{E}^n V) \\ {\scriptstyle \mathrm{p}_{C,V}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{p}_{C,\mathbf{E}^n V}} \\ \boldsymbol{Mod}_k(C, \mathrm{End}_k(V)) & \xrightarrow[f \mapsto f_n]{} & \boldsymbol{Mod}_k(C, \mathrm{End}_k(\mathbf{E}^n V)) \end{array}$$

(iii) *If H is a cocommutative Hopf algebra and $f \in \boldsymbol{Alg}_k(H, \mathrm{End}_k(V))$ then $f_n \in \boldsymbol{Alg}_k(H, \mathrm{End}_k(\mathbf{E}^n V))$.*

1.5.10 Let V be a finite dimensional vector space. If we put $d = \dim_k V$ then

$$\mathrm{End}_k(\mathbf{E}^d V) = k.$$

For an element f of $\boldsymbol{Mod}_k(C, \mathrm{End}_k(V))$, where $C \in \boldsymbol{W}_k$, the element f_d of $\boldsymbol{Mod}_k(C, k) = C^*$ is called *the Larson's character* of f [11, page 151]. We put

$$\mathrm{Larson}_{C,V}: \boldsymbol{Mod}_k(C, \mathrm{End}_k(V)) \to \boldsymbol{Mod}_k(C, k),\ f \mapsto f_d.$$

This map is functorial with respect to $C \in \boldsymbol{W}_k$ and induces a group homomorphism:

$$\mathrm{T}_{\mathrm{End}_k(V)}(C) \to \mathrm{T}_k(C).$$

If H is a cocommutative Hopf algebra then the map $\mathrm{Larson}_{H,V}$ induces a map:

$$\boldsymbol{Alg}_k(H, \mathrm{End}_k(V)) \to \boldsymbol{Alg}_k(H, k).$$

Recall that $\pi'_A: \mathrm{B_m}(\mathrm{End}_k(V)) \to \mathrm{End}_k(V)$ is an algebra map (1.5.5), where we put $A = \mathrm{End}_k(V)$. Hence the map

$$(\pi'_A)_d: \mathrm{B_m}(\mathrm{End}_k(V)) \to k$$

is also an algebra map. Let

$$\mathbf{D}_V : \mathrm{B_m}\,(\mathrm{End}_k\,(V)) \to \mathrm{B_m}(k)$$

be the unique hyperalgebra map such that $\pi'_k \circ \mathbf{D}_V = (\pi'_A)_d$ (1.5.7). The following is an easy consequence of Lemma 1.5.9:

PROPOSITION. *Let V be a d-dimensional vector space with $d < \infty$. Then for any $C \in \boldsymbol{W}_k$ the following diagram commutes:*

$$\begin{array}{ccccc} \boldsymbol{W}_k(C, \mathrm{B_m}\,(\mathrm{End}_k\,(V))) & \simeq & \mathrm{T}_{\mathrm{End}_k(V)}\,(C) & \xrightarrow{\mathrm{p}_{C,V}^{-1}} & \mathrm{End}_{C^*}\,(C^* \otimes V) \\ \Big\downarrow {\scriptstyle \boldsymbol{W}_k(C, \mathbf{D}_V)} & & \Big\downarrow {\scriptstyle \mathrm{Larson}_{C,V}} & & \Big\downarrow {\scriptstyle \mathrm{determinant}} \\ \boldsymbol{W}_k(C, \mathrm{B_m}(k)) & \simeq & \mathrm{T}_k(C) & \xrightarrow{\mathrm{p}_{C,k}^{-1}} & C^* \end{array}.$$

If H is a hyperalgebra we have a commutative diagram

$$\begin{array}{ccc} \boldsymbol{Hopf}_k(H, \mathrm{B_m}\,(\mathrm{End}_k\,(V))) & \simeq & \boldsymbol{Alg}_k(H, \mathrm{End}_k\,(V)) \\ \Big\downarrow {\scriptstyle \boldsymbol{Hopf}_k(H, \mathbf{D}_V)} & & \Big\downarrow {\scriptstyle \mathrm{Larson}_{H,V}} \\ \boldsymbol{Hopf}_k(H, \mathrm{B_m}(k)) & \simeq & \boldsymbol{Alg}_k(H, k) \end{array}.$$

1.5.11 Let A and B be algebras and $C \in \boldsymbol{W}_k$. A linear map

$$f : C \to \boldsymbol{Mod}_k(A, B)$$

is said to measure A to B [11, page 138] if it satisfies

$$f(c)(a\alpha) = \sum [f(c_{(1)})(a)][f(c_{(2)})(\alpha)] \quad \text{and} \quad f(c)(1) = \varepsilon(c)1,$$

for any $a, \alpha \in A$ and $c \in C$. This is equivalent to saying that the map:

$$A \to \boldsymbol{Mod}_k(C, B),\ a \mapsto f(?)(a)$$

is an algebra map, where $f(?)(a) : c \mapsto f(c)(a)$ [11, Proposition 7.0.1].

Let's denote by Meas (C, A, B) the set of elements of $\boldsymbol{Mod}_k(C, \boldsymbol{Mod}_k(A, B))$ which measure A to B. The set-functor on $\boldsymbol{W}_k$

$$C \mapsto \mathrm{Meas}\,(C, A, B)$$

is seen to be representable [11, Theorem 7.0.4]. We denote by $\mathrm{M_c}\,(A, B)$ the representing object in $\boldsymbol{W}_k$.

Now Meas (C, A, A) is a multiplicative submonoid of the algebra $\boldsymbol{Mod}_k(C, \mathrm{End}_k\,(A))$, that is if $f, g \in \mathrm{Meas}\,(C, A, A)$ then $f * g \in \mathrm{Meas}\,(C, A, A)$ and the unit $\eta \circ \varepsilon$ belongs to Meas (C, A, A). Since the coalgebra $\mathrm{M_c}\,(A, A)$ represents the monoid-functor on $\boldsymbol{W}_k$:

$$C \mapsto \mathrm{Meas}\,(C, A, A),$$

it follows that there exists a unique bialgebra structure (μ, η) on $\mathrm{M}_c(A, A)$ which makes the canonical isomorphism: $W_k(C, \mathrm{M}_c(A, A)) \simeq \mathrm{Meas}\,(C, A, A)$ into a monoid isomorphism for any $C \in W_k$. This is also a unique bialgebra structure on $\mathrm{M}_c(A, A)$ which turns the universal measuring

$$\theta : \mathrm{M}_c(A, A) \to \mathrm{End}_k(A)$$

into an algebra map (cf. [11, § 7.0]).

Let $C \in W_k^{\mathrm{f}}$. The algebra isomorphism

$$\mathrm{p}_{C,A} : \mathrm{End}_{C^*}(C^* \otimes A) \xrightarrow{\simeq} \boldsymbol{Mod}_k(C, \mathrm{End}_k(A))$$

induces an isomorphism of multiplicative submonoids:

$$\boldsymbol{Alg}_{C^*}(C^* \otimes A, C^* \otimes A) \xrightarrow{\simeq} \mathrm{Meas}\,(C, A, A),$$

which is the composite of the natural isomorphisms

$$\begin{aligned}\boldsymbol{Alg}_{C^*}(C^* \otimes A, C^* \otimes A) &\simeq \boldsymbol{Alg}_k(A, C^* \otimes A)\\ &\simeq \boldsymbol{Alg}_k(A, \boldsymbol{Mod}_k(C, A)) \simeq \mathrm{Meas}\,(C, A, A).\end{aligned}$$

Thus the bialgebra $\mathrm{M}_c(A, A)$ represents in the sense of (1.3.2) the monoid-functor on W_k^{f}:

$$C \mid \to \boldsymbol{Alg}_{C^*}(C^* \otimes A, C^* \otimes A).$$

1.6 Birkhoff-Witt coalgebras

In this paragraph we give two characterizations of Birkhoff-Witt coalgebras $\mathrm{B}(V)$.

1.6.1 For $C \in W_k^{\mathrm{cn}}$ we denoted by $\mathrm{P}(C)$ the set of primitive elements of C with respect to the unique group-like element g_C, i.e.

$$\mathrm{P}(C) = \{x \in C \mid \Delta(x) = g_C \otimes x + x \otimes g_C\}.$$

Let $B = \mathrm{B}(k)$ and $\{b_n\}$ be the canonical basis of B (1.5.4). For any vector space V we have

$$\mathrm{P}(\mathrm{B}(V)) \simeq W_k(B_1, \mathrm{B}(V)) \simeq \boldsymbol{Mod}_k(B_1/B_0, V) \simeq V.$$

This means that $\pi : \mathrm{P}(\mathrm{B}(V)) \xrightarrow{\simeq} V$, where π is the canonical map (1.5.2).

PROPOSITION. (i) *Let V be a finite dimensional vector space. Then* $\mathrm{B}(V)$ *is of finite type and* $K \dim \mathrm{B}\,(V) = \dim_k V$.

(ii) *Let C be a connected cocommutative coalgebra of finite type. Then* $K\dim C \leqq \dim_k \mathrm{P}(C)$. *The equality holds iff* $C \simeq \mathrm{B}(V)$ *for some vector space* V.

PROOF. (i) Since $\mathrm{P}(\mathrm{B}(V)) \simeq V$ is finite dimensional, $\mathrm{B}(V)$ is of finite type. By [11, page 278, Example-Exercise],

$$B(V)^* \simeq k[[X_1, \cdots, X_d]],$$

where $d = \dim_k V$. Hence by 1.4.4,

$$K\dim \mathrm{B}(V) = K\dim \mathrm{B}(V)^* = d.$$

(ii) Let $u: C \to \mathrm{P}(C)$ be a linear map such that $u|C_0 = 0$ and $u|\mathrm{P}(C) =$ identity. Let $\tilde{u}: C \to \mathrm{B}(\mathrm{P}(C))$ be the coalgebra map such that $\pi \circ \tilde{u} = u$. Since $\tilde{u}|\mathrm{P}(C)$ is injective, u is injective by [11, Lemma 11.0.1]. Hence by 1.4.5, we have

$$K\dim C \leqq K\dim \mathrm{B}(\mathrm{P}(C)) = \dim_k \mathrm{P}(C),$$

and the equality holds iff u is bijective

1.6.2 LEMMA. *Let k be an infinite field. For any integer n, there exists a set $\{f_\alpha\}_{\alpha \in \Phi}$ of algebra maps from $k[[X_1, \cdots, X_n]]$, the power series ring in n variables, to $k[[X]]$ such that*

$$\bigcap\nolimits_{\alpha \in \Phi} \mathrm{Ker}\,(f_\alpha) = 0.$$

PROOF. Put $\Phi = k^n$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Phi$, let

$$f_\alpha : k[[X_1, \cdots, X_n]] \to k[[X]]$$

be the algebra map determined by $f_\alpha(X_i) = \alpha_i X$. Let ϕ be an element in $\bigcap_\alpha \mathrm{Ker}\,(f_\alpha)$ with ϕ_r the r-th homogeneous part. Then we have

$$f_\alpha(\phi) = \textstyle\sum_r \phi_r(\alpha_1, \cdots, \alpha_n) X^r = 0,$$

for any $\alpha \in \Phi$. Since k is infinite this means $\phi = 0$.

1.6.3 LEMMA. *Let V be a vector space over an infinite field k. Then* $\mathrm{B}(V) = \sum \omega(B)$, *where ω is taken over all elements of $W_k(B, \mathrm{B}(V))$.*

PROOF. Since $\mathrm{B}(V) = \varinjlim_{V \supset U \text{ f.d.}} \mathrm{B}(U)$ by [11, Lemma 12.3.1], we may suppose that V is finite, say n-, dimensional. Then we have

$$\mathrm{B}(V)^* \simeq k[[X_1, \cdots, X_n]] \quad \text{and} \quad B^* \simeq k[[X]].$$

By Lemma above there exists a set $\{f_\alpha\}_{\alpha \in \Phi}$ of algebra maps from $\mathrm{B}(V)^*$ to

B^* such that

$$\bigcap_{\alpha\in\Phi} \operatorname{Ker}(f_\alpha)=0.$$

It follows from 1.4.6, that $f_\alpha={}^t\omega_\alpha$ for some coalgebra map ω_α from B to $\mathrm{B}(V)$, for any $\alpha\in\Phi$. Hence we have

$$(\sum_\alpha \omega_\alpha(B))^\perp=\bigcap_\alpha \operatorname{Ker}(f_\alpha)=0.$$

This means

$$\mathrm{B}(V)=\sum_\alpha \omega_\alpha(B).$$

1.6.4 Let $C\in \boldsymbol{W}_k$. A sequence of elements of C

$$\{d_0, d_1, \cdots, d_n\}$$

is called an n-sequence of divided powers in C if

$$\Delta(d_j)=\sum_{i=0}^{j} d_i\otimes d_{j-i} \quad \text{and} \quad \varepsilon(d_j)=\delta_{0,j},$$

for $j\leqq n$. The set of n-sequences of divided powers in C is identified with $\boldsymbol{W}_k(B_n, C)$.

PROPOSITION. *Let k be an infinite field. A cocommutative connected coalgebra C is isomorphic to* $\mathrm{B}(V)$ *for some vector space V iff any n-sequence of divided powers in C can be extended to an $(n+1)$-sequence of divided powers in C, for any $n<\infty$.*

PROOF. The latter condition is equivalent to the fact that the canonical map

$$\boldsymbol{W}_k(B_n, C)\leftarrow \boldsymbol{W}_k(B_{n+1}, C)$$

is surjective for any n. If $C=\mathrm{B}(V)$, then this map is identified with

$$\boldsymbol{Mod}_k(B_n/B_0, V)\leftarrow \boldsymbol{Mod}_k(B_{n+1}/B_0, V),$$

which is clearly surjective. Assume that C satisfies the latter condition in Proposition. As is shown in Proof of Proposition 1.6.1, there exists an injective coalgebra map $u: C\to \mathrm{B}(V)$ such that

$$u: \mathrm{P}(C)\xrightarrow{\simeq}\mathrm{P}(\mathrm{B}(V))\xrightarrow{\simeq} V.$$

We identify C with $\operatorname{Im}(u)$, a subcoalgebra of $\mathrm{B}(V)$. First we show that

$$\boldsymbol{W}_k(B_n, C)=\boldsymbol{W}_k(B_n, \mathrm{B}(V)), \qquad \text{for } n\geqq 0.$$

Indeed this is valid when $n=1$, for $\boldsymbol{W}_k(B_1, C)$ is identified with $\mathrm{P}(C)$. Assume

that $W_k(B_{n-1}, C)=W_k(B_{n-1}, \mathrm{B}(V))$, where $n>1$. Let $\phi \in W_k(B_n, \mathrm{B}(V))$. Since $\phi(B_{n-1})\subset C$, there exists an element $\psi \in W_k(B_n, C)$ such that

$$\phi \mid B_{n-1}=\psi \mid B_{n-1}$$

by hypothesis. Then it is clear that $\phi(b_n)-\psi(b_n) \in \mathrm{P}(\mathrm{B}(V))=\mathrm{P}(C)$. Hence $\phi(b_n) \in C$. This shows $\phi \in W_k(B_n, C)$. Thus we have

$$W_k(B_n, C)=W_k(B_n, \mathrm{B}(V)),$$

by induction. Since $B=\varinjlim_n B_n$, we have

$$W_k(B, C)=W_k(B, \mathrm{B}(V)).$$

This implies in view of Lemma 1.6.3 that $C=\mathrm{B}(V)$.

1.7 Flatness

Let $\phi: A\to B$ be a map of finitely generated commutative k-algebras. The main purpose of this paragraph is to prove that B is a flat (resp. faithfully flat) A-module iff B^0 is an injective A^0-comodule (resp. an injective cogenerator in the category of A^0-comodules).

1.7.1 Let A (resp. C) be a k-algebra (resp. a k-coalgebra). By an A-module (resp. by a C-comodule) we mean a *left* A-module (resp. a *right* C-comodule). The category of A-modules (resp. of C-comodules) will be denoted by $\boldsymbol{Mod}_A$(resp. by $\boldsymbol{Comod}_C$).

An A-module is said to be *locally finite* if it is a union of finite dimensional submodules. The category of locally finite A-modules is denoted by $\boldsymbol{Mod}_A^{\mathrm{lf}}$. Any A-module M contains the largest locally finite submodule denoted by M^{lf}, i.e.,

$$M^{\mathrm{lf}}=\{m \in M \mid Am \text{ is finite dimensional}\}.$$

For any locally finite A-module N we have

$$\boldsymbol{Mod}_A^{\mathrm{lf}}(N, M^{\mathrm{lf}})=\boldsymbol{Mod}_A(N, M).$$

Let V be a C-comodule. The comodule structure map on V will be denoted by

$$\rho_V: V\to V\otimes C, v \mapsto \sum v_{(0)}\otimes v_{(1)}$$

or simply by ρ. It follows from [11, Corollary 2.1.4] that the operation

$$X\cdot v=\sum \langle X, v_{(1)}\rangle v_{(0)},$$

where $X \in C^*$ and $v \in V$, makes V into a *locally finite* C^*-module.

In particular by pull back along the canonical map: $A \to (A^0)^*$, every A^0-comodule becomes a locally finite A-module.

LEMMA. *The functor*

$$\alpha : V \mapsto V,\ \boldsymbol{Comod}_{A^0} \to \boldsymbol{Mod}_A^{\mathrm{lf}}$$

is an isomorphism of categories.

PROOF. We shall construct a functor

$$\beta : \boldsymbol{Mod}_A^{\mathrm{lf}} \to \boldsymbol{Comod}_{A^0}$$

such that $\alpha\beta$ and $\beta\alpha$ are identities. Let M be a locally finite A-module. Let $\rho : M \to \boldsymbol{Mod}_k(A, M)$ be the map defined by

$$\rho(m)(a) = am.$$

Let $M \otimes A^0 \hookrightarrow \boldsymbol{Mod}_k(A, M)$ be the composite of natural maps

$$M \otimes A^0 \subset M \otimes A^* \hookrightarrow \boldsymbol{Mod}_k(A, M).$$

We show that $\rho(M) \subset M \otimes A^0$. Since M is a union of finite dimensional submodules we may assume that M is finite dimensional. Then the annihilator $I = \mathrm{Ann}_A(M)$ of M is a cofinite ideal in A and we have

$$\rho(M) \subset \boldsymbol{Mod}_k(A/I, M) \subset \boldsymbol{Mod}_k(A, M).$$

Since $\boldsymbol{Mod}_k(A/I, M) \simeq M \otimes (A/I)^* \subset M \otimes A^0$, we have

$$\rho(M) \subset M \otimes A^0.$$

It is easy to see that ρ defines an A^0-comodule structure on M and that the correspondence $M \mapsto (M, \rho)$ induces a functor

$$\beta : \boldsymbol{Mod}_A^{\mathrm{lf}} \to \boldsymbol{Comod}_{A^0},$$

which is the inverse of α.

In the following we shall identify the category $\boldsymbol{Comod}_{A^0}$ with the category $\boldsymbol{Mod}_A^{\mathrm{lf}}$ via the isomorphism α.

1.7.2 Throughout the rest of this paragraph A will denote a *commutative* k-algebra.

For any A-module M, $M^* = \boldsymbol{Mod}_k(M, k)$ has an A-module structure defined by

$$\langle aX, m \rangle = \langle X, am \rangle$$

for $a \in A, X \in M^*$ and $m \in M$. M^* equipped with this A-module structure is called the transpose module of M [11, page 98].

Let M and $N \in \boldsymbol{Mod}_A$. Then the natural isomorphism

$$\boldsymbol{Mod}_k(M, N^*) \simeq \boldsymbol{Mod}_k(N, M^*)$$

induces clearly a natural isomorphism (of A-modules)

$$\boldsymbol{Mod}_A(M, N^*) \simeq \boldsymbol{Mod}_A(N, M^*).$$

LEMMA. *For any $M \in \boldsymbol{Mod}_A$, the largest locally finite submodule $(M^*)^{\mathrm{lf}}$ of M^* coincides with*

$$M^0 = \{X \in M^* \mid \langle X, N \rangle = 0 \text{ for some cofinite submodule } N \text{ of } M\}.$$

PROOF. Let P be a finite dimensional submodule of M^*. Then the canonical map $M \to P^*$ is A-linear. Its kernel N is therefore a cofinite submodule of M and satisfies $\langle N, P \rangle = 0$. This implies $(M^*)^{\mathrm{lf}} \subset M^0$. The opposite inclusion $M^0 \subset (M^*)^{\mathrm{lf}}$ is clear.

REMARK. In particular A^0 (defined in [11, page 109]) is equal to $(A^*)^{\mathrm{lf}}$.

1.7.3 For an A-module M, M^0 can be considered as an A^0-comodule by 1.7.1. If V is an A^0-comodule, then since $M^0 = (M^*)^{\mathrm{lf}}$, we have a natural isomorphism (of A^0-modules)

$$\boldsymbol{Comod}_{A^0}(V, M^0) = \boldsymbol{Mod}_A(V, M^*) \simeq \boldsymbol{Mod}_A(M, V^*).$$

In this sense the functors

$$\boldsymbol{Mod}_A \to \boldsymbol{Comod}_{A^0}, M \mapsto M^0 \quad \text{and} \quad \boldsymbol{Comod}_{A^0} \to \boldsymbol{Mod}_A, V \mapsto V^*$$

are adjoint to one another (cf. [11, Theorem 6.0.5]).

In particular the functor $M \mapsto M^0$ is left exact, i.e. if

$$M' \to M \to M'' \to 0$$

is exact in $\boldsymbol{Mod}_A$, then

$$M'^0 \leftarrow M^0 \leftarrow M''^0 \leftarrow 0$$

is exact in $\boldsymbol{Comod}_{A^0}$.

1.7.4 We denote by $\boldsymbol{Mod}_A^{\mathrm{fg}}$ the category of *finitely generated* A-modules.

LEMMA. *If A is a commutative Noetherian k-algebra then the functor*

$$M \mapsto M^0$$

is exact on $\boldsymbol{Mod}_A^{\mathrm{fg}}$.

PROOF. It is enough to show the right exactness. In other words for any $M \in \boldsymbol{Mod}_A^{\mathrm{fg}}$ and any submodule N of M we have only to show the surjectivity of the canonical map $M^0 \to N^0$. Let $\{N_i\}$ be the set of all cofinite submodules of N. Then we have

$$N^0 = \varinjlim_i (N/N_i)^* \quad \text{and} \quad M^0 = \varinjlim_i (M/N_i)^0.$$

Since the surjectivity of the canonical maps

$$(M/N_i)^0 \to (N/N_i)^*$$

implies the desired surjectivity we can suppose that N is finite dimensional. Put $I = \mathrm{Ann}_A(N)$ the annihilator of N. Then $I^n M$ is a cofinite submodule of M for any integer n. By Krull's theorem we have

$$\bigcap_n I^n M = \{m \in M \mid (1-y)m = 0 \qquad \text{for some } y \in I\}.$$

This means in particular $N \cap (\bigcap_n I^n M) = 0$. Since N is finite dimensional $N \cap I^n M = 0$ for some n, or equivalently $N \hookrightarrow M/(I^n M)$. This implies the surjectivity of the composite

$$(M/(I^n M))^* \hookrightarrow M \to N^*.^0$$

and completes Proof.

1.7.5 LEMMA. *For any finitely generated A-module M, we have a natural isomorphism of A-modules*

$$\boldsymbol{Mod}_A(M, A^0) \simeq M^0.$$

In particular if A is Noetherian, A^0 is an injective *A-module.*

PROOF. First we show that $\boldsymbol{Mod}_A(M, A^0)$ is locally finite. Indeed if f is an element in $\boldsymbol{Mod}_A(M, A^\circ)$ then $f(M)$ is a finite dimensional submodule, since A^0 is locally finite and M is finitely generated. Hence $f(M)$ is annihilated by some cofinite ideal I of A. Since I annihilates f, Af is finite dimensional. Thus $\boldsymbol{Mod}_A(M, A^0)$ is locally finite.

Let N be an A^0-comodule. Then we have natural isomorphisms

$$\begin{aligned} \boldsymbol{Comod}_{A^0}(N, \boldsymbol{Mod}_A(M, A^0)) &\simeq \boldsymbol{Mod}_A(M, \boldsymbol{Comod}_{A^0}(N, A^0)) \\ &\simeq \boldsymbol{Mod}_A(M, N^*) \\ &\simeq \boldsymbol{Comod}_{A^0}(N, M^0), \end{aligned}$$

in view of

$$\boldsymbol{Comod}_{A^0}(N, A^0) \simeq \boldsymbol{Mod}_A(A, N^*) \simeq N^*.$$

This shows the existence of a natural isomorphism of A-modules

$$\boldsymbol{Mod}_A(M, A^0) \simeq M^0.$$

Next we suppose that A is Noetherian. Then by 1.7.4 the functor

$$M \mid \to M^0 \simeq \boldsymbol{Mod}_A(M, A^0)$$

is exact on $\boldsymbol{Mod}_A^{\mathrm{fg}}$. In particular for any ideal I of A the canonical map

$$\boldsymbol{Mod}_A(A, A^0) \to \boldsymbol{Mod}_A(I, A^0)$$

is surjective. This means that A^0 is an injective A-module.

1.7.6 LEMMA. *Let A be Noetherian. Then A^0 is an injective cogenerator in $\boldsymbol{Mod}_A$ iff every maximal ideal of A is cofinite.*

PROOF. "If" part. Let M be a non-zero finitely generated A-module. Take a maximal ideal m of A containing the annihilator $\mathrm{Ann}_A(M)$. By Krull's theorem we have $M \neq mM$. Since mM is cofinite in M, this means $M^0 \neq 0$. Hence the functor $\boldsymbol{Mod}_A(-, A^0)$ is faithful on $\boldsymbol{Mod}_A^{\mathrm{fg}}$. Since A^0 is injective this means that $\boldsymbol{Mod}_A(-A^0)$ is faithful on $\boldsymbol{Mod}_A$. Therefore A^0 is an injective cogenerator in $\boldsymbol{Mod}_A$.

"Only if" part. Let m be a maximal ideal of A. Since $(A/m)^0 \simeq \boldsymbol{Mod}_A(A/m, A^0) \neq 0$, A has a proper cofinite ideal containing m. Hence m is cofinite.

REMARK. If A is a finitely generated k-algebra, then it is well-known that the maximal ideals of A are cofinite. Hence A^0 is then an injective cogenerator in $\boldsymbol{Mod}_A$.

1.7.7 LEMMA. *Suppose that A is Noetherian. Then an A^0-comodule V is injective in $\boldsymbol{Comod}_{A^0}$ iff it is injective in $\boldsymbol{Mod}_A$.*

PROOF. Enough to prove the "only if" part. Suppose that V is an injective object in $\boldsymbol{Comod}_{A^0}$. Let M be a finitely generated A-module and N be a submodule of M. The surjectivity of the canonical map $M^0 \to N^0$ means that for any cofinite submodule P of N, there exists a cofinite submodule Q of M such that

$$Q \cap N = P.$$

Let f be an element in $\boldsymbol{Mod}_A(N, V)$. Since $f(N)$ is finite dimensional, the kernel P of f is cofinite in N. Take a cofinite submodule Q of M such that $Q \cap N = P$. Since $f \in \boldsymbol{Mod}_A(N/P, V)$ and V is injective in $\boldsymbol{Comod}_{A^0}$, f can be extended to an element of $\boldsymbol{Mod}_A(M/Q, V)$. Hence the canonical map

$$\boldsymbol{Mod}_A(M, V) \to \boldsymbol{Mod}_A(N, V)$$

is surjective. In particular if I is an ideal of A the canonical map

$$\boldsymbol{Mod}_A(A, V) \to \boldsymbol{Mod}_A(I, V)$$

is surjective. Therefore V is injective in $\boldsymbol{Mod}_A$.

REMARK. Let C be a coalgebra and V a vector space. The C-comodule $V \otimes C$, whose structure map is $1_V \otimes \Delta_C$, is an injective object in $\boldsymbol{Comod}_C$, since we have a natural isomorphism

$$\boldsymbol{Comod}_C(W, V \otimes C) \simeq \boldsymbol{Mod}_k(W, V)$$

for any $W \in \boldsymbol{Comod}_C$. If V is a C-comodule, the structure map

$$\rho_V : V \to V \otimes C$$

is a C-comodule map. Hence V is injective in $\boldsymbol{Comod}_C$ iff ρ_V splits in $\boldsymbol{Comod}_C$.

Let A be Noetherian. Then A^0 is an injective A-module (1.7.5). Since any direct sum of injective modules is also injective [5, page 17, Exercise 8], $V \otimes A^0$ is injective for any vector space V. This observation gives another proof of the above Lemma.

1.7.8 Let $\phi : A \to B$ be a map of commutative Noetherian k-algebras. We suppose that every maximal ideal of B is cofinite. Any B^0-comodule is viewed also as an A^0-comodule via the coalgebra map $\phi^0 : B^0 \to A^0$. In particular B^0 has a structure of A^0-comodule.

PROPOSITION. *For any finitely generated B-module M, M is a flat (resp. faithfully flat) A-modulo iff M^0 is an injective object (resp. an injective cogenerator) in $\boldsymbol{Comod}_{A^0}$, where M^0 is the largest locally finite B-submodule of M^*.*

PROOF. M is A-flat (resp. faithfully A-flat) iff the functor

$$M \otimes_A (-) : \boldsymbol{Mod}_A \to \boldsymbol{Mod}_B$$

is exact (resp. faithfully exact). Since B^0 is an injective cogenerator in $\boldsymbol{Mod}_B$, this is equivalent to saying that the functor

$$\begin{aligned} &\boldsymbol{Mod}_B(M \otimes_A (-), B^0) \\ &\quad \simeq \boldsymbol{Mod}_A(-, \boldsymbol{Mod}_B(M, B^0)) \\ &\quad \simeq \boldsymbol{Mod}_A(-, M^0) \end{aligned}$$

is exact (resp. faithfully exact), or equivalently that M^0 is an injective object (resp. an injective cogenerator) in $\boldsymbol{Mod}_A$. It follows from 1.7.7 that this

is equivalent to saying that M^0 is an injective object (resp. an injective cogenerator) in $\boldsymbol{Comod}_{A^0}$.

1.7.9 COROLLALY. *B is a flat (resp. faithfully flat) A-module iff B^0 is an injective object (resp. an injective cogenerator) in $\boldsymbol{Comod}_{A^0}$. If B is a faithfully flat A-module, then the induced map*

$$\phi^0 : B^0 \to A^0$$

is surjective.

PROOF. It is enough to show the later part. Suppose that B is faithfully flat over A. Since ϕ is injective, we regard A as a subalgebra of B. Let I be a cofinite ideal of A. Then there exists an ideal J of B such that $J \cap A = I$ [2, I, § 3, n°5, Proposition 9]. We claim that B/J is *proper*, that is the canonical map

$$B/J \to (B/J)^{0*}$$

is injective [11, § 6.1]. But it follows from Lemma 1.7.6 that $(B/J)^0$ is a faithful B/J-module, since B/J is Noetherian and its every maximal ideal is cofinite by assumption. Hence B/J is proper. Therefore J is the intersection of the cofinite ideals of B containing J. Since A/I is a finite dimensional subalgebra of B/J, there exists a cofinite ideal J' of B containing J such that $J' \cap A = I$. This proves the assertion.

1.8 Theorem of smoothness

1.8.1 THEOREM. *Let $f: C \to D$ be a map of connected cocommutative coalgebras. Let $E = \mathrm{Ker}_0(f)$ (1.2.8) be the largest subcoalgebra of C contained in $f^{-1}(kg_D)$, where g_D is the unique group-like element in D. Then the following conditions are equivalent.*

(i) *C is an injective D-comodule and $E \simeq \mathrm{B}(U)$ for some vector space U.*

(iii) *For any connected commutative coalgebra F, any subcoalgebra F' of F and any cocommutative diagram*

$$\begin{tikzcd} C \arrow[r, "f"] & D \\ F' \arrow[u, "\phi"] \arrow[r, phantom, "\subset"] & F \arrow[u, "\psi"'] \end{tikzcd},$$

where ϕ and ψ are coalgebra maps, there exists a coalgebra map $\chi: C \to F$ such that

$$f \circ \chi = \psi \quad \textit{and} \quad \chi | F' = \phi.$$

(v) *There exist a vector space U and an isomorphism of coalgebras*

$$\theta: C \xrightarrow{\simeq} D\otimes \mathrm{B}(U)$$

such that $(1\otimes\varepsilon)\circ\theta=f$.

DEFINITION. *A map* $f: C\to D$ *in* W_k^{cn} *is said to be smooth if it satisfies one of the equivalent conditions* (i), (iii) *and* (v) *above.*

One sees latter that the conditions (i), (iii) and (v) are obtained respectively by translating the conditions (i), (iii) and (v) of Theorem of smoothness [7, I, § 4, 4.2] into the coalgebra language (cf. Introduction). Thus the above theorem is, in a sense, a coalgebra-theoretical version of Theorem of smoothness. The condition (ii), which is concerned with the concept of being étale, will be treated later. The condition (iv) there seems difficult to be translated into the coalgebra language.[6]

PROOF. (v)$\Rightarrow$(iii). Suppose that we are given a commutative diagram

$$\begin{tikzcd} D\otimes \mathrm{B}(U) \arrow[r, "1\otimes\varepsilon"] & D \\ F' \arrow[u, "\phi"] & F, \arrow[u, "\psi"'] \arrow[l, phantom, "\supset"] \end{tikzcd}$$

where $F\in W_k^{\mathrm{cn}}$ and ϕ and ψ are coalgebra maps. Recall that

$$\mathrm{B}(U) \xleftarrow{\varepsilon\otimes 1} D\otimes \mathrm{B}(U) \xrightarrow{1\otimes\varepsilon} D$$

is a direct product diagram in W_k^{cn} (1.2.2). Since

$$W_k^{\mathrm{cn}}(F, \mathrm{B}(U)) \simeq \boldsymbol{Mod}_k(F^+, U)$$

naturally (1.5.2), there exists a coalgebra map

$$\xi: F\to \mathrm{B}(U)$$

such that $\xi|F'=(\varepsilon\otimes 1)\circ\phi$. Let

$$\chi: F\to D\otimes \mathrm{B}(U)$$

be the unique coalgebra map determined by

$$(1\otimes\varepsilon)\circ\chi=\psi \quad \text{and} \quad (\varepsilon\otimes 1)\circ\chi=\xi.$$

Then we have $\chi|F'=\phi$. Hence the condition (iii) is valid.

(v)$\Rightarrow$(i). The diagram

$$\begin{array}{ccc} D\otimes \mathrm{B}(U) & \xrightarrow{1\otimes\varepsilon} & \mathrm{D} \\ \cup & & \cup \\ kg_D\otimes \mathrm{B}(U) & \xrightarrow{1\otimes\varepsilon} & kg_D, \end{array}$$

which is of the form

$$\begin{array}{ccc} X\times Z & \xrightarrow{\text{projection}} & X \\ \uparrow{\scriptstyle \alpha\times 1_Z} & & \uparrow{\scriptstyle \alpha} \\ Y\times Z & \xrightarrow{\text{projection}} & Y \end{array}$$

is clearly a pullback diagram in the category W_k^{cn}. Hence by definition we have

$$E \simeq kg_D \otimes \mathrm{B}(U) \simeq \mathrm{B}(U).$$

Since we know that $D\otimes \mathrm{B}(U)$ is an injective D-comodule (1.7.7 Remark), the condition (i) is satisfied.

Let $U=\mathrm{P}(E)$ be the set of primitive elements in E. Let $u: C \to U$ be a linear map such that

$$u|C_0=0 \quad \text{and} \quad u|U=\text{identity}.$$

It follows from (1.5.2) that there exists a coalgebra map $\tilde{u}: C \to \mathrm{B}(U)$ such that $\pi\circ\tilde{u}=u$. Since $D\otimes\mathrm{B}(U)$ is the direct product of D with $B(U)$ there exists a coalgebra map $\theta: C\to D\otimes\mathrm{B}(U)$ such that

$$(1\otimes\varepsilon)\circ\theta=f \quad \text{and} \quad (\varepsilon\otimes 1)\circ\theta=\tilde{u}.$$

By Lemma 1.8.2 below, θ is injective.

(iii)$\Rightarrow$(v). Enough to show that θ is bijective. If we apply the condition (iii) to the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \text{identity}}\uparrow & & \uparrow{\scriptstyle 1\otimes\varepsilon} \\ C & \xhookrightarrow{\theta} & D\otimes\mathrm{B}(U), \end{array}$$

we get a coalgebra map $\chi: D\otimes\mathrm{B}(U)\to C$ such that

$$f\circ\chi=1\otimes\varepsilon \quad \text{and} \quad \chi\circ\theta=\text{identity}.$$

Since $f\circ\chi=1\otimes\varepsilon$ has a section in the category W_k^{cn}, f has also a section. In particular the map

$$\mathrm{P}(f): \mathrm{P}(C)\to\mathrm{P}(D)$$

is surjective. Hence by Lemma 1.8.2 the map

$$\mathrm{P}(\theta): \mathrm{P}(C)\to\mathrm{P}(D\otimes\mathrm{B}(U))$$

is bijective. This means that the map

$$\mathrm{P}(\chi): \mathrm{P}(D \otimes \mathrm{B}(U)) \to \mathrm{P}(C)$$

is also bijective. Hence χ is injective by [11, Lemma 11.0.1]. Therefore $\theta: C \xrightarrow{\simeq} D \otimes \mathrm{B}(U)$ is an isomorphism.

(i)$\Rightarrow$(v). Since the diagram

$$\begin{array}{ccc} D\otimes \mathrm{B}(U) & \xrightarrow{1\otimes\varepsilon} & D \\ \cup & & \cup \\ kg_D\otimes \mathrm{B}(U) & \xrightarrow{1\otimes\varepsilon} & kg_D \end{array}$$

is a pullback diagram in W_k^{cn}, we have a pullback diagram

$$\begin{array}{ccc} C & \xhookrightarrow{\theta} & D\otimes \mathrm{B}(U) \\ \cup & & \cup \\ E & \hookrightarrow & kg_D\otimes \mathrm{B}(U). \end{array}$$

Since E is a Birkhoff-Witt coalgebra, it follows from Lemma 1.8.2 below that the composite

$$E \subset C \xrightarrow{\tilde{u}} \mathrm{B}(U)$$

is an isomorphism. This means that

$$E \xrightarrow{\simeq} kg_D \otimes \mathrm{B}(U).$$

Now the map θ is also a D-comodule map. Since C is an injective D-comodule we have

$$D\otimes \mathrm{B}(U) = \theta(C) \oplus V$$

for some D-subcomodule V. Since $kg_D \otimes \mathrm{B}(U) \subset \theta(C)$, we have

$$V \cap (kg_D \otimes \mathrm{B}(U)) = 0.$$

This implies $V=0$ in view of Lemma 1.8.3 below. Therefore θ is an isomorphism. This proves (v).

1.8.2 Let $f: C \to D$ be a map in W_k^{cn} and E the largest subcoalgebra of C contained in $f^{-1}(kg_D)$. Put $U = \mathrm{P}(E)$. Let $u: C \to U$ be a linear map such that

$$u \mid C_0 = 0 \quad \text{and} \quad u \mid U = 1.$$

Let $\tilde{u}: C \to \mathrm{B}(U)$ be the unique coalgebra map which satisfies $\pi \circ \tilde{u} = u$. Let $\theta: C \to D \otimes \mathrm{B}(U)$ be the unique coalgebra map such that

$$(1 \otimes \varepsilon) \circ \theta = f \quad \text{and} \quad (\varepsilon \otimes 1) \circ \theta = \tilde{u}.$$

LEMMA. *The map* $\mathrm{P}(\theta): \mathrm{P}(C) \to \mathrm{P}(D \otimes \mathrm{B}(U))$ *is injective. Hence* θ *is itself injective. If the map* $\mathrm{P}(f): \mathrm{P}(C) \to \mathrm{P}(D)$ *is surjective, then* $\mathrm{P}(\theta)$ *is also surjective. If* $E \simeq \mathrm{B}(V)$ *for some vector space* V, *then the composite*

$$E \subset C \xrightarrow{\tilde{u}} \mathrm{B}(U)$$

is an isomorphism.

PROOF. Recall that the functor

$$\mathrm{P}: F \mapsto \mathrm{P}(F),\ W_k^{\mathrm{cn}} \to \boldsymbol{Mod}_k$$

commutes with finite limits (1.2.8). In particular for any F and $F' \in W_k^{\mathrm{cn}}$, we have

$$\mathrm{P}(F \otimes F') \xrightarrow{\simeq} \mathrm{P}(F) \times \mathrm{P}(F').$$

Now we prove Lemma. Since the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \cup & & \cup \\ E & \longrightarrow & kg_D \end{array}$$

is a pullback diagram in W_k^{cn}, it follows from above that

$$U = \mathrm{P}(E) = \mathrm{Ker}\,(\mathrm{P}(f): \mathrm{P}(C) \to \mathrm{P}(D)).$$

Consider the composite

$$\mathrm{P}(C) \xrightarrow{\mathrm{P}(\theta)} \mathrm{P}(D \otimes \mathrm{B}(U)) \xrightarrow{\simeq} \mathrm{P}(D) \times \mathrm{P}(\mathrm{B}(U))$$
$$\xrightarrow{\simeq} \mathrm{P}(D) \times U.$$

This is easily seen to equal the map

$$(\mathrm{P}(f), u): x \mapsto (f(x), u(x)).$$

Since u=identity on $U = \mathrm{Ker}\,(\mathrm{P}(f))$, this map is always injective and becomes surjective when $\mathrm{P}(f)$ is surjective. In particular θ is always injective by [11, Lemma 11.0.1].

Let $\xi: E \to \mathrm{B}(U)$ be the composite

$$E \subset C \xrightarrow{\tilde{u}} \mathrm{B}(U).$$

Since $u: \mathrm{P}(E) \xrightarrow{\simeq} U$, we have

$$\mathrm{P}(\xi): \mathrm{P}(E) \xrightarrow{\simeq} \mathrm{P}(\mathrm{B}(U)).$$

In particular ξ is injective. For any vector space V the map induced by ξ

$$W_k^{\mathrm{cn}}(E, \mathrm{B}(V)) \leftarrow W_k^{\mathrm{cn}}(\mathrm{B}(U), \mathrm{B}(V))$$

is therefore surjective, since it is identified with

$$\boldsymbol{Mod}_k(E^+, V) \leftarrow \boldsymbol{Mod}_k(\mathrm{B}(U)^+, V).$$

In particular if $E \simeq \mathrm{B}(\mathrm{V})$, then there exists a coalgebra map

$$\chi : \mathrm{B}(U) \to E$$

such that $\chi \circ \xi = 1$. Since $\mathrm{P}(\xi)$ is bijective, $\mathrm{P}(\chi)$ is also bijective. Hence χ is injective. This means that

$$\xi : E \xrightarrow{\simeq} \mathrm{B}(U).$$

1.8.3 Let D be a coalgebra and V a vector space. Let $V \otimes D$ be the D-comodule with $1_V \otimes \Delta_D$ its structure map.

LEMMA. *For any non-zero D-subcomodule W of $V \otimes D$, we have*

$$W \cap (V \otimes D_0) \neq 0.$$

PROOF. It follows from Dual Nakayama Lemma [11, Theorem 9.0.3] that $0 \wedge D_0 \neq 0$ in W. This means the existence of an element $x \neq 0$ in W such that $\rho_W(x) \in W \otimes D_0$, where ρ_W denotes the comodule structure map on W. Since

$$x = (1 \otimes \varepsilon \otimes 1)(1 \otimes \Delta)(x) \in V \otimes D_0,$$

we have $W \cap (V \otimes D_0) \neq 0$.

1.9 The reduced part of a hyperalgebra

Given a ring homomorphism $\phi : R \to S$ and an R-module M, we put

$$S \otimes_\phi M = S \otimes_R M.$$

1.9.1 Let p be the characteristic exponent of k, i.e.

$$p = \mathrm{Max}\,(1, \mathrm{char}\,(k)).$$

Then the map

$$f : k \to k, \lambda \mapsto \lambda^p$$

is a ring homomorphism. For any vector space V over k, we put

$$V^{(p)} = k \otimes_f V.$$

If A is a commutative k-algebra, then the map

$$\phi_A : A^{(p)} \to A, \lambda \otimes a \mapsto \lambda a^p$$

is clearly a k-algebra map.

PROPOSITION. *Let C be a cocommutative k-coalgebra. Then there exists a unique k-linear map*

$$\Upsilon_C : C \to C^{(p)}$$

such that the composite

$$C^{*(p)} \hookrightarrow C^{(p)*} \xrightarrow{{}^t\Upsilon_C} C^*$$

coincides with ϕ_{C^*}. In particular Υ_C is a k-coalgebra map.

PROOF. Let V be a vector space. If U is a dense subspace of V^*, then given two elements x and y in V, it is easy to see that $x=y$ if and only if $\langle X, x\rangle = \langle X, y\rangle$ for all $X \in U$. From this the uniqueness of Υ_C follows, since $C^{*(p)}$ is dense in $C^{(p)*}$.

Now C is a directed union of finite dimensional subcoalgebras. Let D be a finite dimensional subcoalgebra of C. Since $D^{*(p)} = D^{(p)*}$, there exists a unique k-coalgebra map $\Upsilon_D : D \to D^{(p)}$ such that ${}^t\Upsilon_D = \phi_{D^*}$. Since Υ_D is functorial with respect to D, we can well define

$$\Upsilon_C = \varinjlim_{C \supset D \text{ f.d.}} \Upsilon_D : C \to C^{(p)},$$

which is clearly a k-coalgebra map and satisfies

$$\langle \lambda \otimes X, \Upsilon_C(c)\rangle = \langle \lambda X^p, c\rangle$$

for any $\lambda \in k$, $X \in C^*$ and $c \in C$. This proves Proposition.

1.9.2 PROPOSITION. *Let C and D be cocommutative coalgebras and $\xi : C \to D$ a coalgebra map. Then the following diagrams commute.*

$$\begin{array}{ccc} C & \xrightarrow{\xi} & D \\ {\scriptstyle \Upsilon_C}\downarrow & & \downarrow{\scriptstyle \Upsilon_D} \\ C^{(p)} & \xrightarrow{\xi^{(p)}} & D^{(p)} \end{array} \qquad \begin{array}{ccc} & C \otimes D & \\ {\scriptstyle \Upsilon_{C\otimes D}}\swarrow & & \searrow{\scriptstyle \Upsilon_C \otimes \Upsilon_D} \\ (C\otimes D)^{(p)} & \simeq & C^{(p)} \otimes D^{(p)} \end{array}$$

PROOF. For example, in order to see $\Upsilon_D \circ \xi = \xi^{(p)} \circ \Upsilon_C$, it is enough to show that

$$ {}^t\xi \circ {}^t\Upsilon_D = {}^t\Upsilon_C \circ {}^t\xi^{(p)} \quad \text{on} \quad D^{*(p)}, $$

since $D^{*(p)}$ is dense in $D^{(p)*}$. But this is valid in view of the following commutative diagram:

$$\begin{array}{ccc} C^* & \xleftarrow{\ {}^t\xi\ } & D^* \\ {\scriptstyle \phi_{C^*}}\uparrow & & \uparrow{\scriptstyle \phi_{D^*}} \\ C^{*(p)} & \xleftarrow{({}^t\xi)^{(p)}} & D^{*(p)}. \end{array}$$

Similarly if we notice that $(C^*\otimes D^*)^{(p)}$ is dense in $(C^{(p)}\otimes D^{(p)})^*$, the commutativity of the latter diagram in Proposition follows immediately.

1.9.3 COROLLARY. *If H is a cocommutative bialgebra then the map*

$$ \Upsilon_H : H \to H^{(p)} $$

is bialgebra map.

1.9.4 Let C be a cocommutative k-coalgebra. Heyneman and Sweedler define in [9, II, page 277] a $1/p$-linear map

$$ \mathscr{V} : C \to k^{1/p} \otimes_k C $$

and proves that

$$ f^p(c) = f(\mathscr{V}(c))^p $$

for any $c \in C$ and $f \in \boldsymbol{Mod}_{k^{1/p}}(k^{1/p}\otimes C, k^{1/p})$ [9, Theorem 4.1.7]. In particular we have

$$ \langle \lambda^p X^p, c\rangle = \langle \lambda \otimes X, \mathscr{V}(c)\rangle^p $$

for any $\lambda \in k^{1/p}$, $X \in C^*$ and $c \in C$. Hence the composition

$$\begin{array}{ccccc} C & \xrightarrow{\mathscr{V}} & k^{1/p}\otimes C & \xrightarrow{\simeq} & C^{(p)} \\ & & \lambda\otimes c & \longmapsto & \lambda^p \otimes c \end{array}$$

coincides with Υ_C. Thus Corollaries 4.2.7 and 4.2.8 in [9] can be read as follows:

THEOREM (Heynemann and Sweedler). *For any hyperalgebra H, the following conditions are equivalent.*

(i) *$H \simeq \mathrm{B}(U)$ as a coalgebra, for some vector space U.*

(ii) *$\Upsilon_H : H \to H^{(p)}$ is surjective.*
(iii) *$H^{(p)*}$ is a domain.*
(iv) *$H^{(p)*}$ is reduced.*
In particular if $p=1$ the above four conditions are always valid.

1.9.5 A connected cocommutative coalgebra C is said to be *smooth* if $C \simeq \mathrm{B}(U)$ for some vector space U. This is equivalent to saying that the map

$$\varepsilon : C \to k$$

is *smooth* in the sense of (1.8.1). C is said to be *reduced* if the algebra C^* is reduced. It follows from Theorem 1.9.4 that a hyperalgebra H over a *perfect* field k is *smooth* iff reduced.

THEOREM. *Assume that k is perfect and H is a k-hyperalgebra of finite type. Then H contains the largest reduced subhyperalgebra, which we shall denote by H_{red} and call* the reduced part *of H. H_{red} contains all co-prime subcoalgebras of H* (1.4.2). *Hence we have*

$$K \dim H = K \dim H_{\mathrm{red}} = \dim_k \mathrm{P}(H_{\mathrm{red}}).$$

PROOF. Since $f : k \xrightarrow{\simeq} k$, every subspace of $H^{(p)}$ is of the form $V^{(p)}$, where V is a uniquely determined subspace of H. In particular the image of the map

$$\Upsilon_H : H \to H^{(p)},$$

which is a hyperalgebra map, is of the form $H'^{(p)}$, where H' is a subhyper-algebra of H. If we put inductively

$$H_{(0)} = H, H_{(n+1)} = (H_{(n)})' \qquad (\text{i.e. } \Upsilon_H(H_{(n)}) = (H_{(n+1)})^{(p)}),$$

we obtain a descending chain of subcoalgebras

$$H = H_{(0)} \supset H_{(1)} \supset H_{(2)} \supset \cdots.$$

Since H^* is Noetherian (1.4.1) and since

$$0 = H_{(0)}^{\perp} \subset H_{(1)}^{\perp} \subset H_{(2)}^{\perp} \subset \cdots$$

is an ascending chain of ideal of H^*, there is an integer $n < \infty$ such that

$$H_{(n)} = H_{(n+1)} = H_{(n+2)} = \cdots.$$

Then it is clear that $H_{(n)}$ is the largest subhyperalgebra of H such that

$$\Upsilon : H_{(n)} \to (H_{(n)})^{(p)}$$

is surjective. Hence $H_{(n)}$ is the largest reduced subhyperalgebra of H by Theorem 1.9.4.

Let P be a coprime subcoalgebra of H. We show that the map

$$\Upsilon_P : P \to P^{(p)}$$

is surjective. Indeed, since $f : k \xrightarrow{\simeq} k$, we have

$$P^{(p)*} = P^{*(p)} = \{1 \otimes X \mid X \in P^*\}.$$

Thus the transpose ${}^t\Upsilon_P$, which is identified with the map

$$P^{*(p)} \to P^*, 1 \otimes X \mapsto X^p,$$

is injective because P^* is a domain (1.4.2). Therefore the map Υ_P is surjective. This means by definition $P \subset H_{(n)}$ for any n. Hence $P \subset H_{\mathrm{red}}$. Since H_{red} is *smooth* it follows from (1.6.1) that

$$K \dim H = K \dim H_{\mathrm{red}} = \dim_k \mathrm{P}(H_{\mathrm{red}}).$$

1.9.6 Assume that k is perfect. Let H be a hyperalgebra of finite type. In view of the equality

$$K \dim H = \dim_k \mathrm{P}(H_{\mathrm{red}}),$$

it is an important question when an element l in $\mathrm{P}(H)$ belongs to $\mathrm{P}(H_{\mathrm{red}})$. But this has already been answered by Sweedler [12, Theorem 2] (cf. [14, Theorem 2]) as follows:

THEOREM (Sweedler). *Assume that k is perfect and H is a hyperalgebra of finite type. Then we have*

$$\mathrm{P}(H_{\mathrm{red}}) = \{l \in \mathrm{P}(H) \mid \textit{There is an } \infty\textit{-sequence of divided powers in } H \textit{ lying over } l\}.$$

1.10 Actions of hyperalgebras on hyperalgebras

1.10.1 Let G be a hyperalgebra. Recall that an algebra A (resp. a coalgebra C) which is a (left) G-module is said to be *a G-module algebra* (resp. *a G-module coalgebra*) if the map

$$A \to \boldsymbol{Mod}_k(G, A), a \mapsto (?)a,$$

where

$$(?)a : x \mapsto xa, G \to A,$$

(resp. the map

$$G \otimes C \to C,\ x \otimes c \mapsto xc)$$

is an algebra map (resp. a coalgebra map). This is equivalent to saying that the structure maps

$$\mu : A \otimes A \to A \quad \text{and} \quad \eta : k \to A$$

(resp.

$$\Delta : C \to C \otimes C \quad \text{and} \quad \varepsilon : C \to k)$$

are G-module maps, where $A \otimes A$ (resp. $C \otimes C$) is viewed as a G-module via the algebra map

$$\Delta : G \to G \otimes G$$

and k via

$$\varepsilon : G \to k$$

[13, page 207, Definition].

A hyperalgebra H which is a G-module is called a *G-module hyperalgebra* if it is a G-module algebra and a G-module coalgebra at the same time, or equivalently if the following four conditions hold:

(i) $x \cdot (ab) = \sum (x_{(1)} \cdot a)(x_{(2)} \cdot b)$

(ii) $x \cdot 1 = \varepsilon(x) 1$

(iii) $\Delta(x \cdot a) = \sum x_{(1)} \cdot a_{(1)} \otimes x_{(2)} \cdot a_{(2)}$

(iv) $\varepsilon(x \cdot a) = \varepsilon(x)\varepsilon(a)$

for *all* $x \in G$ and $a, b \in H$. In this case we also say that *G acts on H*.

LEMMA. *If H is a G-module hyperalgebra, then the antipode S of H is G-linear.*

PROOF. Let $x \in G$ and $a \in H$. Since

$$\varepsilon(a) 1 = \sum a_{(1)} \cdot S(a_{(2)}),$$

we have

$$\varepsilon(x)\varepsilon(a) 1 = x \cdot (\varepsilon(a) 1) = \sum (x_{(1)} \cdot a_{(1)})(x_{(2)} \cdot S(a_{(2)})).$$

Hence

$$\begin{aligned} S(x \cdot a) &= \sum S(x_{(1)} \cdot a_{(1)}) \varepsilon(x_{(2)}) \varepsilon(a_{(2)}) \\ &= \sum S(x_{(1)} \cdot a_{(1)})(x_{(2)} \cdot a_{(2)})(x_{(3)} \cdot S(a_{(3)})) \\ &= \sum \varepsilon(x_{(1)} \cdot a_{(1)})(x_{(2)} \cdot S(a_{(2)})) \\ &= x \cdot S(a). \end{aligned}$$

1.10.2 Let G be a hyperalgebra and A a G-module algebra. Let's recall the definition of $A\sharp G$, the smash product of A with G [11, page 155]. $A\sharp G$ is an algebra defined as follows:

1) As a vector space $A\sharp G=A\otimes G$. We write $a\otimes x=a\sharp x$.
2) Multiplication is defined by

$$(a\sharp x)(b\sharp y)=\sum a(x_{(1)}\cdot b)\sharp x_{(2)}y.$$

Then unit of $A\sharp G$ is $1\sharp 1$.

Let G be a hyperalgebra and H a G-module hyperalgebra. *The semi-direct product* of H with G, written $H\underset{s}{\cdot}G$, is a hyperalgebra defined as follows:

1) As an algebra $H\underset{s}{\cdot}G$ is $H\sharp G$.
2) As a coalgebra $H\underset{s}{\cdot}G$ is $H\otimes G$, the direct product of H with G in the category W_k^{cn} (1.2.2).

LEMMA. *$H\underset{s}{\cdot}G$ is actually a hyperalgebra.*

PROOF. Since H is cocommutative the maps

$$\begin{gathered}\eta\colon k\to H\underset{s}{\cdot}G \quad \text{and}\\ \mu\colon (H\underset{s}{\cdot}G)\otimes(H\underset{s}{\cdot}G)\to H\underset{s}{\cdot}G,\\ (a\sharp x)\otimes(b\sharp y)\mapsto\sum a(x_{(1)}\cdot b)\sharp x_{(3)}y\end{gathered}$$

are clearly coalgebra maps. Hence $H\underset{s}{\cdot}G$ is a bialgebra. Because $H\otimes G$ is cocommutative and connected $H\underset{s}{\cdot}G$ is a hyperalgebra by definition.

1.10.3 EXAMPLES

1) *The adjoint action* (cf. [1, Example 1.1.3]). Let H be a hyperalgebra. The adjoint action of H, written

$$\mathrm{ad}\colon H\to\mathrm{End}_k(H),$$

is defined by

$$\mathrm{ad}(x)(a)=\sum x_{(1)}\cdot a\cdot S(x_{(2)})$$

for any x and $a\in H$. This action is easily seen to make H into an H-module hyperalgebra.

2) Let G be a hyperalgebra and V a (left) G-module. It follows from (1.5.2) that there exists a unique coalgebra map

$$\phi: G \otimes \mathrm{B_a}(V) \to \mathrm{B_a}(V)$$

such that

$$\pi(\phi(x \otimes a)) = x \cdot \pi(a)$$

for all $x \in G$ and $a \in \mathrm{B_a}(V)$. We show that ϕ makes $\mathrm{B_a}(V)$ into a G-module hyperalgebra.

We write $\phi(x \otimes a) = x \cdot a$. It is enough to check the following conditions:

(0) $xy \cdot a = x \cdot (y \cdot a)$ and $1 \cdot a = a$

(i) $x \cdot ab = \sum (x_{(1)} \cdot a)(x_{(2)} \cdot b)$

(ii) $x \cdot 1 = \varepsilon(x) 1$

for $x, y \in G$ and $a, b \in \mathrm{B_a}(V)$. For example in order to see (i) it is enough to prove

$$\pi(x \cdot (ab)) = \sum \pi((x_{(1)} \cdot a)(x_{(2)} \cdot b)),$$

since the maps

$$x \otimes a \otimes b \mapsto x \cdot ab \quad \text{and} \quad x \otimes a \otimes b \mapsto \sum (x_{(1)} \cdot a)(x_{(2)} \cdot b)$$

are coalgebra maps from $G \otimes \mathrm{B_a}(V) \otimes \mathrm{B_a}(V)$ to $\mathrm{B_a}(V)$ (1.5.2). But we have

$$\pi(ab) = \pi(a)\varepsilon(b) + \varepsilon(a)\pi(b).$$

Hence

$$\begin{aligned} &\sum \pi((x_{(1)} \cdot a)(x_{(2)} \cdot b)) \\ &= \sum \pi(x_{(1)} \cdot a)\varepsilon(x_{(2)} \cdot b) + \sum \varepsilon(x_{(1)} \cdot a)\pi(x_{(2)} \cdot b) \\ &= \sum (x_{(1)} \cdot \pi(a))\varepsilon(x_{(2)})\varepsilon(b) + \sum \varepsilon(x_{(1)})\varepsilon(a)(x_{(2)} \cdot \pi(b)) \\ &= (x \cdot \pi(a))\varepsilon(b) + \varepsilon(a)(x \cdot \pi(b)) \\ &= x \cdot \pi(ab) = \pi(x \cdot ab). \end{aligned}$$

The proof of (0) and (ii) is similar and left to the reader as an exercise.

3) Let G be a hyperalgebra and A a G-module algebra. It follows from (1.5.5) that there exists a unique coalgebra map

$$\psi: G \otimes \mathrm{B_m}(A) \to \mathrm{B_m}(A)$$

such that

$$\pi'(\psi(x \otimes a)) = x \cdot \pi'(a).$$

In the same way as in 2) ψ is easily seen to turn $\mathrm{B_m}(A)$ into a G-module hyperalgebra.

4) Let H be a hyperalgebra. The set of primitive elements $\mathrm{P}(H)$ is easily seen to be H-stable under the adjoint action. Hence the H-module $\mathrm{P}(H)$ induces an H-module hyperalgebra structure on $\mathrm{B}_a(\mathrm{P}(H))$.

5) *Lie algebras*. Let L_1 and L_2 be Lie algebras over k. We denote by $\mathrm{Der}_k(L_2)$ the Lie algebra of k-derivations of L_2. An *action* of L_1 on L_2 is by definition a Lie algebra map $\alpha: L_1 \to \mathrm{Der}_k(L_2)$ (cf. [7, III, § 6, 8.2]). In this case the semi-direct product of L_2 with L_1, written $L_2 \underset{s}{\cdot} L_1$, is a Lie algebra defined as follows:

1) As a vector space $L_2 \underset{s}{\cdot} L_1 = L_2 \oplus L_1$.

2) Bracket product is defined by

$$[(a, x), (b, y)] = ([a, b] + \alpha(x)(b) - \alpha(y)(a), [x, y]).$$

It is well known that $L_2 \underset{s}{\cdot} L_1$ is actually a Lie algebra [4, § 1, n° 8].

Let L be a Lie algebra. The universal enveloping algebra $\mathbf{U}(L)$ has a bialgebra structure (Δ, ε) determined by

$$\Delta(l) = l \otimes 1 + 1 \otimes l \quad \text{and} \quad \varepsilon(l) = 0$$

for $l \in L$ [11, Proposition 3.2.2]. Since $\mathbf{U}(L)$ is connected cocommutative [11, Proposition 11.0.9 and page 66, Examples 2)], $\mathbf{U}(L)$ is a hyperalgebra.

Let L_1 and L_2 be Lie algebras. We assume that L_1 acts on L_2 via

$$\alpha: L_1 \to \mathrm{Der}_k(L_2).$$

We show that $\mathbf{U}(L_1)$ acts on $\mathbf{U}(L_2)$ naturally and that

$$\mathbf{U}(L_2 \underset{s}{\cdot} L_1) \simeq \mathbf{U}(L_2) \underset{s}{\cdot} \mathbf{U}(L_1)$$

as hyperalgebras.

Indeed, since every derivation of L_2 can be uniquely extended to a derivation of $\mathbf{U}(L_2)$ [4, § 2, n° 8, Proposition 7], we obtain a natural map of Lie algebras:

$$\mathrm{Der}_k(L_2) \to \mathrm{Der}_k(\mathbf{U}(L_2)).$$

Let $\beta: L_1 \to \mathrm{Der}_k(\mathbf{U}(L_2))$ be the composite:

$$L_1 \xrightarrow{\alpha} \mathrm{Der}_k(L_2) \to \mathrm{Der}_k(\mathbf{U}(L_2)).$$

Extend β to an algebra map

$$\gamma: \mathbf{U}(L_1) \to \mathrm{End}_k(\mathbf{U}(L_2)).$$

It follows from [11, page 154, Examples c)] that γ turns $\mathbf{U}(L_2)$ into a $\mathbf{U}(L_1)$-*module algebra*. We show that $\mathbf{U}(L_2)$ is in fact a $\mathbf{U}(L_1)$-*module hyperalgebra*. To see this it suffices to show that the structure maps Δ and ε on $\mathbf{U}(L_2)$ commute with the operation of L_1, that is

$$(\beta(x)\otimes 1+1\otimes\beta(x))\circ\Delta=\Delta\circ\beta(x): \mathbf{U}(L_2)\to\mathbf{U}(L_2)\otimes\mathbf{U}(L_2)$$
$$\text{and}\quad 0=\varepsilon\circ\beta(x): \mathbf{U}(L_2)\to k$$

for all $x\in L_1$. To begin note that $(\beta(x)\otimes 1+1\otimes\beta(x))\circ\Delta$ and $\Delta\circ\beta(x)$ are derivations with respect to Δ, where by *a derivation with respect to* Δ we mean a linear map

$$D: \mathbf{U}(L_2)\to\mathbf{U}(L_2)\otimes\mathbf{U}(L_2)$$

which satisfies

$$D(xy)=D(x)\Delta(y)+\Delta(x)D(y).$$

It is easy to see that two derivations

$$D_1 \quad\text{and}\quad D_2: \mathbf{U}(L_2)\to\mathbf{U}(L_2)\otimes\mathbf{U}(L_2)$$

coincide iff the restrictions $D_1|L_2$ and $D_2|L_2$ coincide, since L_2 generates $\mathbf{U}(L_2)$ as an algebra. Since we can easily verify that

$$(\beta(x)\otimes 1+1\otimes\beta(x))\circ\Delta\,|\,L_2=\Delta\circ\beta(x)\,|\,L_2$$

by calculation and definition, it follows that

$$(\beta(x)\otimes 1+1\otimes\beta(x))\circ\Delta=\Delta\circ\beta(x).$$

Similarly we have

$$0=\varepsilon\circ\beta(x).$$

It remains to prove

$$\mathbf{U}(L_2)\underset{s}{\cdot}\mathbf{U}(L_1)\simeq\mathbf{U}(L_2\underset{s}{\cdot}L_1)$$

as hyperalgebras. Our proof below is only sketchy. Details are left to the reader. Now let H be a hyperalgebra. Then we have natural isomorphism:

$$\begin{aligned}
&\boldsymbol{Hopf}_k(\mathbf{U}(L_2)\underset{s}{\cdot}\mathbf{U}(L_1), H)\\
&\simeq\{(\phi_2,\phi_1)\in\boldsymbol{Hopf}_k(\mathbf{U}(L_2), H)\times\boldsymbol{Hopf}_k(\mathbf{U}(L_1), H)\,|\\
&\qquad\qquad \phi_2(\gamma(x)(a))=\mathrm{ad}\,(\phi_2(x))(\phi_2(a))\quad\text{for all } x\in\mathbf{U}(L_1)\quad\text{and}\quad a\in\mathbf{U}(L_2)\}\\
&\simeq\{(f_2,f_1)\in\boldsymbol{Lie}_k(L_2, \mathrm{P}(H))\times\boldsymbol{Lie}_k(L_1, \mathrm{P}(H))\,|\\
&\qquad\qquad\qquad\qquad f_2(\alpha(x)(a))=[f_1(x), f_2(a)]\quad\text{for all } x\in L_1\quad\text{and}\quad a\in L_2\}
\end{aligned}$$

$$\simeq \boldsymbol{Lie}_k(L_2 \underset{s}{\cdot} L_1, \mathrm{P}(H)) \simeq \boldsymbol{Hopf}_k(\mathbf{U}(L_2 \underset{s}{\cdot} L_1), H).$$

This means the existence of an isomorphism of hyperalgebras

$$\mathbf{U}(L_2) \underset{s}{\cdot} \mathbf{U}(L_1) \simeq \mathbf{U}(L_2 \underset{s}{\cdot} L_1).$$

Let L be a Lie algebra. Then L acts on L itself via

$$\mathrm{ad}: L \to \mathrm{Der}_k(L), x \mapsto \mathrm{ad}(x) = [x, -].$$

This is called *the adjoint action* of L. It is easy to show that the action of $\mathbf{U}(L)$ on $\mathbf{U}(L)$ obtained from the adjoint action of L by the process as above coincides with the adjoint action of $\mathbf{U}(L)$.

Let G be a hyperalgebra and H a G-module hyperalgebra. Then we have

$$\mathrm{P}(G) \cdot \mathrm{P}(H) \subset \mathrm{P}(H).$$

For

$$\begin{aligned}\Delta(x \cdot a) &= \Delta(x) \cdot \Delta(a) = (x \otimes 1 + 1 \otimes x)(a \otimes 1 + 1 \otimes a)\\ &= x \cdot a \otimes 1 + 1 \otimes x \cdot a + x \cdot 1 \otimes a + a \otimes x \cdot 1\\ &= x \cdot a \otimes 1 + 1 \otimes x \cdot a\end{aligned}$$

for any $a \in \mathrm{P}(H)$ and $x \in \mathrm{P}(G)$. It is easy to see that this induces an action of $\mathrm{P}(G)$ on $\mathrm{P}(H)$ as Lie algebras. In particular if the canonical maps

$$\mathbf{U}(\mathrm{P}(G)) \to G \quad \text{and} \quad \mathbf{U}(\mathrm{P}(H)) \to H$$

are isomorphisms, then the action of G on H is obtained by the above process from the action of $\mathrm{P}(G)$ on $\mathrm{P}(H)$ as is easily verified. This occurs for example if k is of characteristic 0 [11, Theorem 13.0.1].[7)]

1.10.4 Let G be a hyperalgebra and H a G-module hyperalgebra. We put

$$[x, a] = \sum (x \cdot a_{(1)}) S(a_{(2)})$$

for $x \in G$ and $a \in H$.

PROPOSITION. *Let C be a subcoalgebra and A a subalgebra of H. If $A \cdot C \subset C$, then there exists the largest subcoalgebra $G_{A,C}$ of G which satisfies*

$$[G_{A,C}, C] \subset A.$$

$G_{A,C}$ is a subhyperalgebra of G.

PROOF. Let $\mathscr{X}$ be the set of all subcoalgebras D of G which satisfy

$$[D, C] \subset A.$$

We show that

(i) $k \in \mathscr{X}$

(ii) $D, E \in \mathscr{X} \Rightarrow D+E, D \cdot E \in X.$

Indeed (i) is trivial. Let D and $E \in \mathscr{X}$. Then clearly $D+E \in \mathscr{X}$. Since we have

$$[xy, a] = \sum [x, y_{(1)} \cdot a_{(1)}][y_{(2)}, a_{(2)}] \quad \text{and} \quad x \cdot a = \sum [x, a_{(1)}] a_{(2)}$$

for $x, y \in G$ and $a \in H$, we have

$$E \cdot C \subset [E, C]C \subset AC \subset C \quad \text{and}$$
$$[DE, C] \subset [D, E \cdot C][E, C] \subset [D, C][E, C] \subset A \cdot A \subset A.$$

Hence $D \cdot E \in \mathscr{X}$. Therefore

$$G_{A,C} = \bigcup_{D \in \mathscr{X}} D$$

is a subhyperalgebra and the largest element in $\mathscr{X}$.

1.10.5 Let H be a G-module hyperalgebra, where G is a hyperalgebra. For a subhyperalgebra H' of H, we put

$$\mathrm{N}_G(H') = G_{H', H'} \quad \text{and} \quad \mathrm{C}_G(H') = G_{k, H'}.$$

$\mathrm{N}_G(H')$ (resp. $\mathrm{C}_G(H')$) is called *the normalizer* (resp. *the centralizer*) of H' in G.

In particular under the adjoint action of H we have

$$[x, y] = \sum x_{(1)} y_{(1)} S(x_{(2)}) S(y_{(2)})$$

for all x and $y \in H$. For a subhyperalgebra H' of H, the normalizer $\mathrm{N}_H(H')$ (resp. the centralizer $\mathrm{C}_H(H')$) of H' in H is the largest subcoalgebra of H such that

$$[\mathrm{N}_H(H'), H'] \subset H'$$

(resp.

$$[\mathrm{C}_H(H'), H'] \subset k).$$

H' is said to be *normal* (resp. *central*) in H if

$$\mathrm{N}_H(H') = H \text{ (resp. } \mathrm{C}_H(H') = H).$$

2. Underlying coalgebras and tangent coalgebras

This chapter, and the next also, depends on the theory of k-functors and k-schemes which is contained in [7].

A *k-functor* is a covariant functor from $\boldsymbol{M}_k$ to $\boldsymbol{E}$ and the category of k-functors is denoted by $\boldsymbol{M}_k\boldsymbol{E}$. For $A \in \boldsymbol{M}_k$, *the affine k-scheme* of A, written $\mathfrak{Sp}\, A$, is the k-functor represented by A, that is $\boldsymbol{M}_k(A, -)$. If $\mathfrak{X}$ is a k-functor, we have a natural isomorphism:

$$\boldsymbol{M}_k\boldsymbol{E}(\mathfrak{Sp}\, A, \mathfrak{X}) \simeq \mathfrak{X}(A)$$

via which we treat an element x of $\mathfrak{X}(A)$ as if it were a map of k-functors: $\mathfrak{Sp}\, A \to \mathfrak{X}$ and vice versa.

For the definition of *a k-scheme* refer to [7, I, § 1, 3.11]. The category of k-schemes, denoted by $\boldsymbol{Sch}_k$, is a full subcategory of $\boldsymbol{M}_k\boldsymbol{E}$.

We define in § 2.1 a covariant functor $\mathbf{T}$ from a full subcategory of $\boldsymbol{M}_k\boldsymbol{E}$ which contains $\boldsymbol{Sch}_k$ to $\boldsymbol{W}_k$ and call $\mathbf{T}(\mathfrak{X})$ *the underlying coalgebra* of $\mathfrak{X}$. *The tangent coalgebra* $\mathbf{T}_x(\mathfrak{X})$ to $\mathfrak{X}$ at a point x is defined as the underlying coalgebra of a subfunctor of $\mathfrak{X}$.

In § 2.2 we investigate some of the basic properties of the functor $\mathbf{T}$ and in § 2.3 translate some concepts in Algebraic Geometry such as being flat, non-ramified, étale or *smooth* into the coalgebra language.

2.1 Definitions and simple properties

2.1.1 Let $\mathfrak{X}$ be a k-functor [7, I, § 1, 6.1], that is a covariant functor from $\boldsymbol{M}_k$ to $\boldsymbol{E}$. Recall that if C is a cocommutative coalgebra, then C^* is a commutative k-algebra. If in addition C is finite dimensional, then C^* is considered as an object in $\boldsymbol{M}_k$ (, since k is assumed to be small).

Now consider the *set-functor* on $\boldsymbol{W}_k^{\mathrm{f}}$ (1.3.1):

$$C \mapsto \mathfrak{X}(C^*).$$

If this functor is represented by a cocommutative coalgebra H in the sense of (1.3.2), we say that H is *the underlying coalgebra* of $\mathfrak{X}$ and put $H = \mathbf{T}(\mathfrak{X})$. Since the cocommutative coalgebras which represent the above functor are isomorphic to one another, the coalgebra $\mathbf{T}(\mathfrak{X})$, if it exists, is well-defined and we have a natural bijection

$$\boldsymbol{W}_k(C, \mathbf{T}(\mathfrak{X})) \simeq \mathfrak{X}(C^*)$$

for any $C \in \boldsymbol{W}_k^{\mathrm{f}}$.

Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a map of k-functors. If both $\mathfrak{X}$ and $\mathfrak{Y}$ have the underlying coalgebra, it follows from Lemma 1.3.2 that there exists a unique coalgebra map $\mathbf{T}(\mathfrak{f}) : \mathbf{T}(\mathfrak{X}) \to \mathbf{T}(\mathfrak{Y})$ which makes the diagram

$$\begin{tikzcd} W_k(C, \mathbf{T}(\mathfrak{X})) \simeq \mathfrak{X}(C^*) \arrow[d, "W_k(C,\mathbf{T}(\mathfrak{f}))"'] & \\ W_k(C, \mathbf{T}(\mathfrak{Y})) \simeq \mathfrak{Y}(C^*) & \end{tikzcd}$$

$$\begin{array}{ccc} W_k(C, \mathbf{T}(\mathfrak{X})) & \simeq & \mathfrak{X}(C^*) \\ {\scriptstyle W_k(C,\mathbf{T}(\mathfrak{f}))}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}(C^*)} \\ W_k(C, \mathbf{T}(\mathfrak{Y})) & \simeq & \mathfrak{Y}(C^*) \end{array}$$

commute for any $C \in W_k^{\mathrm{f}}$. Thus $\mathbf{T}$ is a covariant functor from some full subcategory of $\boldsymbol{M_k E}$ to $\boldsymbol{W_k}$.

2.1.2 Let $\mathfrak{X}$ be a k-functor. Let $\boldsymbol{Fld_k}$ be the full subcategory of $\boldsymbol{M_k}$ consisting of *field extensions* of k. It is known [7, I, § 1, 4.9] that *the underlying set* $|\mathfrak{X}|$ of *the geometric realization* of $\mathfrak{X}$ [7, I, § 1, 4.2] is canonically isomorphic to the set $\varinjlim (\mathfrak{X} \mid \boldsymbol{Fld_k})$. We shall denote by $[a]$ the *point* of $\mathfrak{X}$, that is an element in $|\mathfrak{X}|$, determined by $a \in \mathfrak{X}(K)$, where K is an object in $\boldsymbol{Fld_k}$. Recall that for a subset P of $|\mathfrak{X}|$, the subfunctor $\mathfrak{X}_P$ of $\mathfrak{X}$ is defined as follows [7, I, § 1, 4.10]: For any $R \in \boldsymbol{M_k}$, $\mathfrak{X}_P(R)$ is the set of $\rho \in \mathfrak{X}(R)$ such that

$$[\mathfrak{X}(\phi)(\rho)] \in P$$

for any $\phi: R \rightarrow K$ in $\boldsymbol{M_k}$, where $K \in \boldsymbol{Fld_k}$. In particular if x is a point of $\mathfrak{X}$, we shall write

$$\mathfrak{X}_x = \mathfrak{X}_{\{x\}}.$$

DEFINITION. *Let $\mathfrak{X}$ be a k-functor and x a point of $\mathfrak{X}$. The underlying coalgebra $\mathbf{T}(\mathfrak{X}_x)$ of $\mathfrak{X}_x$, if it exists, is called the* tangent coalgebra *to $\mathfrak{X}$ at x and denoted by $\mathbf{T}_x(\mathfrak{X})$.*

Let $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a map of k-functors. Then $\mathfrak{f}$ induces a map of sets

$$|\mathfrak{f}|: |\mathfrak{X}| \rightarrow |\mathfrak{Y}|.$$

Let $x \in \mathfrak{X}$ and $y = \mathfrak{f}(x)$, that is $y = |\mathfrak{f}|(x)$. Then it is easy to see that $\mathfrak{f}$ induces a map of k-functors:

$$\mathfrak{X}_x \rightarrow \mathfrak{Y}_y.$$

Hence if both $\mathbf{T}_x(\mathfrak{X})$ and $\mathbf{T}_y(\mathfrak{Y})$ exist, this morphism induces a map of coalgebras, written

$$\mathbf{T}_x(\mathfrak{f}): \mathbf{T}_x(\mathfrak{X}) \rightarrow \mathbf{T}_y(\mathfrak{Y}).$$

Thus the correspondence

$$(\mathfrak{X}, x) \mapsto \mathbf{T}_x(\mathfrak{X})$$

gives rise to a functor.

2.1.3 Let $\mathfrak{X}$ be a k-functor. For a map $\phi: R \to S$ in $\boldsymbol{M}_k$ and $f \in \mathfrak{X}(R)$, we shall write

$$f_S = \mathfrak{X}(\phi)(f) \in \mathfrak{X}(S).$$

If $C \in \boldsymbol{W}_k^{\mathrm{f}}$, $K \in \boldsymbol{Fld}_k$ and $\phi \in \boldsymbol{M}_k(C^*, K)$, then $\phi(C^*)$ is a subfield of K, since it is a finite dimensional subalgebra of K. Hence $\phi(C^*)$ is canonically isomorphic to D^* for some simple subcoalgebra D of C. Conversely if D is a simple subcoalgebra of C, then D^* is a field and in particular $[f_{D^*}]$ is a well-defined point of $\mathfrak{X}$ for $f \in \mathfrak{X}(C^*)$, where f_{D^*} is taken with respect to the canonical map: $C^* \to D^*$.

Let $x \in \mathfrak{X}$ and suppose that $\mathbf{T}_x(\mathfrak{X})$ exists. Then it follows from above that $\mathbf{T}_x(\mathfrak{X})$ is a cocommutative coalgebra such that for any $C \in \boldsymbol{W}_k^{\mathrm{f}}$ the set $\boldsymbol{W}_k(C, \mathbf{T}_x(\mathfrak{X}))$ is naturally isomorphic to the set

$$\{f \in \mathfrak{X}(C^*) \mid [f_{D^*}] = x \text{ for all simple subcoalgebras } D \text{ of } C\}.$$

In particular we have a canonical imbedding for $C \in \boldsymbol{W}_k^{\mathrm{f}}$

$$\boldsymbol{W}_k(C, \mathbf{T}_x(\mathfrak{X})) \hookrightarrow \mathfrak{X}(C^*).$$

Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of k-functors. Let $x \in \mathfrak{X}$ and $y = \mathfrak{f}(x)$ and suppose that both $\mathbf{T}_x(\mathfrak{X})$ and $\mathbf{T}_y(\mathfrak{Y})$ exist. One easily checks that the following diagram commutes for any $C \in \boldsymbol{W}_k^{\mathrm{f}}$:

$$\begin{tikzcd}
\boldsymbol{W}_k(C, \mathbf{T}_x(\mathfrak{X})) \arrow[r, hook] \arrow[d, "\boldsymbol{W}_k(C, \mathbf{T}_x(\mathfrak{f}))"'] & \mathfrak{X}(C^*) \arrow[d, "\mathfrak{f}(C^*)"] \\
\boldsymbol{W}_k(C, \mathbf{T}_y(\mathfrak{Y})) \arrow[r, hook] & \mathfrak{Y}(C^*).
\end{tikzcd}$$

2.1.4 Let $\boldsymbol{Fld}_k^{\mathrm{f}}$ be the full subcategory of $\boldsymbol{M}_k$ consisting of all *finite* field extensions of k. Let $\mathfrak{X}$ be a k-functor. We put $\|\mathfrak{X}\| = \varinjlim (\mathfrak{X} \mid \boldsymbol{Fld}_k^{\mathrm{f}})$. For each $\alpha \in \mathfrak{X}(K)$ with $K \in \boldsymbol{Fld}_k^{\mathrm{f}}$, we denote by $\langle \alpha \rangle$ the element of $\|\mathfrak{X}\|$ determined by α. Let $\omega: \|\mathfrak{X}\| \to |\mathfrak{X}|$ be the canonical map.

PROPOSITION. (i) *If $x \in |\mathfrak{X}| - \omega(\|\mathfrak{X}\|)$, then $\mathbf{T}_x(\mathfrak{X})$ exists iff $\mathfrak{X}(0)$ consists of only one element and in this case $\mathbf{T}_x(\mathfrak{X}) = 0$.*

(ii) *If $\mathbf{T}(\mathfrak{X})$ exists, then the set $\|\mathfrak{X}\|$ can be canonically identified with the simple subcoalgebras of $\mathbf{T}(\mathfrak{X})$. Let $\mathbf{T}(\mathfrak{X})_\alpha$ denotes the irreducible component of $\mathbf{T}(\mathfrak{X})$ containing the simple subcoalgebra associated with $\alpha \in \|\mathfrak{X}\|$. Then $\mathbf{T}_x(\mathfrak{X})$ exists for all $x \in |\mathfrak{X}|$ and*

$$\mathbf{T}_x(\mathfrak{X}) \simeq \bigoplus_{\omega(\alpha) = x} \mathbf{T}(\mathfrak{X})_\alpha.$$

In particular the inclusions $\mathfrak{X}_x \hookrightarrow \mathfrak{X}$ *induce an isomorphism*

$$\bigoplus_{x \in |\mathfrak{X}|} \mathbf{T}_x(\mathfrak{X}) \xrightarrow{\sim} \mathbf{T}(\mathfrak{X}).$$

(iii) *Suppose that* $\mathfrak{X}$ *commutes with finite products. If* $\mathbf{T}_x(\mathfrak{X})$ *exists for all* $x \in \omega(\|\mathfrak{X}\|)$, *then* $\mathbf{T}(\mathfrak{X})$ *exists and is isomorphic to* $\bigoplus_{x \in \omega(\|\mathfrak{X}\|)} \mathbf{T}_x(\mathfrak{X})$.

PROOF. (i) Let $x \in |\mathfrak{X}| - \omega(\|\mathfrak{X}\|)$ and $C \in W_k^f$. Then the set

$$\{f \in \mathfrak{X}(C^*) \mid [f_{D^*}] = x \text{ for all } D \subset C \text{ simple}\}$$

is $\emptyset$ if $C \neq 0$ and $\mathfrak{X}(0)$ if $C=0$. Hence if $\mathbf{T}_x(\mathfrak{X})$ exists, it is necessarily zero and hence $\mathfrak{X}(0)$ consists of only one element. The converse is clear.

(ii) Suppose that $\mathbf{T}(\mathfrak{X})$ exists. Since $\mathfrak{X}(0) \simeq W_k(0, \mathbf{T}(\mathfrak{X}))$ consists of only one element, $\mathbf{T}_x(\mathfrak{X})$ exists and is zero for $x \in |\mathfrak{X}| - \omega(\|\mathfrak{X}\|)$.

Next we establish a bijection between the set $\|\mathfrak{X}\|$ and the set of simple subcoalgebras of $\mathbf{T}(\mathfrak{X})$.

Let K be an object in $\boldsymbol{Fld}_k^f$. Since K^* belongs to W_k^f, to each element a of $\mathfrak{X}(K)$ there corresponds a unique coalgebra map $\bar{a}: K^* \to \mathbf{T}(\mathfrak{X})$. Since K^* is simple and cocommutative, $\bar{a}(K^*)$ is also simple. If we are given an $\boldsymbol{Fld}_k^f$-map $\sigma: K \to L$, the map $\overline{a_L}$ which corresponds to $a_L \in \mathfrak{X}(L)$, where $a \in \mathfrak{X}(K)$, factors as

$$\overline{a_L}: L^* \xrightarrow{{}^t\sigma} K^* \xrightarrow{\bar{a}} \mathbf{T}(\mathfrak{X}).$$

Since ${}^t\sigma$ is surjective, $\overline{a_L}(L^*)$ is equal to $\bar{a}(K^*)$. This implies the existence of a map

$$\Phi: \|\mathfrak{X}\| \to (\text{the set of simple subcoalgebras of } \mathbf{T}(\mathfrak{X}))$$

such that

$$\Phi(\langle a \rangle) = \bar{a}(K^*)$$

for all $a \in \mathfrak{X}(K)$, where $K \in \boldsymbol{Fld}_k^f$.

Conversely let D be a simple subcoalgebra of $\mathbf{T}(\mathfrak{X})$ with

$$i: D \hookrightarrow \mathbf{T}(\mathfrak{X})$$

the inclusion. Since D is finite dimensional there corresponds a unique element $\bar{i}$ of $\mathfrak{X}(D^*)$. Since D^* is a finite field extension of k, we can well-define

$$\Psi(D) = \langle \bar{i} \rangle \in \|\mathfrak{X}\|.$$

The reader may easily verify that Φ and Ψ establish a bijection between $\|\mathfrak{X}\|$ and the set of all simple subcoalgebras of $\mathbf{T}(\mathfrak{X})$.

For an element α of $\|\mathfrak{X}\|$, let $\mathbf{T}(\mathfrak{X})_\alpha$ be the *irreducible component* of $\mathbf{T}(\mathfrak{X})$ which contains $\Phi(\alpha)$ [11, page 163, Definition]. It follows from [11, Theorem 8.0.5c)] that

$$\mathbf{T}(\mathfrak{X})=\bigoplus_{\alpha\in\|\mathfrak{X}\|}\mathbf{T}(\mathfrak{X})_\alpha.$$

Let $\phi: C\to\mathbf{T}(\mathfrak{X})$ be a coalgebra map, where $C\in W_k^{\mathrm{f}}$. Let $x\in|\mathfrak{X}|$. As an easy application of [11, Theorem 8.0.8d)] we see that $\phi(C)\subset\bigoplus_{\omega(\alpha)=x}\mathbf{T}(\mathfrak{X})_\alpha$ iff for each simple subcoalgebra D of C there exists an $\alpha\in\omega^{-1}(x)$ with $\phi(D)=\Phi(\alpha)$. If $\overline{\phi|D}$ denotes the element of $\mathfrak{X}(D^*)$ corresponding to $\phi|D: D\to\mathbf{T}(\mathfrak{X})$, then $\phi(D)=\Phi(\alpha)$ for some $\alpha\in\omega^{-1}(x)$ iff $[\overline{\phi|D}]=x$. Thus we have for all $x\in|\mathfrak{X}|$ and $C\in W_k^{\mathrm{f}}$

$$W_k(C,\bigoplus_{\omega(\alpha)=x}\mathbf{T}(\mathfrak{X})_\alpha)\simeq\{f\in\mathfrak{X}(C^*)\,|\,[f_{D^*}]=x\text{ for all }D\subset C\text{ simple}\}.$$

In view of (2.1.3) this proves (ii).

(iii) We put $H=\bigoplus_{x\in\omega(\|\mathfrak{X}\|)}\mathbf{T}_x(\mathfrak{X})$. Let C be a cocommutative irreducible coalgebra. We claim that

$$W_k(C,H)=\coprod_{x\in\omega(\|\mathfrak{X}\|)}W_k(C,\mathbf{T}_x(\mathfrak{X})).$$

Indeed let $\phi\in W_k(C,H)$. Since $\phi(C)$ is irreducible [11, Theorem 8.0.8d)], it is contained in a unique $\mathbf{T}_x(\mathfrak{X})$, e.g. by [11, Lemma 9.0.1b)]. This means the above equality.

Suppose further that C is finite dimensional. Let D be the unique simple subcoalgebra of C. We have by (2.1.3)

$$W_k(C,\mathbf{T}_x(\mathfrak{X}))\simeq\{f\in\mathfrak{X}(C^*)\,|\,[f_{D^*}]=x\}.$$

It follows that

$$W_k(C,H)=\coprod_{x\in\omega(\|\mathfrak{X}\|)}W_k(C,\mathbf{T}_x(\mathfrak{X}))\simeq\mathfrak{X}(C^*).$$

Now let $C\in W_k^{\mathrm{f}}$. Then C is the direct sum of its irreducible components $C_1,\cdots,C_n$, that is

$$C=C_1\oplus\cdots\oplus C_n.$$

Since

$$C^*=C_1^*\times\cdots\times C_n^*$$

and $\mathfrak{X}$ commutes with finite products by hypothesis, we have

$$\begin{aligned}\mathfrak{X}(C^*) &\simeq \mathfrak{X}(C_1^*)\times\cdots\times\mathfrak{X}(C_n^*)\\ &\simeq W_k(C_1,H)\times\cdots\times W_k(C_n,H)\\ &\simeq W_k(C,H).\end{aligned}$$

Since this isomorphism is clearly functiorial with respect to C, we have by definition

$$\mathbf{T}(\mathfrak{X})\simeq H=\bigoplus_{x\in\omega(\|\mathfrak{X}\|)}\mathbf{T}_x(\mathfrak{X}).$$

2.1.5 For $A\in \boldsymbol{M}_k$, we shall denote by $\mathfrak{Sp}\,A$ the affine scheme of A, that is the k-functor $\boldsymbol{M}_k(A,-)$. It follows directly from definition and [11, Theorem 6.0.5] that

$$\mathbf{T}(\mathfrak{Sp}\,A)\simeq A^0.$$

2.1.6 Let $\mathfrak{X}$ be a k-scheme [7, I, § 1, 3.11 and 6.1]. For a point x of $\mathfrak{X}$, we shall denote by $\mathcal{O}_x$ the fibre of the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ at x. $\mathcal{O}_x$ is a local k-algebra with maximal ideal m_x. The residue field at x is $\boldsymbol{\kappa}(x)=\mathcal{O}_x/m_x$. It is easy to see that[8)]

$$x\in\|\mathfrak{X}\|\Leftrightarrow[\boldsymbol{\kappa}(x):k]<\infty.$$

It follows from (1.1.2) that

$$(\mathcal{O}_x)^0\neq 0\Leftrightarrow[\boldsymbol{\kappa}(x):k]<\infty$$

and that $(\mathcal{O}_x)^0$ is irreducible with coradial filtration

$$\{(\mathcal{O}_x)^0\cap(m_x^{i+1})^\perp\}_{i\geqq 0}$$

if $[\boldsymbol{\kappa}(x):k]<\infty$.

PROPOSITION. *Let $\mathfrak{X}$ be a k-scheme. Then the tangent coalgebra $\mathbf{T}_x(\mathfrak{X})$ exists and is canonically isomorphic to $(\mathcal{O}_x)^0$ for all $x\in\mathfrak{X}$. Since $\mathfrak{X}$ commutes with finite products* [7, I, § 1, 3.11], *it follows that*

$$\mathbf{T}(\mathfrak{X})=\bigoplus_{x\in\mathfrak{X}}(\mathcal{O}_x)^0.$$

PROOF. For a point x of $\mathfrak{X}$, let $\boldsymbol{\varepsilon}_x:\mathfrak{Sp}\,\mathcal{O}_x\to\mathfrak{X}$ be the canonical map [7, I, § 1, 5.6]. Let R be a *finite dimensional* commutative local k-algebra with maximal ideal m. We show that $\boldsymbol{\varepsilon}_x$ induces a bijection

$$\boldsymbol{M}_k(\mathcal{O}_x,R)\to\mathfrak{X}_x(R).$$

Indeed let σ be an $\boldsymbol{M}_k$-map: $\mathcal{O}_x\to R$. We must show that $\mathfrak{X}(\sigma)(\boldsymbol{\varepsilon}_x)$ belongs

to $\mathfrak{X}_x(R)$, where we view $\boldsymbol{\varepsilon}_x$ as an element of $\mathfrak{X}(\mathcal{O}_x)$. Let $\tau: R \to K$ be an $\boldsymbol{M}_k$-map, where $K \in \boldsymbol{Fld}_k$. Since R is finite dimensional, σ and τ are local. In particular the composite

$$\tau \circ \sigma: \mathcal{O}_x \to K$$

is necessarily local and hence factors as

$$\tau \circ \sigma: \mathcal{O}_x \to \boldsymbol{\kappa}(x) \to K.$$

This means that

$$[\mathfrak{X}(\tau \circ \sigma)(\boldsymbol{\varepsilon}_x)] = x$$

and hence

$$\mathfrak{X}(\sigma)(\boldsymbol{\varepsilon}_x) \in \mathfrak{X}_x(R).$$

Conversely let $f \in \mathfrak{X}(R)$. We can view f as a map of k-schemes

$$f: \mathfrak{Sp}\, R \to \mathfrak{X}.$$

This induces a local homomorphism of local k-algebras

$$f_m: \mathcal{O}_{f(m)} \to R_m = R.$$

Since $f = \mathfrak{X}(f_m)(\boldsymbol{\varepsilon}_{f(m)})$, it follows that

$$f \in \mathfrak{X}_x(R) \Leftrightarrow f(m) = x$$

and that the correspondence

$$f \mapsto f_m$$

gives rise to the inverse of the map

$$\boldsymbol{M}_k(\mathcal{O}_x, R) \to \mathfrak{X}_x(R),\ \sigma \mapsto \mathfrak{X}(\sigma)(\boldsymbol{\varepsilon}_x).$$

In particular if C is a cocommutative finite dimensional irreducible coalgebra, then since C^* is local by [11, Lemma 8.0.2], we have a natural bijections

$$\mathfrak{X}_x(C^*) \simeq \boldsymbol{M}_k(\mathcal{O}_x, C^*) \simeq \boldsymbol{W}_k(C, (\mathcal{O}_x)^0).$$

Since $\mathfrak{X}$, hence $\mathfrak{X}_x$ also, commutes with finite products, these isomorphisms hold for all $C \in \boldsymbol{W}_k^{\mathrm{f}}$, as in Proof of (2.1.4 (iii)). This means that

$$\mathbf{T}_x(\mathfrak{X}) \simeq (\mathcal{O}_x)^0.$$

REMARK. By the assertion that $\mathbf{T}_x(\mathfrak{X})$ is *canonically* isomorphic to

$(\mathcal{O}_x)^0$, we mean the following property, which the reader may easily verify: Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of k-schemes. If $x \in \mathfrak{X}$ and $y = \mathfrak{f}(x)$, $\mathfrak{f}$ induces a local homomorphism of local k-algebras

$$\mathfrak{f}_x : \mathcal{O}_y \to \mathcal{O}_x.$$

Then the diagram

$$\begin{tikzcd} \mathbf{T}_x(\mathfrak{X}) \arrow[r, "\simeq"] \arrow[d, "\mathbf{T}_x(\mathfrak{f})"'] & (\mathcal{O}_x)^0 \arrow[d, "(\mathfrak{f}_x)^0"] \\ \mathbf{T}_y(\mathfrak{Y}) \arrow[r, "\simeq"] & (\mathcal{O}_y)^0 \end{tikzcd}$$

commutes.

2.1.7 COROLLARY. *For $A \in \boldsymbol{M}_k$ we have*

$$A^0 \simeq \bigoplus_{P \in \operatorname{Spec} A} (A_P)^0.$$

2.1.8 Let $\mathfrak{X}$ be a k-functor. If $\mathbf{T}(\mathfrak{X})$ exists, the set of group-like elements $\mathrm{G}(\mathbf{T}(\mathfrak{X}))$, which can be identified with $\boldsymbol{W}_k(k, \mathbf{T}(\mathfrak{X}))$, is canonically isomorphic to the set of *rational points* $\mathfrak{X}(k)$.[9)]

In particular if $\mathfrak{X}$ has the coalgebra $\mathbf{S}_e(\mathfrak{X})$ at a rational point $e \in \mathfrak{X}(k)$, then $\mathbf{S}_e(\mathfrak{X})$ is connected, since it is irreducible and contains at least one group-like element e.

We fix for the time being a rational point $e \in \mathfrak{X}(k)$ and assume that the coalgebra $\mathbf{S}_e(\mathfrak{X})$ exists at e. We view each set $\mathfrak{X}(R)$, where $R \in \boldsymbol{M}_k$, as a pointed set with e_R the original point. In particular for an $\boldsymbol{M}_k$-map $\phi: R \to S$, we put

$$\operatorname{Ker}_e(\mathfrak{X}(\phi)) = \{f \in \mathfrak{X}(R) \mid f_S = e_S\}.$$

It follows easily that there exists a natural isomorphism

$$\boldsymbol{W}_k(C, \mathbf{S}_e(\mathfrak{X})) \simeq \operatorname{Ker}_e(\mathfrak{X}(C^*) \to \mathfrak{X}(C_0^*))$$

for all $C \in \boldsymbol{W}_k^{\mathrm{f}}$, where C_0 is the coradical of C and the map

$$\mathfrak{X}(C^*) \to \mathfrak{X}(C_0^*)$$

is the one induced by the canonical surjection: $C^* \to C_0^*$.[10)]

Recall that the category $\boldsymbol{W}_R$ (§ 1.2), where $R \in \boldsymbol{M}_k$, is as follows: The class of objects in $\boldsymbol{W}_R$ is the same as $\boldsymbol{W}_k$; The set of $\boldsymbol{W}_R$-maps from C to D,

where $C, D \in W_k$, consists of all R-coalgebra maps $f: R\otimes C \to R\otimes D$ such that

$$f(R\otimes C_0) \subset R\otimes D_0.$$

We shall denote by W_R^{f} the full subcategory of W_R whose class of objects is the same as W_k^{f}.

The category W^{f} is defined as follows (1.2.9): The class of objects in W^{f} is equal to $M_k \times W_k^{\mathrm{f}}$; A W^{f}-map from (R, C) to (S, D), where $R, S \in M_k$ and $C, D \in W_k^{\mathrm{f}}$, is a pair (ϕ, σ) with $\phi \in M_k(S, R)$ and $\sigma \in W_R(C, D)$. The composite of two W^{f}-maps

$$(R, C) \xrightarrow{(\phi,\sigma)} (S, D) \xrightarrow{(\psi,\tau)} (T, E)$$

is $(\phi\circ\psi, W_\phi(\tau)\circ\sigma)$.

We have defined a contravariant functor

$$(R, C) \mapsto R\otimes C^*, \ W^{\mathrm{f}} \to M_k$$

in (1.3.9). This induces a set-functor on W^{f}:

$$(R, C) \mapsto \mathrm{Ker}_e\, (\mathfrak{X}(R\otimes C^*) \to \mathfrak{X}(R\otimes C_0^*)),$$

where the map: $\mathfrak{X}(R\otimes C^*) \to \mathfrak{X}(R\otimes C_0^*)$ is induced by the canonical projection: $R\otimes C^* \to R\otimes C_0^*$.

We say that an object H in W_k is a tangent coalgebra to $\mathfrak{X}$ at e *in the strong sense* if the above set-functor on W^{f} is isomorphic to the representable functor Θ_H (1.3.8). Since such an H is unique up to W_k-isomorphism by Lemma 1.3.8, if it exists, we say that H is *the* tangent coalgebra to $\mathfrak{X}$ at e in the strong sense and put

$$H = \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}).$$

If $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ exists, then it coincides of course with $\mathbf{S}_e(\mathfrak{X})$.

Suppose that $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ exists. Then we have a natural bijection:

$$W_R(C, \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})) \simeq \mathrm{Ker}_e\, (\mathfrak{X}(R\otimes C^*) \to \mathfrak{X}(R\otimes C_0^*))$$

for $(R, C) \in W^{\mathrm{f}}$. In the following we shall often denote by

$$\sigma \mapsto \exp\, (\sigma, R, C)$$

the above isomorphism. This exponential notation will be found useful later. The fact that the above map is functorial with respect to $(R, C) \in W^{\mathrm{f}}$ is equivalent to that the following two conditions hold at the same time, as

is easily verified:

(i) If $\phi \in \boldsymbol{M}_k(R, S), C \in W_k^{\mathrm{f}}$ and $\sigma \in W_R(C, \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}))$, then we have

$$\exp(W_\phi(\sigma), S, C) = \exp(\sigma, R, C)_{S\otimes C^*},$$

where the right hand side is taken with respect to the map

$$\phi\otimes 1 : R\otimes C^* \to S\otimes C^*.$$

(ii) If $R \in \boldsymbol{M}_k, \tau \in W_R^{\mathrm{f}}(D, C)$ and $\sigma \in W_R(C, \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}))$, then we have

$$\exp(\sigma\circ\tau, R, D) = \exp(\sigma, R, C)_{R\otimes D^*},$$

where the right hand side is taken with respect to the map

$$\boldsymbol{Mod}_R(\tau, R) : R\otimes C^* \to R\otimes D^*.$$

Let C be a finite dimensional subcoalgebra of $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ with $i : C \hookrightarrow \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ the inclusion. We shall put

$$\exp(R, C) = \exp(W_\eta(i), R, C),$$

for $R \in \boldsymbol{M}_k$ with $\eta : k \to R$ the structure map. Then we have

$$\exp(R, C) = \exp(k, C)_{R\otimes C^*}$$

with respect to the map

$$\eta\otimes 1 : C^* \to R\otimes C^*.$$

Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a map of k-functors. Let $e' = \mathfrak{f}(e)$. Suppose that $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ and $\mathbf{T}_{e'}^{\mathrm{st}}(\mathfrak{Y})$ exist. It follows from Lemma 1.3.8 that there exists a unique coalgebra map $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{f}) : \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}) \to \mathbf{T}_{e'}^{\mathrm{st}}(\mathfrak{Y})$ such that the diagram:

$$\begin{array}{ccc} W_R(C, \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})) & \xrightarrow{W_R(C, W_\eta(\mathbf{T}_e^{\mathrm{st}}(\mathfrak{f})))} & W_R(C, \mathbf{T}_{e'}^{\mathrm{st}}(\mathfrak{Y})) \\ \big\downarrow{\scriptstyle \mathrm{cano.}} & & \big\downarrow{\scriptstyle \mathrm{cano.}} \\ \mathfrak{X}(R\otimes C^*) & \xrightarrow{\mathfrak{f}(R\otimes C^*)} & \mathfrak{Y}(R\otimes C^*) \end{array}$$

commutes for all $(R, C) \in W^{\mathrm{f}}$. Thus the correspondence:

$$(\mathfrak{X}, e) \mapsto \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$$

can be viewed as a functor.

2.1.9 PROPOSITION. *Let $\mathfrak{X}$ be a k-scheme and $e \in \mathfrak{X}(k)$. If $\mathfrak{X}$ is locally algebraic* [7, I, § 3,2.1], *then $(\mathcal{O}_e)^0$ is the tangent coalgebra to $\mathfrak{X}$ at e in the strong sense.*

PROOF. Let

$$\mathfrak{i}: \mathfrak{Z} \to \mathfrak{Y}$$

be an immersion of k-schemes [7, I, § 2, 5.1]. We review first the definition of *the n-th neighborhood of* $\mathfrak{i}$ *in* $\mathfrak{Y}$ [7, II, § 4, 5.5], where n is an integer $\geqq 0$. It is defined as follows: By definition $\mathfrak{i}$ can be decomposed as

$$\mathfrak{i}: \mathfrak{Z} \xrightarrow{\mathfrak{j}} \mathfrak{U} \xrightarrow{\mathfrak{h}} \mathfrak{Y},$$

where $\mathfrak{j}$ (resp. $\mathfrak{h}$) is a closed (resp. an open) immersion. We regard $\mathfrak{U}$ as an open subscheme of $\mathfrak{Y}$ via $\mathfrak{h}$. Let $\mathscr{I}$ be the quasi-coherent ideal of $\mathscr{O}_{\mathfrak{U}}$ the structure sheaf of $\mathfrak{U}$ which corresponds to the closed immersion $\mathfrak{j}$ [7, I, § 2, 4.8]. Let $\mathfrak{Y}_{\mathfrak{i}}^{n}$ be the closed subscheme of $\mathfrak{U}$ determined by the quasi-coherent ideal $\mathscr{I}^{n+1}$ of $\mathscr{O}_{\mathfrak{U}}$. Since $\mathfrak{U}$ is an open subscheme of $\mathfrak{Y}$, $\mathfrak{Y}_{\mathfrak{i}}^{n}$ is a subscheme of $\mathfrak{Y}$. As a subscheme of $\mathfrak{Y}$, $\mathfrak{Y}_{\mathfrak{i}}^{n}$ is known to be independent of the choice of decomposition $\mathfrak{i}=\mathfrak{h}\circ\mathfrak{j}$ and called the n-th neighborhood of $\mathfrak{i}$ in $\mathfrak{Y}$ [7, II, § 4, 5.5].

The following simple properties may be easily verified:

(i) Given a commutative diagram consisting of k-schemes and k-scheme maps

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, "\mathfrak{f}"] & \mathfrak{Y}' \\ \mathfrak{Z} \arrow[u, "\mathfrak{i}"] \arrow[r, "\mathfrak{g}"] & \mathfrak{Z}' \arrow[u, "\mathfrak{i}'"'] \end{tikzcd},$$

where $\mathfrak{i}$ and $\mathfrak{i}'$ are immersions, the morphism $\mathfrak{f}$ transforms $\mathfrak{Y}_{\mathfrak{i}}^{n}$ the n-th neighborhood of $\mathfrak{i}$ in $\mathfrak{Y}$ into $\mathfrak{Y}'^{n}_{\mathfrak{i}'}$ the n-th neighborhood of $\mathfrak{i}'$ in $\mathfrak{Y}'$.

(ii) Let $\mathfrak{i}: \mathfrak{Z} \to \mathfrak{Y}$ be a closed immersion of k-schemes. If the quasi-coherent ideal of $\mathscr{O}_{\mathfrak{Y}}$ corresponding to $\mathfrak{i}$ is nilpotent, then $\mathfrak{Y}$ is the n-th neighborhood of $\mathfrak{i}$ in $\mathfrak{Y}$ for sufficiently large $n<\infty$.

(iii) Let $\mathfrak{X}$ be a k-scheme and $e \in \mathfrak{X}(k)$ a rational point. We regard as usual e as a map of k-schemes

$$e: \mathfrak{Sp}\, k \to \mathfrak{X}$$

which is known to be an immersion [7, I, § 2, 5.6]. Then the n-th neighborhood of e in $\mathfrak{X}$ is the image of the composite

$$\mathfrak{Sp}\,(\mathscr{O}_e/m_e^{n+1}) \hookrightarrow \mathfrak{Sp}\,\mathscr{O}_e \xrightarrow{\varepsilon_e} \mathfrak{X}.$$

Now let's begin Proof. Let $C \in W_k^{\mathrm{f}}$ and $R \in M_k$. Let $\mathscr{X}$ be the set of k-schemes maps $\mathfrak{f}: \mathfrak{Sp}\,(R \otimes C^*) \to \mathfrak{X}$ which make the following diagram commute:

$$\begin{array}{ccc} \mathfrak{Sp}\,(R\otimes C^*) & \xrightarrow{\mathfrak{f}} & \mathfrak{X} \\ \uparrow\iota & & \uparrow e \\ \mathfrak{Sp}(R\otimes C_0^*) & \xrightarrow{\mathfrak{Sp}\,\eta} & \mathfrak{Sp}\,k, \end{array}$$

where C_0 is the coradical of C, ι is the closed immersion induced by the canonical projection: $R\otimes C^*\to R\otimes C_0^*$ and $e\in\mathfrak{X}(k)$ is regarded as a map: $\mathfrak{Sp}\,k\to\mathfrak{X}$. It follows from above that any element $\mathfrak{f}$ of $\mathscr{X}$ factors as

$$\mathfrak{f}:\mathfrak{Sp}\,(R\otimes C^*)\longrightarrow\mathfrak{Sp}\,(\mathcal{O}_e/m_e^{n+1})\hookrightarrow\mathfrak{Sp}\,\mathcal{O}_e\xrightarrow{\varepsilon_e}\mathfrak{X}$$

for sufficiently large $n<\infty$, since the kernel of the canonical projection: $R\otimes C^*\to R\otimes C_0^*$ is nilpotent.

Hence if we put $\mathscr{P}_n$ the set of k-algebra maps

$$f:\mathcal{O}_e/m_e^{n+1}\to R\otimes C^*$$

such that the diagram

$$\begin{array}{ccc} \mathcal{O}_e/m_e^{n+1} & \xrightarrow{f} & R\otimes C^* \\ \downarrow\text{cano.} & & \downarrow\text{cano.} \\ k & \xrightarrow{\eta} & R\otimes C_0^* \end{array}$$

commutes, we can naturally identify

$$\mathscr{X}\simeq\bigcup_n\mathscr{P}_n.$$

By the way $\mathfrak{X}$ is assumed to be locally algebraic. This means in particular that $\mathcal{O}_e$ is Noetherian and hence m_e^{n+1}'s are all cofinite in $\mathcal{O}_e$. Therefore we have

$$\begin{aligned} &\boldsymbol{M}_k(\mathcal{O}_e/m_e^{n+1},R\otimes C^*)\\ &\simeq\boldsymbol{M}_R(R\otimes(\mathcal{O}_e/m_e^{n+1}),R\otimes C^*)\\ &\simeq\boldsymbol{Coalg}_R(R\otimes C,R\otimes(\mathcal{O}_e/m_e^{n+1})^*),\end{aligned}$$

where $\boldsymbol{Coalg}_R$ denotes the category of R-coalgebras. The reader may easily verify that this isomorphism induces a bijection

$$\mathscr{P}_n\simeq\boldsymbol{W}_R(C,(\mathcal{O}_e/m_e^{n+1})^*).$$

Since $\mathbf{T}_e(\mathfrak{X})=(\mathcal{O}_e)^0=\cup(\mathcal{O}_e/m_e^{n+1})^*$ by (1.1.2) and that C is finite dimensional, we have

$$\begin{aligned}\boldsymbol{W}_R(C,\mathbf{T}_e(\mathfrak{X}))&=\cup\ \boldsymbol{W}_R(C,(\mathcal{O}_e/m_e^{n+1})^*)\\ &\simeq\cup\ \mathscr{P}_n\simeq\mathscr{X}.\end{aligned}$$

But under the canonical isomorphism

$$\mathfrak{X}(R\otimes C^*)\simeq \boldsymbol{M_kE}(\mathfrak{Sp}\,(R\otimes C^*), \mathfrak{X})$$

the subset of $\mathfrak{X}(R\otimes C^*)$

$$\mathrm{Ker}_e\,(\mathfrak{X}(R\otimes C^*)\to\mathfrak{X}(R\otimes C_0^*))$$

clearly corresponds to the subset $\mathscr{X}$ of $\boldsymbol{M_kE}(\mathfrak{Sp}\,(R\otimes C^*), \mathfrak{X})$. Thus we obtain a natural isomorphism

$$\boldsymbol{W}_R(C, \mathbf{T}_e(\mathfrak{X}))\simeq \mathrm{Ker}_e\,(\mathfrak{X}(R\otimes C^*)\to\mathfrak{X}(R\otimes C_0^*))$$

for all $(R, C)\in \boldsymbol{W}^{\mathrm{f}}$.

It remains to verify that this isomorphism is functorial. We leave it to the reader as an exercise.

2.1.10 Let $\mathfrak{X}$ be a k-functor and $e\in\mathfrak{X}(k)$ a rational point. Let $\mathfrak{G}$ be a k-group-functor, that is a covariant functor from $\boldsymbol{M}_k$ to $\boldsymbol{Gr}$, or equivalently a group object in the category $\boldsymbol{M_kE}$ of k-functors [7, II, § 1, 1.1]. Suppose that $\mathfrak{G}$ acts on $\mathfrak{X}$ [7, II, § 1, 3.1]. This means that we are given a map of k-functors

$$\mathfrak{u}: \mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$$

such that for any $R\in\boldsymbol{M}_k$, the group $\mathfrak{G}(R)$ acts on the set $\mathfrak{X}(R)$ via

$$\mathfrak{u}(R): \mathfrak{G}(R)\times\mathfrak{X}(R)\to\mathfrak{X}(R).$$

In the following we shall write as usual

$$\mathfrak{u}(R)(a, x)=a\cdot x$$

for $a\in\mathfrak{G}(R)$ and $x\in\mathfrak{X}(R)$. Recall that the subfunctor $\mathfrak{X}^{\mathfrak{G}}$ of $\mathfrak{X}$ [7, II, § 1, 3.4] is defined as follows: For any $R\in\boldsymbol{M}_k$, $\mathfrak{X}^{\mathfrak{G}}(R)$ is the set of $\sigma\in\mathfrak{X}(R)$ such that

$$a\cdot\sigma_S=\sigma_S$$

for any $\boldsymbol{M}_k$-map $\phi: R\to S$ and $a\in\mathfrak{G}(S)$. It is the largest subfunctor of $\mathfrak{X}$ on which $\mathfrak{G}$ acts trivially.

Let V be a vector space. Recall that V_a is the k-functor: $R\mid\!\to R\otimes V$ [7, II, § 1, 2.1]. An action of $\mathfrak{G}$ on V_a is said to be *linear* if each action of $g\in\mathfrak{G}(R)$ on $V_a(R)=R\otimes V$ is R-linear. A pair $(V, \mathfrak{u})$, where $\mathfrak{u}$ is a linear action of $\mathfrak{G}$ on V_a, is called a k-$\mathfrak{G}$-module [7, II, § 2, 1.1]. It is known [7, II, § 2, 1.6] that for any k-$\mathfrak{G}$-module V we have

$$(V_a)^{\mathfrak{G}}=(V^{\mathfrak{G}})_a,$$

where $V^{\mathfrak{G}}$ is the set of $v \in V$ such that

$$g \cdot (1 \otimes v) = 1 \otimes v$$

for any $R \in \boldsymbol{M}_k$ and any $g \in \mathfrak{G}(R)$.

PROPOSITION. *Let $\mathfrak{X}$ be a k-functor on which a k-group-functor $\mathfrak{G}$ acts. Suppose that $e \in \mathfrak{X}(k)$ is a rational point which is $\mathfrak{G}$-stable, that is $e \in \mathfrak{X}^{\mathfrak{G}}(k)$, and that the tangent coalgebra $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ in the strong sense exists at e. Then $\mathfrak{G}$ acts naturally and linearly on $(\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}))_{\mathrm{a}}$ and the largest subcoalgebra of $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ which is contained in $(\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}))^{\mathfrak{G}}$ is the tangent coalgebra to $\mathfrak{X}^{\mathfrak{G}}$ at e in the* strong sense.

PROOF. Put $H = \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$. Let $R \in \boldsymbol{M}_k$ and $g \in \mathfrak{G}(R)$. We claim that there exists a unique W_R-automorphism $\alpha(g)$ of H such that

$$\exp(\alpha(g) \circ \sigma, R, C) = g_{R \otimes C^*} \cdot \exp(\sigma, R, C)$$

for any $C \in W_k^{\mathrm{f}}$ and $\sigma \in W_R(C, H)$, where $g_{R \otimes C^*}$ is taken with respect to the canonical map $1 \otimes \eta : R \to R \otimes C^*$.

Indeed let $\mathfrak{G}(R)$ act on each $\mathfrak{X}(R \otimes C^*)$, where $C \in W_k^{\mathrm{f}}$, via $1 \otimes \eta : R \to R \otimes C^*$. Since $e \in \mathfrak{X}(k)$ is $\mathfrak{G}$-stable, the subset

$$\mathrm{Ker}_e\,(\mathfrak{X}(R \otimes C^*) \to \mathfrak{X}(R \otimes C_0^*))$$

is $\mathfrak{G}(R)$-stable. Hence we obtain a natural action of $\mathfrak{G}(R)$ on the set $W_R(C, H)$, which is canonically identified with the above set. Since this action is functional with respect to $C \in W_R^{\mathrm{f}}$, it follows from Lemma 1.3.7 that the action of $g \in \mathfrak{G}(R)$ on $W_R(C, H)$ is of the form $W_R(C, \alpha(g))$, where $\alpha(g)$ is a uniquely determined W_R-automorphism of H. This proves the assertion.

By definition, W_R-aut (H) the group of W_R-automorphisms of H is a subgroup of $\mathbf{GL}_R\,(R \otimes H)$. The reader may easily verify that the correspondence

$$g \mapsto \alpha(g), \ \mathfrak{G}(R) \to W_R\text{-aut}\,(H)$$

is a group homomorphism and functorial with respect to $R \in \boldsymbol{M}_k$. Thus a natural linear action of $\mathfrak{G}$ on H_{a} is determined. Let H' be the largest subcoalgebra of H contained in $H^{\mathfrak{G}}$.

It remains to show that H' is the tangent coalgebra to $\mathfrak{X}^{\mathfrak{G}}$ at e in the strong sense. More precisely we shall prove that under the canonical isomorphism

$$W_R(C, H) \xrightarrow{\simeq} \mathrm{Ker}_e\,(\mathfrak{X}(R \otimes C^*) \longrightarrow \mathfrak{X}(R \otimes C_0^*))$$
$$\sigma \mapsto \exp(\sigma, R, C),$$

the subset $W_R(C, H')$ and the subset $\mathrm{Ker}_e\,(\mathfrak{X}^{\mathfrak{G}}(R\otimes C^*)\to\mathfrak{X}^{\mathfrak{G}}(R\otimes C_0^*))$ correspond to each other, where $(R, C)\in W^{\mathrm{f}}$.

Indeed let $\sigma\in W_R(C, H)$. Then $\exp(\sigma, R, C)$ belongs to $\mathfrak{X}^{\mathfrak{G}}(R\otimes C^*)$ iff for any M_k-map $\phi: R\to S$ and $g\in\mathfrak{G}(S)$ we have

$$g_{S\otimes C^*}\cdot\exp(\sigma, R, C)_{S\otimes C^*}=\exp(\sigma, R, C)_{S\otimes C^*}$$

in view of [7, II, § 1, 3.5]. Therefore we have

$$\begin{aligned}
&\exp(\sigma, R, C)\in\mathfrak{X}^{\mathfrak{G}}(R\otimes C^*)\\
&\Leftrightarrow\exp(\alpha(g)\circ W_\phi(\sigma), S, C)=\exp(W_\phi(\sigma), S, C)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for any } \phi\in M_k(R, S) \text{ and } g\in\mathfrak{G}(S)\\
&\Leftrightarrow\alpha(g)\circ W_\phi(\sigma)=W_\phi(\sigma)\qquad\text{for any } \phi\in M_k(R, S) \text{ and } g\in\mathfrak{G}(S)\\
&\Leftrightarrow\sigma(R\otimes C)\subset(H_a)^{\mathfrak{G}}(R)=(H_{\mathfrak{G}})_a(R)=R\otimes H^{\mathfrak{G}}\\
&\Leftrightarrow\sigma\in W_R(C, H')\qquad\text{(by Lemma 1.2.5).}
\end{aligned}$$

This completes Proof.

2.1.11 Let $\mathfrak{X}$ be a locally algebraic k-scheme. For each point x of $\mathfrak{X}$ we have

$$\begin{aligned}
&\mathbf{T}_x(\mathfrak{X})=(\mathcal{O}_x)^0\neq 0\\
&\Leftrightarrow[\boldsymbol{\kappa}(x): k]<\infty\\
&\Leftrightarrow x \text{ is a closed point of } \mathfrak{X}
\end{aligned}$$

by [7, I, § 3, 6.5]. Suppose that x is a closed point of $\mathfrak{X}$. We have by (1.1.2) that

$$\mathbf{T}_x(\mathfrak{X})=(\mathcal{O}_x)^0=\bigcup_n(\mathcal{O}_x/m_x^{n+1})^*.$$

In particular $\mathbf{T}_x(\mathfrak{X})$ is *of finite type* and

$$\mathbf{T}_x(\mathfrak{X})^*=\varprojlim\mathcal{O}_x/m_x^{n+1}$$

is nothing other than the completion $\hat{\mathcal{O}}_x$ of $\mathcal{O}_x$ with respect to the m_x-adic topology. Now we have

$$\begin{aligned}
\dim_x\mathfrak{X}&=K\dim\mathcal{O}_x+\mathrm{tr.\,deg}_k\,\boldsymbol{\kappa}(x)\\
&=K\dim\mathcal{O}_x
\end{aligned}$$

by [7, I, § 3, 6.1] and

$$K\dim\mathcal{O}_x=K\dim\hat{\mathcal{O}}_x=K\dim\mathbf{T}_x(\mathfrak{X})^*=K\dim\mathbf{T}_x(\mathfrak{X})$$

by [7, I, § 3, 5.13] and (1,4,4), since $\mathbf{T}_x(\mathfrak{X})$ is of finite type. Thus we have

proved:

PROPOSITION. *Let $\mathfrak{X}$ be a locally algebraic k-scheme. Let x be a point of $\mathfrak{X}$. If x is closed then the tangent coalgebra $\mathbf{T}_x(\mathfrak{X})$ is irreducible of finite type and the local dimension* $\dim_x \mathfrak{X}$ *at x* [7, I, § 3, 6.1] *is equal to the Krull dimension* $K \dim \mathbf{T}_x(\mathfrak{X})$. *If x is not closed $\mathbf{T}_x(\mathfrak{X})$ is zero.*

2.1.12 Let $\mathfrak{X}$ be a locally algebraic k-scheme and $e \in \mathfrak{X}(k)$ a rational point. The Zariski's tangent space to $\mathfrak{X}$ at e [7, I, § 4, 4,15] is the k-vector space

$$(m_e/m_e^2)^*$$

which is canonically identified with $\mathrm{P}(\mathbf{T}_e(\mathfrak{X}))$ the set of primitive elements with respect to the unique group-like element of $\mathbf{T}_e(\mathfrak{X})$.

Let x be a closed point of $\mathfrak{X}$. We shall show in the next section that the Zariski's tangent space to $\mathfrak{X}$ at x can be canonically identified with $\mathrm{P}_g(\boldsymbol{\kappa}(x)\otimes\mathbf{T}_x(\mathfrak{X}))$ the set of primitive elements of the $\boldsymbol{\kappa}(x)$-coalgebra $\boldsymbol{\kappa}(x)\otimes\mathbf{T}_x(\mathfrak{X})$ with respect to the group-like element g of $\boldsymbol{\kappa}(x)\otimes\mathbf{T}_x(\mathfrak{X})$ which corresponds to the canonical injection: $\boldsymbol{\kappa}(x)^* \hookrightarrow \mathbf{T}_x(\mathfrak{X})$.

2.2 Some more properties of the functor $\mathbf{T}(-)$

2.2.1 $\boldsymbol{M_kE}$ denotes the category of k-functors. Let $\boldsymbol{M_kE}^{\mathrm{uc}}$ be the full subcategory of $\boldsymbol{M_kE}$ consisting of all k-functors $\mathfrak{X}$ which have the underlying coalgebra $\mathbf{T}(\mathfrak{X})$ (2.1.1). We know that the correspondence

$$\mathfrak{X} \mapsto \mathbf{T}(\mathfrak{X})$$

gives rise to a covariant functor from $\boldsymbol{M_kE}^{\mathrm{uc}}$ to $\boldsymbol{W_k}$ and that $\boldsymbol{M_kE}^{\mathrm{uc}}$ contains the category of k-schemes $\boldsymbol{Sch_k}$. The following proposition is clear from definition:

PROPOSITION. *The subcategory $\boldsymbol{M_kE}^{\mathrm{uc}}$ of $\boldsymbol{M_kE}$ is closed under finite limits and the functor*

$$\mathbf{T}: \boldsymbol{M_kE}^{\mathrm{uc}} \to \boldsymbol{W_k}$$

commutes with finite limits. In particular if $\mathfrak{X}$ and $\mathfrak{Y}$ are k-functors having underlying coalgebras then $\mathfrak{X}\times\mathfrak{Y}$ has also the underlying coalgebra and

$$\mathbf{T}(\mathfrak{X}\times\mathfrak{Y}) \xrightarrow{\simeq} \mathbf{T}(\mathfrak{X})\otimes\mathbf{T}(\mathfrak{Y}).$$

2.2.2 Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two k-functors. Let $\mathfrak{Z}=\mathfrak{X}\times\mathfrak{Y}$ be the direct product of $\mathfrak{X}$ and $\mathfrak{Y}$ with the canonical projections

$$\mathfrak{pr}_1 : \mathfrak{Z} \to \mathfrak{X} \quad \text{and} \quad \mathfrak{pr}_2 : \mathfrak{Z} \to \mathfrak{Y}.$$

We say that a point z of $\mathfrak{Z}$ lies over (x, y), where $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$, if $x = \mathfrak{pr}_1(z)$ and $y = \mathfrak{pr}_2(z)$ (cf. (2.1.2)).

COROLLARY. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two k-functors. Let $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$ and suppose that the tangent coalgebras $\mathbf{T}_x(\mathfrak{X})$ and $\mathbf{T}_y(\mathfrak{Y})$ exist. Then for any point z of $\mathfrak{X} \times \mathfrak{Y}$ lying over (x, y), the tangent coalgebra $\mathbf{T}_z(\mathfrak{X} \times \mathfrak{Y})$ exists and*

$$\bigoplus_{z \in \mathfrak{X} \times \mathfrak{Y} \text{ lying over } (x,y)} \mathbf{T}_z(\mathfrak{X} \times \mathfrak{Y}) \xrightarrow{\simeq} \mathbf{T}_x(\mathfrak{X}) \otimes \mathbf{T}_y(\mathfrak{Y}).$$

PROOF. By assumption $\mathfrak{X}_x$ and $\mathfrak{Y}_y$ belong to $\boldsymbol{M_k E}^{\mathrm{uc}}$ and so does $\mathfrak{X}_x \times \mathfrak{Y}_y$. The above assertion will follow from Proposition 2.1.4 (ii) if we show the following two facts:

(i) The inclusion: $\mathfrak{X}_x \times \mathfrak{Y}_y \hookrightarrow \mathfrak{X} \times \mathfrak{Y}$ induces an injection:

$$|\mathfrak{X}_x \times \mathfrak{Y}_y| \hookrightarrow |\mathfrak{X} \times \mathfrak{Y}|$$

whose image coincides with the set of points lying over (x, y).

(ii) For any $z \in \mathfrak{X} \times \mathfrak{Y}$ lying over (x, y), which is identified with a point of $\mathfrak{X}_x \times \mathfrak{Y}_y$, we have

$$(\mathfrak{X}_x \times \mathfrak{Y}_y)_z = (\mathfrak{X} \times \mathfrak{Y})_z.$$

Indeed the injectivity of the map $|\mathfrak{X}_x \times \mathfrak{Y}_y| \to |\mathfrak{X} \times \mathfrak{Y}|$ follows from Lemma below. On the other hand it is known that $|\mathfrak{X}_P| = P$ for any subset P of $|\mathfrak{X}|$ [7, I, § 1, 4.10].

Now let $z \in \mathfrak{X} \times \mathfrak{Y}$. If $z \in \mathfrak{X}_x \times \mathfrak{Y}_y$ then $\mathfrak{pr}_1(z) \in |\mathfrak{X}_x| = \{x\}$ and $\mathfrak{pr}_2(z) \in |\mathfrak{Y}_y| = \{y\}$. If z lies over (x, y), then since

$$(\mathfrak{X} \times \mathfrak{Y})_z \subset \mathfrak{X}_x \times \mathfrak{Y}_y,$$

we have $\{z\} = |(\mathfrak{X} \times \mathfrak{Y})_z| \subset |\mathfrak{X}_x \times \mathfrak{Y}_y|$. This proves (i). In general $\mathfrak{X}_P$ is the largest subfunctor of $\mathfrak{X}$ such that $|\mathfrak{X}_P| \subset P$. Therefore the inclusion $(\mathfrak{X} \times \mathfrak{Y})_z \subset \mathfrak{X}_x \times \mathfrak{Y}_y$ means $(\mathfrak{X} \times \mathfrak{Y})_z = (\mathfrak{X}_x \times \mathfrak{Y}_y)_z$ for any point z of $\mathfrak{X} \times \mathfrak{Y}$ lying over (x, y).

2.2.3 LEMMA. *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a monomorphism in $\boldsymbol{M_k E}$. Then the induced map $|\mathfrak{f}| : |\mathfrak{X}| \to |\mathfrak{Y}|$ is injective* (cf. [7, I, § 1, 5.3]).

PROOF. The category $\boldsymbol{Fld_k}$ has the initial object k and is directed in the following sense: For any objects J, K and L of $\boldsymbol{Fld_k}$ and any $\boldsymbol{Fld_k}$-maps $\alpha : J \to K$ and $\beta : J \to L$ there exist an $\boldsymbol{Fld_k}$-object M and $\boldsymbol{Fld_k}$-maps $\sigma : K \to M$ and $\tau : L \to M$ such that $\sigma \circ \alpha = \tau \circ \beta$.

It follows that for any $a \in \mathfrak{X}(K)$ and $b \in \mathfrak{X}(L)$, where K and $L \in \boldsymbol{Fld_k}$, the equality $[a] = [b]$ holds iff there exist an $\boldsymbol{Fld_k}$-object M and $\boldsymbol{Fld_k}$-maps $\sigma : K$

$\to M$ and $\tau: L \to M$ such that $a_M = b_M$. Lemma follows immediately from this fact.

2.2.4 PROPOSITION. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two k-functors. Let $e \in \mathfrak{X}(k)$ and $e' \in \mathfrak{Y}(k)$. If the coalgebras $\mathbf{S}_e(\mathfrak{X})$ and $\mathbf{S}_{e'}(\mathfrak{Y})$ (resp. the tangent coalgebras in the strong sense $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ and $\mathbf{T}_{e'}^{\mathrm{st}}(\mathfrak{Y})$) exist, then the coalgebra $\mathbf{S}_{(e,e')}(\mathfrak{X} \times \mathfrak{Y})$ (resp. the tangent coalgebra in the strong sense $\mathbf{T}_{(e,e')}^{\mathrm{st}}(\mathfrak{X} \times \mathfrak{Y})$) exists and*

$$\mathbf{S}_{(e,e')}(\mathfrak{X} \times \mathfrak{Y}) \xrightarrow{\simeq} \mathbf{S}_e(\mathfrak{X}) \otimes \mathbf{S}_{e'}(\mathfrak{Y})$$

(resp.

$$\mathbf{T}_{(e,e')}^{\mathrm{st}}(\mathfrak{X} \times \mathfrak{Y}) \xrightarrow{\simeq} \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}) \otimes \mathbf{T}_{e'}^{\mathrm{st}}(\mathfrak{Y})).$$

PROOF. Put $\mathfrak{Z} = \mathfrak{X} \times \mathfrak{Y}$, $e'' = (e, e')$, $H = \mathbf{S}_e(\mathfrak{X})$ and $H' = \mathbf{S}_{e'}(\mathfrak{Y})$ (resp. $\bar{H} = \mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ and $\bar{H}' = \mathbf{T}_{e'}^{\mathrm{st}}(\mathfrak{Y})$). Since $H \otimes H'$ (resp. $\bar{H} \otimes \bar{H}'$) is the direct product of H and H' (resp. of $\bar{H}$ and $\bar{H}'$) in the category W_k (resp. in the category W_R for any $R \in M_k$) by (1.2.2) (resp. by (1.2.4)), we have for all $C \in W_k^{\mathrm{f}}$,

$$\begin{aligned} W_k(C, H \otimes H') &\simeq \mathrm{Ker}_e\,(\mathfrak{X}(C^*) \to \mathfrak{X}(C_0^*)) \\ &\quad \times \mathrm{Ker}_{e'}\,(\mathfrak{Y}(C^*) \to \mathfrak{Y}(C_0^*)) \\ &= \mathrm{Ker}_{e''}\,(\mathfrak{Z}(C^*) \to \mathfrak{Z}(C_0^*)) \end{aligned}$$

(resp.

$$\begin{aligned} W_R(C, \bar{H} \otimes \bar{H}') &\simeq \mathrm{Ker}_e\,(\mathfrak{X}(R \otimes C^*) \to \mathfrak{X}(R \otimes C_0^*)) \\ &\quad \times \mathrm{Ker}_{e'}\,(\mathfrak{Y}(R \otimes C^*) \to \mathfrak{Y}(R \otimes C_0^*)) \\ &= \mathrm{Ker}_{e''}\,(\mathfrak{Z}(R \otimes C^*) \to \mathfrak{Z}(R \otimes C_0^*)). \end{aligned}$$

Since this isomorphism is functorial with respect to $C \in W_k^{\mathrm{f}}$ (resp. with respect to $(R, C) \in W^{\mathrm{f}}$), Proposition follows.

2.2.5 PROPOSITION. *Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of k-functors. Suppose that the underlying coalgebras $\mathbf{T}(\mathfrak{X})$ and $\mathbf{T}(\mathfrak{Y})$ exist. If $\mathfrak{f}$ is a monomorphism in $M_k E$ then the coalgebra map*

$$\mathbf{T}(\mathfrak{f}): \mathbf{T}(\mathfrak{X}) \to \mathbf{T}(\mathfrak{Y})$$

is injective.

PROOF. Let $C \in W_k$ and $\{C_i\}$ be the set of all finite dimensional subcoalgebras of C. The map

$$W_k(C, \mathbf{T}(\mathfrak{f})): W_k(C, \mathbf{T}(\mathfrak{X})) \to W_k(C, \mathbf{T}(\mathfrak{Y}))$$

is injective, since it factors as

$$\begin{aligned} & W_k(C, \mathbf{T}(\mathfrak{X})) \\ \simeq & \varprojlim W_k(C_i, \mathbf{T}(\mathfrak{X})) \\ \simeq & \varprojlim \mathfrak{X}(C_i^*) \xhookrightarrow{f} \varprojlim \mathfrak{Y}(C_i^*) \\ & \qquad \simeq \varprojlim W_k(C_i, \mathbf{T}(\mathfrak{Y})) \\ & \qquad \simeq W_k(C, \mathbf{T}(\mathfrak{Y})). \end{aligned}$$

Hence $\mathbf{T}(\mathfrak{f})$ is a monomorphism in W_k and is injective by (1.2.3).

2.2.6 Let $\mathfrak{X}$ be a k-scheme. For any point x of $\mathfrak{X}$ let

$$\boldsymbol{\varepsilon}(x): \mathfrak{Sp}\ \boldsymbol{\kappa}(x) \to \mathfrak{X}$$

denote the canonical map [7, I, § 1, 5.2]. This is a monomorphism in $M_k E$ and its image is contained in $\mathfrak{X}_x$. Hence it induces an injective coalgebra map

$$\mathbf{T}_x(\boldsymbol{\varepsilon}(x)): \boldsymbol{\kappa}(x)^* \to \mathbf{T}_x(\mathfrak{X})$$

if $[\boldsymbol{\kappa}(x): k] < \infty$. If we identify $\mathbf{T}_x(\mathfrak{X}) = (\mathcal{O}_x)^0$, then this map is induced by the canonical projection

$$\boldsymbol{\kappa}(x) \leftarrow \mathcal{O}_x.$$

In the following the above injection

$$\boldsymbol{\kappa}(x)^* \hookrightarrow \mathbf{T}_x(\mathfrak{X})$$

is called the canonical injection and denoted by $\iota(x)$ (, when of course $[\boldsymbol{\kappa}(x): k] < \infty$).

Now let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of k-schemes. Let $y \in \mathfrak{Y}$. The *fibre* of $\mathfrak{f}$ over y, written $\mathfrak{f}^{-1}(y)$, is defined by the following pullback diagram [7, I, § 1, 5.8]:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Y} \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \varepsilon(y) \\ \mathfrak{f}^{-1}(y) & \longrightarrow & \mathfrak{Sp}\ \boldsymbol{\kappa}(y). \end{array}$$

Let x be a point of $\mathfrak{X}$ such that $\mathfrak{f}(x) = y$. Then we have a pullback diagram

$$\begin{array}{ccc} \mathfrak{X}_x & \xrightarrow{\mathfrak{f}} & \mathfrak{Y}_y \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \varepsilon(y) \\ (\mathfrak{f}^{-1}(y))_x & \longrightarrow & \mathfrak{Sp}\ \boldsymbol{\kappa}(y). \end{array}$$

Hence if in particular $[\kappa(y): k]<\infty$, then the diagram

$$\begin{tikzcd} \mathbf{T}_x(\mathfrak{X}) \arrow[r, "\mathbf{T}_x(\mathfrak{f})"] & \mathbf{T}_y(\mathfrak{Y}) \\ \mathbf{T}_x(\mathfrak{f}^{-1}(y)) \arrow[u, hook] \arrow[r] & \kappa(y)^* \arrow[u, hook, "\iota(y)"'] \end{tikzcd}$$

is a pullback diagram in W_k. In other words this diagram determines the subcoalgebra $\mathbf{T}_x(\mathfrak{f}^{-1}(y))$ of $\mathbf{T}_x(\mathfrak{X})$.

2.2.7 PROPOSITION. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally algebraic k-schemes and $\mathfrak{f}, \mathfrak{g}: \mathfrak{X} \rightrightarrows \mathfrak{Y}$ two k-scheme maps.*

(i) *Let x be a closed point of $\mathfrak{X}$ and suppose that $\mathfrak{f}(x)=y=\mathfrak{g}(x)$. If the induced maps $\mathbf{T}_x(\mathfrak{f}), \mathbf{T}_x(\mathfrak{g}): \mathbf{T}_x(\mathfrak{X}) \rightarrow \mathbf{T}_y(\mathfrak{Y})$ coincide, then there exists an open subscheme $\mathfrak{U}$ of $\mathfrak{X}$ such that $x \in \mathfrak{U}$ and that $\mathfrak{f}|\mathfrak{U}=\mathfrak{g}|\mathfrak{U}$.*

(ii) *If the induced maps $\mathbf{T}(\mathfrak{f}), \mathbf{T}(\mathfrak{g}): \mathbf{T}(\mathfrak{X}) \rightarrow \mathbf{T}(\mathfrak{Y})$ coincide, then $\mathfrak{f}=\mathfrak{g}$.*

(iii) *$\mathfrak{f}$ is a monomorphism in $M_k E$ iff $\mathbf{T}(\mathfrak{f}): \mathbf{T}(\mathfrak{X}) \rightarrow \mathbf{T}(\mathfrak{Y})$ is injective.*

PROOF. (i) Since m_x is cofinite in $\mathcal{O}_x$, $(\mathcal{O}_x)^0=\mathbf{T}_x(\mathfrak{X})$ is dense in $(\mathcal{O}_x)^*$ by Proposition 1.1.2 (iii). Hence the equality $\mathbf{T}_x(\mathfrak{f})=\mathbf{T}_x(\mathfrak{g})$ means that the induced maps $\mathfrak{f}_x, \mathfrak{g}_x: \mathcal{O}_y \rightarrow \mathcal{O}_x$ coincide. The assertion follows immediately from [7, I, § 3, 4.1].

(ii) If we notice that

$$\mathbf{T}(\mathfrak{X})=\bigoplus_{x \in \mathfrak{X} \text{ closed}} \mathbf{T}_x(\mathfrak{X})$$

(2.1.4 (ii)) and that $\mathbf{T}_x(\mathfrak{X}) \neq 0$ for any closed point x of $\mathfrak{X}$, then the equality $\mathbf{T}(\mathfrak{f})=\mathbf{T}(\mathfrak{g})$ is equivalent to saying that $\mathfrak{f}(x)=\mathfrak{g}(x)$ and $\mathbf{T}_x(\mathfrak{f})=\mathbf{T}_x(\mathfrak{g})$ for all closed points x of $\mathfrak{X}$. It follows from (i) that there exists an open subscheme $\mathfrak{U}_x$ containing x such that $\mathfrak{f}|\mathfrak{U}_x=\mathfrak{g}|\mathfrak{U}_x$ for each closed point x of $\mathfrak{X}$. Let $\mathfrak{U}$ be the smallest open subscheme of $\mathfrak{X}$ which contains all $\mathfrak{U}_x$'s. In view of [7, I, § 1, 4.13], one sees that $\mathfrak{f}|\mathfrak{U}=\mathfrak{g}|\mathfrak{U}$. But since $\mathfrak{U}$ contains all closed points of $\mathfrak{X}$, it follows that $\mathfrak{U}=\mathfrak{X}$ from [7, I, § 3, 6.9]. Hence $\mathfrak{f}=\mathfrak{g}$.

(iii) Construct a pullback diagram in $M_k E$ as follows:

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "\mathfrak{pr}_1"] \arrow[d, "\mathfrak{pr}_2"] & \mathfrak{X} \arrow[d, "\mathfrak{f}"] \\ \mathfrak{X} \arrow[r, "\mathfrak{f}"] & \mathfrak{Y} . \end{tikzcd}$$

It is known that $\mathfrak{Z}$ is a locally algebraic k-scheme [7, I, § 3, 1.10]. If $\mathbf{T}(\mathfrak{f})$ is injective, then since

$$\mathbf{T}(\mathfrak{f}) \circ \mathbf{T}(\mathfrak{pr}_1)=\mathbf{T}(\mathfrak{f}) \circ \mathbf{T}(\mathfrak{pr}_2),$$

we have $\mathbf{T}(\mathfrak{pr}_1)=\mathbf{T}(\mathfrak{pr}_2)$. Hence $\mathfrak{pr}_1=\mathfrak{pr}_2$ by (ii). This means that $\mathfrak{f}$ is a monomorphism in $\boldsymbol{M}_k\boldsymbol{E}$. The converse follows from Proposition 2.2.5.

2.2.8 Let $K|k$ be a field extension. Let $\mathfrak{X}$ be a k-functor. Then the K-functor $K\otimes_k \mathfrak{X}$ is defined by

$$K\otimes\mathfrak{X} : \boldsymbol{M}_K\to\boldsymbol{E}, R\mapsto\mathfrak{X}({}_kR),$$

where ${}_kR$ is the underlying k-algebra of R [7, I, § 1, 6.5]. Notice that there exists a natural equivalence between the categories $\boldsymbol{M}_K\boldsymbol{E}$ and $\boldsymbol{M}_k\boldsymbol{E}/(\mathfrak{Sp}\, K)$, where the latter denotes the category of k-functors over $\mathfrak{Sp}\, K$ [7, I, § 1, 6.2]. Then the K-functor $K\otimes\mathfrak{X}$ corresponds to the k-functor over $\mathfrak{Sp}\, K$

$$\mathfrak{pr}_1 : (\mathfrak{Sp}\, K)\times\mathfrak{X}\to\mathfrak{Sp}\, K,$$

where $(\mathfrak{Sp}\, K)\times\mathfrak{X}$ denotes the direct product in $\boldsymbol{M}_k\boldsymbol{E}$ and $\mathfrak{pr}_1$ the canonical projection. In particular $|K\otimes\mathfrak{X}|$ the underlying set of the K-functor $K\otimes\mathfrak{X}$ is equal to $|(\mathfrak{Sp}\, K)\times\mathfrak{X}|$ by definition [7, I, § 1, 6.3]. We say that a point $\tilde{x}$ of $K\otimes\mathfrak{X}$, which is identified with a point of $(\mathfrak{Sp}\, K)\times\mathfrak{X}$, lies over a point x of $\mathfrak{X}$ and write $\tilde{x}|x$ if $\mathfrak{pr}_2(\tilde{x})=x$, where

$$\mathfrak{pr}_2 : (\mathfrak{Sp}\, K)\times\mathfrak{X}\to\mathfrak{X}$$

is the canonical projection.

For a K-functor $\mathfrak{Y}$, we denote by $\mathbf{T}^K(\mathfrak{Y})$ the underlying K-coalgebra of $\mathfrak{Y}$ (, if it exists). In particular $\mathbf{T}^K_y(\mathfrak{Y})$ denotes the tangent K-coalgebra to $\mathfrak{Y}$ at a point y of $\mathfrak{Y}$. Suppose that $\mathbf{T}(\mathfrak{X})$ and $\mathbf{T}^K(K\otimes\mathfrak{X})$ exist. We define a K-coalgebra map

$$\xi : K\otimes\mathbf{T}(\mathfrak{X})\to\mathbf{T}^K(K\otimes\mathfrak{X})$$

as follows: Let C be a finite dimensional subcoalgebra of $\mathbf{T}(\mathfrak{X})$ with $i : C \hookrightarrow\mathbf{T}(\mathfrak{X})$ the inclusion. Let

$$\xi_C : K\otimes C\to\mathbf{T}^K(K\otimes\mathfrak{X})$$

be the K-coalgebra map which corresponds naturally to the element $(\bar{i})_{K\otimes C^*}$ of $\mathfrak{X}(K\otimes C^*)$, where $\bar{i}$ is the element of $\mathfrak{X}(C^*)$ determined by i and $(\bar{i})_{K\otimes C^*}$ is taken with respect to the canonical map

$$C^*\to K\otimes C^*.$$

Since ξ_C is functorial with respect to C, there exists a unique K-coalgebra map ξ such that

$$\xi|K\otimes C=\xi_C$$

for all finite dimensional subcoalgebras C of $\mathbf{T}(\mathfrak{X})$.

Let C be a finite dimensional cocommutative k-coalgebra, then the diagram below commutes:

$$\begin{array}{ccc} \boldsymbol{Coalg}_k(C, \mathbf{T}(\mathfrak{X})) & \xrightarrow{\bar{\xi}} & \boldsymbol{Coalg}_K(K\otimes C, \mathbf{T}^K(K\otimes\mathfrak{X})) \\ \wr\| & & \wr\| \\ \mathfrak{X}(C^*) & \xrightarrow{\mathfrak{X}(\eta\otimes 1)} & \mathfrak{X}(K\otimes C^*) \end{array},$$

where $\bar{\xi}$ is the composite

$$\boldsymbol{Coalg}_k(C, \mathbf{T}(\mathfrak{X})) \xrightarrow{K\otimes ?} \boldsymbol{Coalg}_K(K\otimes C, K\otimes\mathbf{T}(\mathfrak{X}))$$
$$\xrightarrow{\boldsymbol{Coalg}_K(K\otimes C, \xi)} \boldsymbol{Coalg}_K(K\otimes C, \mathbf{T}^K(K\otimes\mathfrak{X})).$$

One sees that ξ is the unique K-coalgebra map such that the above diagram commutes for all $C \in W_k^{\mathrm{f}}$.

Take $\mathfrak{X}=\mathfrak{Sp}\, A$ for instance, where $A \in M_k$. Let

$$\iota\colon K\otimes A^0 \to (K\otimes A)^0$$

be the canonical K-coalgebra map, where the right hand side denotes the K-coalgebra dual, that is

$$\iota(\lambda\otimes\chi)(\mu\otimes a)=\langle\chi, a\rangle\lambda\mu$$

for $\lambda, \mu \in K, \chi \in A^0$ and $a \in A$. In view of the above characterization of the K-coalgebra map ξ, one may easily verify the commutativity of the following diagram

$$\begin{array}{ccc} K\otimes\mathbf{T}(\mathfrak{X}) & \xrightarrow{\xi} & \mathbf{T}^K(K\otimes\mathfrak{X}) \\ \wr\| & & \wr\| \\ K\otimes A^0 & \xrightarrow{\iota} & (K\otimes A)^0 \end{array}.$$

PROPOSITION. *Let $\mathfrak{X}$ be a locally algebraic k-scheme and $K|k$ a field extension.*

(i) *Let x be a closed point of $\mathfrak{X}$. There exists an isomorphism of K-coalgebras $K\otimes\mathbf{T}_x(\mathfrak{X}) \simeq \bigoplus_{\bar{x}|x} \mathbf{T}_{\bar{x}}^K(K\otimes\mathfrak{X})$ such that the diagram*

$$\begin{array}{ccc} K\otimes\mathbf{T}(\mathfrak{X}) & \xrightarrow{\xi} & \mathbf{T}^K(K\otimes\mathfrak{X}) \\ \uparrow\cup & & \uparrow\cup \\ K\otimes\mathbf{T}_x(\mathfrak{X}) & \simeq & \bigoplus_{\bar{x}|x} \mathbf{T}_{\bar{x}}^K(K\otimes\mathfrak{X}) \end{array}$$

commutes.

(ii) *In particular ξ is injective and*

$$\xi(K\otimes\mathbf{T}(\mathfrak{X}))=\bigoplus_{\mathfrak{pr}_2(\tilde{x})\text{ is closed}}\mathbf{T}^K_{\tilde{x}}(K\otimes\mathfrak{X}).$$

(iii) *If $K|k$ is an algebraic extension, then ξ is an isomorphism of K-coalgebras.*

PROOF. Let x be a point of $\mathfrak{X}$. We can identify $\mathbf{T}_x(\mathfrak{X})=(\mathcal{O}_x)^0$. Then the canonical inclusion

$$(\mathcal{O}_x)^\circ=\mathbf{T}_x(\mathfrak{X})\hookrightarrow\mathbf{T}(\mathfrak{X})$$

is clearly obtained by applying the functor $\mathbf{T}(-)$ to the canonical monomorphism

$$\boldsymbol{\varepsilon}_x:\mathfrak{Sp}\,\mathcal{O}_x\hookrightarrow\mathfrak{X}.$$

Since the K-coalgebra map ξ is clearly functorial, the diagram below commutes:

$$\begin{array}{ccc} K\otimes\mathbf{T}(\mathfrak{X}) & \xrightarrow{\xi} & \mathbf{T}^K(K\otimes\mathfrak{X}) \\ \Big\uparrow{\scriptstyle K\otimes\mathbf{T}(\boldsymbol{\varepsilon}_x)} & & \Big\uparrow{\scriptstyle \mathbf{T}^K(K\otimes\boldsymbol{\varepsilon}_x)} \\ K\otimes(\mathcal{O}_x)^0 & \xrightarrow{\iota} & (K\otimes\mathcal{O}_x)^0 \end{array},$$

where ι is the canonical map defined before this Proposition.

On the other hand we have a canonical bijection

$$\{\tilde{x}\in K\otimes\mathfrak{X}\,|\,\tilde{x}\text{ lies over }x\}\simeq\operatorname{Spec}(K\otimes_k\boldsymbol{\kappa}(x))$$

[7, I, § 1, 5.2]. We denote by $P(\tilde{x})$ the prime ideal of $K\otimes\boldsymbol{\kappa}(x)$ which corresponds to $\tilde{x}$. If we regard $P(\tilde{x})$ as a prime ideal of $K\otimes\mathcal{O}_x$ naturally, then the K-algebra $\mathcal{O}_{\tilde{x}}$ can be canonically identified with the localization $(K\otimes\mathcal{O}_x)_{P(\tilde{x})}$ and the canonical monomorphism

$$\boldsymbol{\varepsilon}_{\tilde{x}}:\mathfrak{Sp}_K\,\mathcal{O}_{\tilde{x}}\to K_{\tilde{x}}\otimes\mathfrak{X}$$

is obtained as the composite

$$\mathfrak{Sp}_K\,(K\otimes\mathcal{O}_x)_{P(\tilde{x})}\hookrightarrow\mathfrak{Sp}_K\,(K\otimes\mathcal{O}_x)\xrightarrow{K\otimes\boldsymbol{\varepsilon}_x}K\otimes\mathfrak{X}.$$

Suppose that x is a closed point of $\mathfrak{X}$. Then $K\otimes\boldsymbol{\kappa}(x)$ is Artinian. Hence its prime ideals are all maximal. Then Corollary 1.1.4 means that the map

$$\iota:K\otimes(\mathcal{O}_x)^0\to(K\otimes\mathcal{O}_x)^0$$

is injective and its image is

$$\bigoplus_{\tilde{x}|x} ((K\otimes\mathcal{O}_x)_{P(\tilde{x})})^0 = \bigoplus_{\tilde{x}|x} (\mathcal{O}_{\tilde{x}})^0.$$

Thus we get a commutative diagram

$$\begin{tikzcd}
K\otimes\mathbf{T}(\mathfrak{X}) \arrow[r] & \mathbf{T}^K(K\otimes\mathfrak{X}) \\
K\otimes(\mathcal{O}_x)^0 \arrow[r, "\simeq"] \arrow[u, hook, "\text{cano.}"'] & \bigoplus_{\tilde{x}|x} (\mathcal{O}_{\tilde{x}})^0 \arrow[u, hook, "\text{cano.}"']
\end{tikzcd} .$$

This proves (i).

(ii) follows immediately from (i).

Next suppose that $K|k$ is algebraic. It is enough to show that if $\tilde{x}$ is a closed point of $K\otimes\mathfrak{X}$, then $\mathfrak{pr}_2(\tilde{x})=x$ is a closed point of $\mathfrak{X}$. But this follows from the diagram below:

$$\begin{tikzcd}
 & \boldsymbol{\kappa}(\tilde{x}) \arrow[dl, dash, "\text{alg.}"'] \arrow[dr, dash] & \\
K \arrow[dr, dash, "\text{alg.}"'] & & \boldsymbol{\kappa}(x) \arrow[dl, dash] \\
 & k &
\end{tikzcd}$$

2.2.9 For a map of fields $\sigma: k\to K$ and a k-functor $\mathfrak{X}$, we put

$$K\otimes_\sigma \mathfrak{X} = K\otimes_k \mathfrak{X}.$$

Let p the characteristic exponent of k. We put

$$f: k\to k, \lambda \mapsto \lambda^p.$$

For a k-functor $\mathfrak{X}$, we put

$$\mathfrak{X}^{(p)} = k\otimes_f \mathfrak{X}.$$

Thus this is the following k-functor

$$\mathfrak{X}^{(p)}: \boldsymbol{M}_k\to\boldsymbol{E}, R\mapsto\mathfrak{X}({}_fR),$$

where ${}_fR$ is the k-algebra obtained by pulling back the k-algebra R along f [10, III, 6, page 90]. The Frobenius map on $\mathfrak{X}$ [7, II, § 7, 1.1], written $\mathfrak{F}_{\mathfrak{X}}$: $\mathfrak{X}\to\mathfrak{X}^{(p)}$, is defined as follows: For each $R\in\boldsymbol{M}_k$, the map

$$\mathfrak{F}_{\mathfrak{X}}(R): \mathfrak{X}(R)\to\mathfrak{X}^{(p)}(R)=\mathfrak{X}({}_fR)$$

is obtained by applying the functor $\mathfrak{X}$ to the $\boldsymbol{M}_k$-map

$$f_R : R \to {}_fR,\ a \mapsto a^p.$$

PROPOSITION. *Let $\mathfrak{X}$ be a k-functor. Suppose that both $\mathfrak{X}$ and $\mathfrak{X}^{(p)}$ have the underlying coalgebras. Then the composite*

$$\mathbf{T}(\mathfrak{X}) \xrightarrow{\Upsilon_{\mathbf{T}(\mathfrak{X})}} \mathbf{T}(\mathfrak{X})^{(p)} \xrightarrow{\ \xi\ } \mathbf{T}(\mathfrak{X}^{(p)})$$

is equal to $\mathbf{T}(\mathfrak{F}_{\mathfrak{X}})$, where $\Upsilon_{\mathbf{T}(\mathfrak{X})}$ is defined in § 1.9.

PROOF. Let $C \in W_k^{\mathrm{f}}$. Then by definition we have a commutative diagram:

$$\begin{tikzcd}
\mathfrak{X}(C^*) \arrow[r, phantom, "\simeq"] \arrow[ddd, "\mathfrak{X}(\eta\otimes 1)"'] & W_k(C, \mathbf{T}(\mathfrak{X})) \arrow[d, "(-)^{(p)}"] \arrow[dr, "W_k(C{,}\Upsilon_{\mathbf{T}(\mathfrak{X})})"] & \\
& W_k(C^{(p)}, \mathbf{T}(\mathfrak{X})^{(p)}) \arrow[r, "W_k(\Upsilon_C{,}\mathbf{T}(\mathfrak{X})^{(p)})"] \arrow[d, "W_k(C^{(p)}{,}\xi)"] & W_k(C, \mathbf{T}(\mathfrak{X})^{(p)}) \arrow[d, "W_k(C{,}\xi)"] \\
& W_k(C^{(p)}, \mathbf{T}(\mathfrak{X}^{(p)})) \arrow[r, "W_k(\Upsilon_C{,}\mathbf{T}(\mathfrak{X}^{(p)}))"] \arrow[dl, phantom, "\simeq" sloped] & W_k(C, \mathbf{T}(\mathfrak{X}^{(p)})) \arrow[d, phantom, "\wr"] \\
\mathfrak{X}({}_f(C^{*(p)})) \arrow[rr, "\mathfrak{X}({}_f\phi)"] & & \mathfrak{X}({}_fC^*)
\end{tikzcd}$$

where we identify as usual $C^{(p)*} = C^{*(p)}$ and

$${}_f\phi : {}_f(C^{*(p)}) \to {}_fC^*$$

is the same as the map

$$\phi : C^{*(p)} \to C^*,\ \lambda\otimes a \mapsto \lambda a^p.$$

Since the composite

$$C^* \xrightarrow{\eta\otimes 1} {}_f(C^{*(p)}) \xrightarrow{{}_f\phi} {}_fC^*$$

is equal to the map: $a \mapsto a^p$, we have

$$W_k(C, \xi) \circ W_k(C, \Upsilon_{\mathbf{T}(\mathfrak{X})}) = W_k(C, \mathbf{T}(\mathfrak{F}_{\mathfrak{X}})).$$

This means

$$\xi \circ \Upsilon_{\mathbf{T}(\mathfrak{X})} = \mathbf{T}(\mathfrak{F}_{\mathfrak{X}}).$$

REMARK. Notice that the field map

$$f : k \to k,\ \lambda \mapsto \lambda^p$$

can be regarded as an algebraic extension of fields. Hence if $\mathfrak{X}$ is a locally algebraic k-scheme, the map

$$\xi : \mathbf{T}(\mathfrak{X})^{(p)} \to \mathbf{T}(\mathfrak{X}^{(p)})$$

is an isomorphism. Thus the above Proposition permits us to identify the coalgebra map $\mathbf{T}(\mathfrak{F}_{\mathfrak{X}})$ with the Υ-map $\Upsilon_{\mathbf{T}(\mathfrak{X})}$.

2.2.10 Let $K|k$ be a field extension and $\mathfrak{X}$ a locally algebraic k-scheme. We put

$$\mathfrak{X}(K)_{\mathrm{cl}} = \{\alpha \in \mathfrak{X}(K) \mid [\alpha] \text{ is a closed point of } \mathfrak{X}\}.$$

We establish a bijection between $\mathfrak{X}(K)_{\mathrm{cl}}$ and $\mathrm{G}(K \otimes \mathbf{T}(\mathfrak{X}))$ the set of group-like elements of the K-coalgebra $K \otimes \mathbf{T}(\mathfrak{X})$.

By (2.1.8) we have

$$\mathrm{G}(\mathbf{T}^K(K \otimes \mathfrak{X})) \simeq (K \otimes \mathfrak{X})(K) = \mathfrak{X}(K).$$

Let $\Phi : \mathrm{G}(K \otimes \mathbf{T}(\mathfrak{X})) \hookrightarrow \mathfrak{X}(K)$ be the composite:

$$\mathrm{G}(K \otimes \mathbf{T}(\mathfrak{X})) \xhookrightarrow{\mathrm{G}(\xi)} \mathrm{G}(\mathbf{T}^K(K \otimes \mathfrak{X})) \simeq \mathfrak{X}(K),$$

where ξ is defined in (2.2.8). Then the image of Φ is contained in $\mathfrak{X}(K)_{\mathrm{cl}}$. Indeed let $g \in \mathrm{G}(K \otimes \mathbf{T}(\mathfrak{X}))$. There exists a finite dimensional subcoalgebra C of $\mathbf{T}(\mathfrak{X})$ such that $g \in \mathrm{G}(K \otimes C)$. We view g as a K-coalgebra map

$$g : K \to K \otimes C.$$

By definition we have $\Phi(g) = (\bar{i})_K \in \mathfrak{X}(K)$, where $\bar{i}$ is the element of $\mathfrak{X}(C^*)$ which is determined by the inclusion

$$i : C \hookrightarrow \mathbf{T}(\mathfrak{X})$$

and $(\bar{i})_K$ is taken with respect to the composite:

$$\tau : C^* \xrightarrow{\eta \otimes 1} K \otimes C^* \xrightarrow{\mathbf{Mod}_K(g,K)} K.$$

Since $\tau(C^*) = F$ is a finite dimensional subfield of K, we have

$$[(\bar{i})_K] = [(\bar{i})_F] \in \|\mathfrak{X}\|.$$

Next we define a map

$$\Psi : \mathfrak{X}(K)_{\mathrm{cl}} \to \mathrm{G}(K \otimes \mathbf{T}(\mathfrak{X}))$$

as follows: Let $\alpha \in \mathfrak{X}(K)_{\mathrm{cl}}$ and put $x = [\alpha] \in |\mathfrak{X}|$. We regard as usual α as a map of k-schemes

$$\alpha : \mathfrak{Sp}\, K \to \mathfrak{X}.$$

This induces an $\boldsymbol{Fld}_k$-map

$$\sigma : \boldsymbol{\kappa}(x) \to K.$$

Extend this to a K-algebra map

$$1 \otimes \sigma : K \otimes \boldsymbol{\kappa}(x) \to K.$$

Let $g : K \to K \otimes \mathbf{T}_x(\mathfrak{X})$ be the composite:

$$K \xrightarrow{\boldsymbol{Mod}_K(1 \otimes \sigma, K)} K \otimes \boldsymbol{\kappa}(x)^* \xrightarrow{K \otimes \boldsymbol{\iota}(x)} K \otimes \mathbf{T}_x(\mathfrak{X}),$$

where $\boldsymbol{\iota}(x)$ is canonic (2.2.6). Clearly g is a K-coalgebra map. Hence $g(1)$ is a group-like element of $K \otimes \mathbf{T}_x(\mathfrak{X})$, and of $K \otimes \mathbf{T}(\mathfrak{X})$ too. We put

$$\Psi(\alpha) = g(1) \in \mathrm{G}(K \otimes \mathbf{T}(\mathfrak{X})).$$

We claim that $\Phi\Psi = 1$. Notice that the map

$$\alpha : \mathfrak{Sp}\, K \to X$$

factors as

$$\mathfrak{Sp}\, K \xrightarrow{\mathfrak{Sp}\,\sigma} \mathfrak{Sp}\, \boldsymbol{\kappa}(x) \xrightarrow{\boldsymbol{\varepsilon}(x)} \mathfrak{X},$$

where $\boldsymbol{\varepsilon}(x)$ is canonic. Let

$$\bar{\alpha} : \mathfrak{Sp}_K\, K \to K \otimes \mathfrak{X}$$

be the map which is naturally identified with the element

$$\alpha \in (K \otimes \mathfrak{X})(K) = \mathfrak{X}(K).$$

The map $\bar{\alpha}$ factors as

$$\bar{\alpha} : \mathfrak{Sp}_K\, K \xrightarrow{\mathfrak{Sp}_K(1 \otimes \sigma)} \mathfrak{Sp}_K\, (K \otimes \boldsymbol{\kappa}(x)) \xrightarrow{K \otimes \boldsymbol{\varepsilon}(x)} K \otimes \mathfrak{X}.$$

Apply the functor $\mathbf{T}^K(-)$ to the above sequence. Then we have

$$\mathbf{T}^K(\bar{\alpha}) : K \xrightarrow{\boldsymbol{Mod}_K(1 \otimes \sigma, K)} K \otimes \boldsymbol{\kappa}(x)^* \xrightarrow{K \otimes \boldsymbol{\iota}(x)} K \otimes \mathbf{T}(\mathfrak{X}) \xrightarrow{\xi} \mathbf{T}^K(K \otimes \mathfrak{X}),$$

that is

$$\mathbf{T}^K(\bar{\alpha}) = \xi \circ g.$$

This means that under the canonical bijection

$$\mathrm{G}(\mathbf{T}^K(K \otimes \mathfrak{X})) \simeq \mathfrak{X}(K),$$

the elements $\xi(g(1)) \in \mathrm{G}(\mathbf{T}^K(K\otimes\mathfrak{X}))$ and $\alpha \in \mathfrak{X}(K)$ correspond to one another. Hence we have $\Phi\Psi=1$. Since Φ is injective, we have also $\Psi\Phi=1$.

Let $\alpha \in \mathfrak{X}(K)_{\mathrm{cl}}$ and put $x=[\alpha]$. We view α as a rational point of $K\otimes\mathfrak{X}$. Then it lies clearly over x. It follows from above that $\Psi(\alpha)$ is a group-like element of $K\otimes\mathbf{T}_x(\mathfrak{X})$ and $\xi(\Psi(\alpha))$ is the unique group-like element of $\mathbf{T}^K_\alpha(K\otimes\mathfrak{X})$ the tangent K-coalgebra to $K\otimes\mathfrak{X}$ at the rational point α of $K\otimes\mathfrak{X}$. Hence Proposition 2.2.8 (i) means that the canonical injection

$$\xi : K\otimes\mathbf{T}(\mathfrak{X}) \hookrightarrow \mathbf{T}^K(K\otimes\mathfrak{X})$$

induces an isomorphism of K-coalgebras

$$(K\otimes\mathbf{T}_x(\mathfrak{X}))^{\Psi(\alpha)} \xrightarrow{\simeq} \mathbf{T}^K_\alpha(K\otimes\mathfrak{X}),$$

where the left hand side denotes the irreducible component of the K-coalgebra $K\otimes\mathbf{T}_x(\mathfrak{X})$ containing the group-like element $\Psi(\alpha)$.

For a coalgebra C and a group-like element g of C, we put (§ 1.0)

$$\mathrm{P}_g(C)=\{x \in C \mid \Delta(x)=g\otimes x+x\otimes g\}.$$

This is equal to $\mathrm{P}(C^g)$ the set of primitive elements of C^g the irreducible component of C containing g. Then we have a canonical isomorphism of K-coalgebras

$$\mathrm{P}_{\Psi(\alpha)}(K\otimes\mathbf{T}_x(\mathfrak{X})) \xrightarrow{\simeq} \mathrm{P}(\mathbf{T}^K_\alpha(K\otimes\mathfrak{X})).$$

Notice that the right hand side can be canonically identified with the Zariski's tangent space to $K\otimes\mathfrak{X}$ at the rational point α (2.1.12).

In general let x be a closed point of $\mathfrak{X}$. We view the canonical map

$$\boldsymbol{\varepsilon}(x) : \mathfrak{Sp}\ \boldsymbol{\kappa}(x) \rightarrow \mathfrak{X}$$

as an element $\boldsymbol{\varepsilon}(x) \in \mathfrak{X}(\boldsymbol{\kappa}(x))$. Then $[\boldsymbol{\varepsilon}(x)]=x$. We put

$$g_x=\Psi(\boldsymbol{\varepsilon}(x)) \in \mathrm{G}(\boldsymbol{\kappa}(x)\otimes\mathbf{T}_x(\mathfrak{X})).$$

This is the element which corresponds to the canonical map

$$\boldsymbol{\iota}(x) : \boldsymbol{\kappa}(x)^* \rightarrow \mathbf{T}_x(\mathfrak{X})$$

under the canonical bijection

$$\boldsymbol{\kappa}(x)\otimes\mathbf{T}_x(\mathfrak{X}) \simeq \boldsymbol{Mod}_k(\boldsymbol{\kappa}(x)^*, \mathbf{T}_x(\mathfrak{X})).$$

It follows from above that

$$(\boldsymbol{\kappa}(x)\otimes\mathbf{T}_x(\mathfrak{X}))^{g_x} \simeq \mathbf{T}^{\boldsymbol{\kappa}(x)}_{\boldsymbol{\varepsilon}(x)}(\boldsymbol{\kappa}(x)\otimes\mathfrak{X}) \quad \text{and}$$

$$\mathrm{P}_{g_x}(\kappa(x)\otimes\mathbf{T}_x(\mathfrak{X}))\simeq\mathrm{P}(\mathbf{T}^{\kappa(x)}_{\varepsilon(x)}(\kappa(x)\otimes\mathfrak{X})),$$

where $\varepsilon(x)$ is regarded as a rational point of $\kappa(x)\otimes\mathfrak{X}$. Since the Zariski's tangent space to $\mathfrak{X}$ at x is canonically isomorphic to that to $\kappa(x)\otimes\mathfrak{X}$ at $\varepsilon(x)$ by [7, I, § 4, 4.1 and 4.15] and the latter is identified with $\mathrm{P}(\mathbf{T}^{\kappa(x)}_{\varepsilon(x)}(\kappa(x)\otimes\mathfrak{X}))$, we have:

PROPOSITION. *Let $\mathfrak{X}$ be a locally algebraic k-scheme and x a closed point of $\mathfrak{X}$. Let $g_x\in\kappa(x)\otimes\mathbf{T}_x(\mathfrak{X})$ be the element which corresponds naturally to the canonical map $\iota(x):\kappa(x)^*\to\mathbf{T}_x(\mathfrak{X})$. Then g_x is a group-like element of the $\kappa(x)$-coalgebra $\kappa(x)\otimes\mathbf{T}_x(\mathfrak{X})$ and the $\kappa(x)$-vector space*

$$\mathrm{P}_{g_x}(\kappa(x)\otimes\mathbf{T}_x(\mathfrak{X}))$$

can be canonically identified with the Zariski's tangent space to $\mathfrak{X}$ at x.

2.3 Flatness and smoothness

2.3.1 A map of k-schemes $\mathfrak{f}:\mathfrak{X}\to\mathfrak{Y}$ is said to be *flat* at a point x of $\mathfrak{X}$ if the induced map $\mathfrak{f}_x:\mathcal{O}_{\mathfrak{f}(x)}\to\mathcal{O}_x$ makes $\mathcal{O}_x$ into a flat $\mathcal{O}_{\mathfrak{f}(x)}$-module [7, I, § 2, 2.4]. In the following we shall view $\mathbf{T}_x(\mathfrak{X})$ as a $\mathbf{T}_{\mathfrak{f}(x)}(\mathfrak{Y})$-*comodule* via the induced coalgebra map $\mathbf{T}_x(\mathfrak{f}):\mathbf{T}_x(\mathfrak{X})\to\mathbf{T}_{\mathfrak{f}(x)}(\mathfrak{Y})$.

PROPOSITION. *Let $\mathfrak{f}:\mathfrak{X}\to\mathfrak{Y}$ be a map of locally algebraic k-schemes and x a closed point of $\mathfrak{X}$. Then $\mathfrak{f}$ is flat at x iff $\mathbf{T}_x(\mathfrak{X})$ is an injective object in **Comod**$_{\mathbf{T}_{\mathfrak{f}(x)}}(\mathfrak{Y})$. If $\mathfrak{f}$ is flat at x, then the induced map $\mathbf{T}_x(\mathfrak{f}):\mathbf{T}_x(\mathfrak{X})\to\mathbf{T}_{\mathfrak{f}(x)}(\mathfrak{Y})$ is surjective.*

PROOF. $\mathcal{O}_x$ and $\mathcal{O}_{\mathfrak{f}(x)}$ are local Noetherian and their maximal ideals are cofinite. Since $\mathcal{O}_x$ is flat over $\mathcal{O}_{\mathfrak{f}(x)}$ iff $\mathcal{O}_x$ is faithfully flat over $\mathcal{O}_{\mathfrak{f}(x)}$ [2, I, § 3, n°5, Proposition 9 e)], the assertion follows directly from Corollary 1.7.9.

2.3.2 A map of k-schemes $\mathfrak{f}:\mathfrak{X}\to\mathfrak{Y}$ is said to be *faithfully flat* if it is flat at every point of $\mathfrak{X}$ and the induced map

$$|\mathfrak{f}|:|\mathfrak{X}|\to|\mathfrak{Y}|$$

is surjective [7, I, § 2, 2.4].

COROLLARY. *Let $\mathfrak{f}:\mathfrak{X}\to\mathfrak{Y}$ be a map of locally algebraic k-schemes. If $\mathfrak{f}$ is faithfully flat, then the induced coalgebra map*

$$\mathbf{T}(\mathfrak{f}):\mathbf{T}(\mathfrak{X})\to\mathbf{T}(\mathfrak{Y})$$

is surjective.

PROOF. Let $x \in \|\mathfrak{X}\|$. Then the map

$$\mathbf{T}_x(\mathfrak{f}) : \mathbf{T}_x(\mathfrak{X}) \to \mathbf{T}_{\mathfrak{f}(x)}(\mathfrak{Y})$$

is surjective by (2.3.1). Since $\mathbf{T}(\mathfrak{X}) = \bigoplus_{x \in \|\mathfrak{X}\|} \mathbf{T}_x(\mathfrak{X})$, it is enough to show that the map

$$\|\mathfrak{f}\| : \|\mathfrak{X}\| \to \|\mathfrak{Y}\|$$

is surjective.

Indeed let $y \in \|\mathfrak{Y}\|$. Then $\mathfrak{f}^{-1}(y) = \mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Sp}\, \boldsymbol{\kappa}(y))$, which is also locally algebraic, has the non-empty underlying set, since

$$|\mathfrak{X}| \times_{|\mathfrak{Y}|} |\mathfrak{Sp}\, \boldsymbol{\kappa}(y)| \neq \emptyset$$

[7, I, § 1, 5.4]. Hence we have $\|\mathfrak{f}^{-1}(y)\| \neq \emptyset$ by [7, I, § 3, 6.8 and 6.9]. This proves our assertion.

2.3.3 A map of k-schemes $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is said to be *quasi-compact* if for any open quasi-compact subset V of $|\mathfrak{Y}|$, $\mathfrak{f}^{-1}(V)$ is quasi-compact [7, I, § 2, 2.1]. A locally algebraic k-scheme is quasi-compact iff it has a finite open affine covering. Such a locally algebraic k-scheme is called an *algebraic* k-scheme [7, I, § 3, 2.1].

PROPOSITION. *Let $\mathfrak{f}$: $\mathfrak{X} \to \mathfrak{Y}$ be a map of locally algebraic k-schemes. Suppose that $\mathfrak{f}$ is quasi-compact. If $\mathbf{T}(\mathfrak{f}) : \mathbf{T}(\mathfrak{X}) \xrightarrow{\simeq} \mathbf{T}(\mathfrak{Y})$ then $\mathfrak{f} : \mathfrak{X} \xrightarrow{\simeq} \mathfrak{Y}$.*[11)]

PROOF. The assumption that $\mathbf{T}(\mathfrak{f})$ is an isomorphism means that $\|\mathfrak{f}\| : \|\mathfrak{X}\| \xrightarrow{\simeq} \|\mathfrak{Y}\|$ and that $\mathbf{T}_x(\mathfrak{f}) : \mathbf{T}_x(\mathfrak{X}) \xrightarrow{\simeq} \mathbf{T}_{\mathfrak{f}(x)}(\mathfrak{Y})$ for any closed point x of $\mathfrak{X}$. In particular $\mathfrak{f}$ is flat at every closed point of $\mathfrak{X}$. Since

$$U = \{x \in \mathfrak{X} \mid \mathfrak{f} \text{ is flat at } x\}$$

is open [7, I, § 3, 3.13] and contains all closed points of $\mathfrak{X}$, we have $U = |\mathfrak{X}|$ by [7, I, § 3, 6.8 and 6.9]. Therefore $\mathfrak{f}$ is flat at every point of $\mathfrak{X}$.

Next we show that $\mathfrak{f}$ is surjective. Let $\mathfrak{V}$ be an open algebraic subscheme of $\mathfrak{Y}$. Then we have

$$\mathbf{T}(\mathfrak{f}) : \mathbf{T}(\mathfrak{f}^{-1}(\mathfrak{V})) \xrightarrow{\simeq} \mathbf{T}(\mathfrak{V}),$$

since $\mathbf{T}$ commutes with finite limits (2.2.1). Since $\mathfrak{Y}$ is covered by a family of open algebraic subschemes, we can assume without loss of generality that $\mathfrak{Y}$, and $\mathfrak{X}$ also, is algebraic. Then the image $\mathfrak{f}(|\mathfrak{X}|)$ is a constructible subset of $|\mathfrak{Y}|$ by [7, I, § 3, 3.9] and contains all closed points of $\mathfrak{Y}$. Hence $\mathfrak{f}(|\mathfrak{X}|) = |\mathfrak{Y}|$ by [7, I, § 3, 6.8 and 6.9] again.

We have proved that $\mathfrak{f}$ is faithfully flat. By Theorem of *fpqc* descent [7, I, § 2, 2.7], the following is a coequalizer diagram in $\boldsymbol{Sch}_k$ the category of k-schemes:

$$\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{X}\underset{\mathfrak{pr}_2}{\overset{\mathfrak{pr}_1}{\rightrightarrows}}\mathfrak{X}\xrightarrow{\mathfrak{f}}\mathfrak{Y}$$

But it follows from Proposition 2.2.7 (iii) that $\mathfrak{f}$ is a monomorphism. Hence $\mathfrak{pr}_1=\mathfrak{pr}_2$ and $\mathfrak{f}$ is an isomorphism.

REMARK. Author does not know whether Proposition holds without the assumption "$\mathfrak{f}$ is quasi-compact".

2.3.4 Let $\mathfrak{f}: \mathfrak{X}\to\mathfrak{Y}$ be a map of schemes. We denote by $\Omega_{\mathfrak{X}/\mathfrak{Y}}$ *the module of differentials* of $\mathfrak{X}$ over $\mathfrak{Y}$ [7, I, § 4, 2.1], which is a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module, and put for any point x of $\mathfrak{X}$

$$\Omega_{\mathfrak{X}/\mathfrak{Y}}(x)=\boldsymbol{\kappa}(x)\otimes_{\mathcal{O}_x}(\Omega_{\mathfrak{X}/\mathfrak{Y}})_x.$$

If $\mathfrak{f}$ is *locally finitely presented* [7, I, § 3, 1.6] then $\Omega_{\mathfrak{X}/\mathfrak{Y}}(x)$ is finite dimensional over $\boldsymbol{\kappa}(x)$ and

$$\dim_x \mathfrak{f}^{-1}(\mathfrak{f}(x))\leqq[\Omega_{\mathfrak{X}/\mathfrak{Y}}(x):\boldsymbol{\kappa}(x)]$$

by [7, I, § 4, 2.10].

For simplicity we assume that $\mathfrak{f}$ is locally finitely presented. Let $x\in\mathfrak{X}$ and $y=\mathfrak{f}(x)$. $\mathfrak{f}$ is said to be *non-ramified* (resp. étale, resp. *smooth*) at x if $\Omega_{\mathfrak{X}/\mathfrak{Y}}(x)=0$ (resp. if $\mathfrak{f}$ is non-ramified and flat at x, resp. if $\mathfrak{f}$ is flat at x and $\dim_x \mathfrak{f}^{-1}(\mathfrak{f}(x))=[\Omega_{\mathfrak{X}/\mathfrak{Y}}(x):\boldsymbol{\kappa}(x)]$ [7, I, § 4, 3.2 and 4.1].

The purpose of the rest of this section is to translate the concepts of being non-ramified, étale or smooth into the coalgebra language (cf. Introduction). In other words we want to find out classes $\mathscr{P}_n$, $\mathscr{P}_e$ and $\mathscr{P}_l$ consisting of maps of cocommutative irreducible k-coalgebras such that for any map $\mathfrak{f}: \mathfrak{X}\to\mathfrak{Y}$ of locally algebraic k-schemes and any closed point x of $\mathfrak{X}$, $\mathfrak{f}$ is non-ramified (resp. étale, resp. *smooth*) at x iff the induced coalgebra map $\mathbf{T}_x(\mathfrak{f}): \mathbf{T}_x(\mathfrak{X})\to\mathbf{T}_{\mathfrak{f}(x)}(\mathfrak{Y})$ belongs to $\mathscr{P}_n$ (resp. $\mathscr{P}_e$, resp. $\mathscr{P}_l$).

We reduce the problem to the *rational* case. Let $K|k$ be an extension of fields and $\mathfrak{f}: \mathfrak{X}\to\mathfrak{Y}$ a map of locally algebraic k-schemes. Let $\tilde{x}$ be a point of $K\otimes\mathfrak{X}$ lying over a point x of $\mathfrak{X}$. Then $\mathfrak{f}$ is non-ramified (resp. étale, resp. smooth) at x iff $K\otimes\mathfrak{f}$ is non-ramified (resp. étale, resp. smooth) at $\tilde{x}$ [7, I, § 4, 4.1]. Suppose that for any finite field extension $K|k$, there exist classes $\mathscr{P}'_n(K)$, $\mathscr{P}'_e(K)$ and $\mathscr{P}'_l(K)$ consisting of maps of cocommutative *connected* K-coalgebras such that for any map $\mathfrak{f}: \mathfrak{X}\to\mathfrak{Y}$ of locally algebraic K-schemes and any rational point $x\in\mathfrak{X}(K)$, $\mathfrak{f}$ is non-ramified (resp. étale, resp. smooth)

at x iff the induced K-coalgebra map $\mathbf{T}_x^K(\mathfrak{f}): \mathbf{T}_x^K(\mathfrak{X}) \to \mathbf{T}_{\mathfrak{f}(x)}^K(\mathfrak{Y})$ belongs to $\mathscr{P}'_n(K)$ (resp. $\mathscr{P}'_e(K)$, resp. $\mathscr{P}'_l(K)$).

Now let $\phi: C \to D$ be a map of cocommutative irreducible k-coalgebras. Then $K = C_0^*$ is a finite field extension of k, where C_0 is the coradical of C. Let $g \in K \otimes C$ be the element which corresponds naturally to the inclusion: $C_0 \hookrightarrow C$. Then g is a group-like element of the K-coalgebra $K \otimes C$ (cf. (2.2.10)). The K-coalgebra map $K \otimes \phi: K \otimes C \to K \otimes D$ induces a map of connected cocommutative K-coalgebras

$$K \otimes \phi: (K \otimes C)^g \to (K \otimes D)^{(K \otimes \phi)(g)},$$

where $(K \otimes C)^g$ is the irreducible component containing g and so is $(K \otimes D)^{(K \otimes \phi)(g)}$. Let $\mathscr{P}_n$ (resp. $\mathscr{P}_e$, resp. $\mathscr{P}_l$) be the class of $\phi: C \to D$ such that the induced map

$$K \otimes \phi: (K \otimes C)^g \to (K \otimes D)^{(K \otimes \phi)(g)}$$

belongs to $\mathscr{P}'_n(K)$ (resp. $\mathscr{P}'_e(K)$, resp. $\mathscr{P}'_l(K)$).

If he notices that a closed point x of a locally algebraic k-scheme $\mathfrak{X}$ determines a rational point $\varepsilon(x)$ of $\kappa(x) \otimes \mathfrak{X}$ and that the tangent coalgebra $\mathbf{T}_{\varepsilon(x)}^{\kappa(x)}(\kappa(x) \otimes \mathfrak{X})$ is equal to $(\kappa(x) \otimes \mathbf{T}_x(\mathfrak{X}))^{g_x}$, the reader may easily verify that the classes $\mathscr{P}_n, \mathscr{P}_e$ and $\mathscr{P}_l$ satisfy the conditions described above. Thus we have only to find out the classes $\mathscr{P}'_n(K)$, $\mathscr{P}'_e(K)$ and $\mathscr{P}'_l(K)$ as above.[12)]

2.3.5 Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of locally algebraic k-schemes. Let $x \in \mathfrak{X}(k)$ and $y = \mathfrak{f}(x)$. The following is a pullback diagram in W_k^{cn} (2.2.6):

$$\begin{tikzcd}
\mathbf{T}_x(\mathfrak{X}) \arrow[r, "\mathbf{T}_x(\mathfrak{f})"] & \mathbf{T}_y(\mathfrak{Y}) \\
\mathbf{T}_x(\mathfrak{f}^{-1}(y)) \arrow[u, hook] \arrow[r] & \kappa(y)^* \arrow[u, hook, "\varepsilon(y)"']
\end{tikzcd}.$$

Since the functor $\mathrm{P}: W_k^{\mathrm{cn}} \to \boldsymbol{Mod}_k$ commutes with finite limits by (1.2.8), we have an exact sequence of vector spaces:

$$0 \longrightarrow \mathrm{P}(\mathbf{T}_x(\mathfrak{f}^{-1}(y))) \longrightarrow \mathrm{P}(\mathbf{T}_x(\mathfrak{X})) \longrightarrow \mathrm{P}(\mathbf{T}_y(\mathfrak{Y})).$$

On the other hand, we have an exact sequence of vector spaces:

$$0 \longleftarrow \Omega_{\mathfrak{X}/\mathfrak{Y}}(x) \longleftarrow \Omega_{\mathfrak{X}/k}(x) \xleftarrow{\Omega_{\mathfrak{f}/}} \Omega_{\mathfrak{Y}/k}(y)$$

[7, I, § 4, 2.9]. Notice that $\Omega_{\mathfrak{X}/k}(x) = m_x/m_x^2$ and $\Omega_{\mathfrak{Y}/k}(y) = m_y/m_y^2$ and that $\Omega_{\mathfrak{f}/}$ is induced by the canonical map $\mathfrak{f}_x: m_x \leftarrow m_y$ [7, I, § 4, 2.2]. But we have naturally

$$\mathrm{P}(\mathbf{T}_x(\mathfrak{X})) \simeq (m_x/m_x^2)^*,$$

since $\{(\mathcal{O}_x/m_x^{n+1})^*\}$ is the coradical filtration of $\mathbf{T}_x(\mathfrak{X})$ (Proposition 1.1.1). Thus we get a commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow \mathrm{P}(\mathbf{T}_x(\mathfrak{f}^{-1}(y)) & \longrightarrow & \mathrm{P}(\mathbf{T}_x(\mathfrak{X})) & \xrightarrow{\mathbf{T}_x(\mathfrak{f})} & \mathrm{P}(\mathbf{T}_y(\mathfrak{Y})) \\ & & \wr\| & & \wr\| \\ 0 \longrightarrow \Omega_{\mathfrak{X}/\mathfrak{Y}}(x)^* & \longrightarrow & \Omega_{\mathfrak{X}/k}(x)^* & \xrightarrow{{}^t(\Omega_{\mathfrak{f}/})} & \Omega_{\mathfrak{Y}/k}(y)^* \end{array}.$$

This implies first that

$$\mathrm{P}(\mathbf{T}_x(\mathfrak{f}^{-1}(y))) \simeq \Omega_{\mathfrak{X}/\mathfrak{Y}}(x)^*$$

and secondly that

$$\Omega_{\mathfrak{X}/\mathfrak{Y}}(x) = 0 \Leftrightarrow \mathbf{T}_x(\mathfrak{f}) \text{ is injective}$$

in view of [11, Lemma 11.0.1].

Recall that $\mathbf{T}_x(\mathfrak{f})$ is smooth (1.8.1) iff $\mathbf{T}_x(\mathfrak{X})$ is an injective object in $\boldsymbol{Comod}_{\mathbf{T}_y(\mathfrak{Y})}$ and $\mathbf{T}_x(\mathfrak{f}^{-1}(y)) \simeq \mathrm{B}(U)$ for some vector space U, where one should notice that $\mathbf{T}_x(\mathfrak{f}^{-1}(y))$ is the largest subcoalgebra of $\mathbf{T}_x(\mathfrak{X})$ contained in $\mathbf{T}_x(\mathfrak{f})^{-1}(\boldsymbol{\kappa}(y)^*)$ and that $\boldsymbol{\kappa}(y)^*$ is the coradical of $\mathbf{T}_y(\mathfrak{Y})$. Since $\mathbf{T}_x(\mathfrak{f}^{-1}(y))$ is of finite type, the latter condition is equivalent to

$$K \dim \mathbf{T}_x(\mathfrak{f}^{-1}(y)) = \dim_k \mathrm{P}(\mathbf{T}_x(\mathfrak{f}^{-1}(y)))$$

by Proposition 1.6.1 (ii). Since $\dim_x \mathfrak{f}^{-1}(y) = K \dim \mathbf{T}_x(\mathfrak{f}^{-1}(y))$ by Proposition 2.1.11 and $\mathrm{P}(\mathbf{T}_x(\mathfrak{f}^{-1}(y)))^* \simeq \Omega_{\mathfrak{X}/\mathfrak{Y}}(x)$, we have

$$\begin{aligned} \mathbf{T}_x(\mathfrak{f}) \text{ is smooth} &\Leftrightarrow \mathfrak{f} \text{ is flat at } x \text{ and} \\ &\qquad \dim_x \mathfrak{f}^{-1}(y) = [\Omega_{\mathfrak{X}/\mathfrak{Y}}(x) : k]. \\ &\Leftrightarrow \mathfrak{f} \text{ is smooth at } x \end{aligned}$$

Thus we have proven:

PROPOSITION. *Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of locally algebraic k-schemes. Let $x \in \mathfrak{X}(k)$ and $y = \mathfrak{f}(x)$. Then $\mathfrak{f}$ is non-ramified (resp. étale, resp. smooth) at x iff the induced coalgebra map $\mathbf{T}_x(\mathfrak{f}): \mathbf{T}_x(\mathfrak{X}) \to \mathbf{T}_y(\mathfrak{Y})$ is injective (resp. bijective, resp. smooth in the sense of* (1.8.1)).

2.3.6 As an application we give here another proof of (i)$\Leftrightarrow$(ii) in Theorem of smoothness [7, I, § 4, 4.2], that is:

PROPOSITION. *Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of locally algebraic k-schemes. Let x be a (not necessarily closed) point of $\mathfrak{X}$ and $y = \mathfrak{f}(x)$. The following are equivalent:*

(i) *$\mathfrak{f}$ is smooth at x.*

(ii) *There exist an open neighboorhood $\mathfrak{U}$ of x, an integer n and a*

map $\mathfrak{g}: \mathfrak{U} \to \mathfrak{Y} \times \mathfrak{Sp}\, k[T_1, \cdots, T_n]$ *such that* $\mathfrak{g}$ *is étale at* x *and that* $\mathfrak{f}|\mathfrak{U} = \mathrm{pr}_1 \circ \mathfrak{g}$, *where* $k[T_1, \cdots, T_n]$ *denotes the polynomial algebra in* n *variables.*

PROOF. (ii)$\Rightarrow$(i). Applying the functor $\boldsymbol{\kappa}(x) \otimes (-)$ and taking the rational point of $\boldsymbol{\kappa}(x) \otimes \mathfrak{X}$ lying over x, we can assume that $x \in \mathfrak{X}(k)$ Then $\mathfrak{g}(x)$ is of the form (y, z), where $z: k[T_1, \cdots, T_n] \to k$ is an algebra map and can be identified with an n-tuple $(\lambda_1, \cdots, \lambda_n)$ of elements of k. Replacing T_i by $T_i - \lambda_i$, we can assume that $\lambda_i = \cdots = \lambda_n = 0$. Since $\mathbf{T}_x(\mathfrak{f})$ factors as

$$\mathbf{T}_x(\mathfrak{f}): \mathbf{T}_x(\mathfrak{X}) = \mathbf{T}_x(\mathfrak{U}) \underset{\simeq}{\xrightarrow{\mathbf{T}_x(\mathfrak{g})}} \mathbf{T}_y(\mathfrak{Y}) \otimes \mathbf{T}_z(\mathfrak{Sp}\, k[T_1, \cdots, T_n])$$
$$\xrightarrow{1 \otimes \varepsilon} \mathbf{T}_y(\mathfrak{Y})$$

(cf. (2.2.4)), we have only to show that

$$\mathbf{T}_z(\mathfrak{Sp}\, k[T_1, \cdots, T_n]) \simeq \mathrm{B}(k^n)$$

in view of Theorem 1.8.1 (v). But we have for any $C \in W_k^{\mathfrak{f}}$,

$$\begin{aligned} &W_k(C, \mathbf{T}_z(\mathfrak{Sp}\, k[T_1, \cdots, T_n]) \\ &\simeq \mathrm{Ker}_z\, (\boldsymbol{M}_k(k[T_1, \cdots, T_n], C^*) \to \boldsymbol{M}_k(k[T_1, \cdots, T_n], C_0^*)) \\ &\simeq \mathrm{Ker}\, ((C^*)^n \to (C_0^*)^n) \\ &\simeq \boldsymbol{Mod}_k(C/C_0, k^n) \simeq W_k(C, \mathrm{B}(k^n)), \end{aligned}$$

where one should notice that the original point of $\boldsymbol{M}_k(k[T_1, \cdots, T_n], C_0^*)$ is $\eta \circ z: k[T_1, \cdots, T_n] \to k \to C_0^*$ and with respect to which Ker$(-)$ is taken. This means that $\mathbf{T}_z(\mathfrak{Sp}\, k[T_1, \cdots, T_n]) \simeq \mathrm{B}(k^n)$.

(i)$\Rightarrow$(ii). Replacing $\mathfrak{X}$ by an open affine subscheme of $\mathfrak{X}$, we can assume that $\mathfrak{X} = \mathfrak{Sp}\, A$ for some $A \in \boldsymbol{M}_k$. Take an extension of fields $K|k$ such that there exists a rational point x' of $K \otimes \mathfrak{X}$ lying over x, for example $K = \boldsymbol{\kappa}(x)$. Put

$$U = \mathrm{Ker}\, (\mathrm{P}(\mathbf{T}_{x'}^K(K \otimes \mathfrak{X})) \xrightarrow{\mathbf{T}_{x'}^K(K \otimes \mathfrak{f})} \mathrm{P}(\mathbf{T}_{y'}^K(K \otimes \mathfrak{Y}))),$$

where $y' = (K \otimes \mathfrak{f})(x')$. U is a finite dimensional K-vector subspace of

$$\mathbf{T}_{x'}^K(K \otimes \mathfrak{X}) \subset \mathbf{T}^K(K \otimes \mathfrak{X}) \subset \boldsymbol{Mod}_K(K \otimes A, K).$$

Hence we have a natural surjection of K-vector spaces:

$$K \otimes A \to \boldsymbol{Mod}_K(U, K).$$

Take elements $a_1, \cdots, a_n \in A$ such that $\{a_1|U, \cdots, a_n|U\}$ form a K-basis of $\boldsymbol{Mod}_K(U, K)$. The k-algebra map

$$\sigma: k[T_1, \cdots, T_n] \to A,\ T_i | \to a_i$$

determines a map of k-schemes

$$\mathfrak{h}=\mathfrak{Sp}\,\sigma : \mathfrak{X}=\mathfrak{Sp}\,A\to\mathfrak{Sp}\,k[T_1,\cdots,T_n].$$

Put $\mathfrak{g}=(\mathfrak{f},\mathfrak{h}):\mathfrak{X}\to\mathfrak{Y}\times\mathfrak{Sp}\,k[T_1,\cdots,T_n]$. We claim that $\mathfrak{g}$ is étale at x. It is enough to show that $K\otimes\mathfrak{g}$ is étale at x'. This argument permits us to assume that $x\in\mathfrak{X}(k)$ from the beginning.

Suppose that $x\in\mathfrak{X}(k)$ and hence take $K=k$. Then we have

$$0\to U\to \mathrm{P}(\mathbf{T}_x(\mathfrak{X}))\xrightarrow{\mathbf{T}_x(\mathfrak{f})}\mathrm{P}(\mathbf{T}_y(\mathfrak{Y}))$$

and $\{a_1|U,\cdots,a_n|U\}$ form a basis of U^*. In particular the composite

$$k[T_1,\cdots,T_n]\xrightarrow{\sigma}A\xrightarrow{\text{cano.}}U^*$$

is surjective. This implies the injectivity of the composite

$$U\subset\mathrm{P}(\mathbf{T}_x(\mathfrak{X}))\xrightarrow{\mathbf{T}_x(\mathfrak{h})}\mathrm{P}(\mathbf{T}_z(\mathfrak{Sp}\,k[T_1,\cdots,T_n])),$$

where $z=\mathfrak{h}(x)$. We have shown during the proof of (ii)$\Rightarrow$(i) that

$$\mathbf{T}_z(\mathfrak{Sp}\,k[T_1,\cdots,T_n])\simeq\mathrm{B}(k^n).$$

Then the composite

$$U\xrightarrow{\mathbf{T}_x(\mathfrak{h})}\mathbf{T}_z(\mathfrak{Sp}\,k[T_1,\cdots,T_n])\simeq\mathrm{B}(k^n)\xrightarrow{\pi}k^n$$

is bijective, because it is injective and U is n-dimensional. Now the map

$$\mathbf{T}_x(\mathfrak{g}):\mathbf{T}_x(\mathfrak{X})\to\mathbf{T}_y(\mathfrak{Y})\otimes\mathbf{T}_z(\mathfrak{Sp}\,k[T_1,\cdots,T_n]))$$

satisfies

$$\mathbf{T}_x(\mathfrak{f})=(1\otimes\varepsilon)\circ\mathbf{T}_x(\mathfrak{g})\quad\text{and}\quad\mathbf{T}_x(\mathfrak{h})=(\varepsilon\otimes 1)\circ\mathbf{T}_x(\mathfrak{g}).$$

Therefore the arguement previous to Proof (iii)$\Leftrightarrow$(v) in Theorem 1.8.1 implies that $\mathbf{T}_x(\mathfrak{g})$ is an isomorphism, since $\mathbf{T}_x(\mathfrak{f})$ is smooth. Hence $\mathfrak{g}$ is "étale" at x.

2.3.7 A locally algebraic k-scheme $\mathfrak{X}$ is said to be k-smooth at a point x of $\mathfrak{X}$, if the canonical projection: $\mathfrak{X}\to\mathfrak{Sp}\,k$ is smooth at x. If e is a rational point of $\mathfrak{X}$, then $\mathfrak{X}$ is k-smooth at e iff $\mathbf{T}_e(\mathfrak{X})\simeq\mathrm{B}(U)$ for some vector space U. As a corollary to Proposition 1.6.4 we have:

PROPOSITION. *Let $\mathfrak{X}$ be a locally algebraic scheme over an infinite field k and $e\in\mathfrak{X}(k)$. Consider the following commutative triangle consisting of canonical maps:*

$$\begin{tikzcd} \mathfrak{X}(k[X]/(X^{n+1})) \arrow[rr, "q_n"] \arrow[dr, "p_{n+1}"'] & & \mathfrak{X}(k[X]/(X^n)) \arrow[dl, "p_n"] \\ & \mathfrak{X}(k) & \end{tikzcd}$$

Then $\mathfrak{X}$ is k-smooth at e iff the map

$$q_n : p_{n+1}^{-1}(e) \to p_n^{-1}(e)$$

is surjective for any integer $n>0$. In particular $\mathfrak{X}$ is k-smooth at all rational points iff q_n is surjective for all $n>0$.

PROOF (cf. [7, I, § 4, 5.11]). Enough to notice that

$$p_n^{-1}(e) \simeq W_k((k[X]/(X^n))^*, \mathbf{T}_e(\mathfrak{X})) \quad \text{and}$$
$$B_n \simeq (k[X]/(X^{n+1}))^*.$$

3. Hyperalgebras

3.1 Basic concepts

3.1.1 A *k-monoid-functor* (resp. *k-group-functor*) means a monoid (resp. group) object in the category $\boldsymbol{M_k E}$, or equivalently a covariant functor from $\boldsymbol{M_k}$ to ***Mon*** (resp. ***Gr***) [7, II, § 1, 1.1]. The category of k-monoid-functors (resp. k-group-functors) is denoted by $\boldsymbol{Mon_k}$ (resp. $\boldsymbol{Gr_k}$). The structure maps of a k-monoid-functor (resp. k-group-functor) $\mathfrak{G}$ will be denoted as follows:

$$\text{(multiplication)} \quad \mathfrak{p}_{\mathfrak{G}} : \mathfrak{G} \to \mathfrak{G}$$
$$\text{(unit)} \quad \mathfrak{e}_{\mathfrak{G}} : \mathfrak{Sp}\ k \to \mathfrak{G}$$

(resp. in addition

$$\text{(inverse)} \quad \mathfrak{i}_{\mathfrak{G}} : \mathfrak{G} \to \mathfrak{G}$$

(cf. (1.3.1)). The index '$\mathfrak{G}$' will be omitted if there is no risk of confusions.

Let $\mathfrak{G}$ be a k-monoid-functor (resp. k-group-functor). Suppose that the underlying k-functor of $\mathfrak{G}$ has the underlying coalgebra $\mathbf{T}(\mathfrak{G})$ (2.1.1). We know that $\mathbf{T}(\mathfrak{G}) \otimes \mathbf{T}(\mathfrak{G})$ is the underlying coalgebra of $\mathfrak{G} \times \mathfrak{G}$ (2.2.1). Hence applying the functor $\mathbf{T}(-)$ to the structure maps of $\mathfrak{G}$, we get coalgebra maps:

$$\mu = \mathbf{T}(\mathfrak{p}) : \mathbf{T}(\mathfrak{G}) \otimes \mathbf{T}(\mathfrak{G}) \to \mathbf{T}(\mathfrak{G})$$
$$\eta = \mathbf{T}(\mathfrak{e}) : k \simeq \mathbf{T}(\mathfrak{Sp}\ k) \to \mathbf{T}(\mathfrak{G})$$

(resp. in addition

$$S=\mathbf{T}(\mathfrak{i}):\mathbf{T}(\mathfrak{G})\to\mathbf{T}(\mathfrak{G}))$$

which make $\mathbf{T}(\mathfrak{G})$ into a cocommutative bialgebra (resp. cocommutative Hopf algebra with antipode S), as is easily verified. We say that $\mathbf{T}(\mathfrak{G})$ is *the underlying bialgebra* (resp. *the underlying Hopf algebra*) of $\mathfrak{G}$. It is easy to see that the structure (μ,η) (resp. (μ,η,S)) is the unique one which turns the canonical isomorphism (2.1.1):

$$W_k(C,\mathbf{T}(\mathfrak{G}))\simeq\mathfrak{G}(C^*)$$

into a monoid (resp. group) isomorphism for all $C\in W_k^{\mathrm{f}}$. (The uniqueness follows from Lemma 1.3.2).

3.1.2 Let $\mathfrak{G}$ be a k-monoid-functor. We identify as usual the structure map $\mathfrak{e}:\mathfrak{Sp}\,k\to\mathfrak{G}$ with an element of $\mathfrak{G}(k)$, which is the unit of the monoid $\mathfrak{G}(k)$. Suppose that the coalgebra $\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})$ to $\mathfrak{G}$ at $\mathfrak{e}$ exists (2.1.8). Since $\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})\otimes\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})$ is the coalgebra $\mathbf{S}_{(\mathfrak{e},\mathfrak{e})}(\mathfrak{G}\times\mathfrak{G})$ by (2.2.4), the structure maps of $\mathfrak{G}$ induce coalgebra maps

$$\mu=\mathbf{S}_{(\mathfrak{e},\mathfrak{e})}(\mathfrak{p}):\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})\otimes\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})\to\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})\quad\text{and}$$
$$\eta=\mathbf{S}_1(\mathfrak{e}):k\simeq\mathbf{S}_1(\mathfrak{Sp}\,k)\to\mathbf{S}_{\mathfrak{e}}(\mathfrak{G}),$$

where 1 is the unique rational point of $\mathfrak{Sp}\,k$. One may easily check that the maps μ and η make $\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})$, which is connected by (2.1.8), into a *hyperalgebra* (1.3.5) and that (μ,η) is the unique bialgebra structure on $\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})$ which turns the canonical isomorphism (2.1.8):

$$W_k(C,\mathbf{S}_{\mathfrak{e}}(\mathfrak{G}))\simeq\operatorname{Ker}(\mathfrak{G}(C^*)\to\mathfrak{G}(C_0^*))$$

into a monoid isomorphism for any $C\in W_k^{\mathrm{f}}$. But the left hand side is in fact a group, since every hyperalgebra has the antipode (1.3.5). Hence the existence of $\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})$ means that $\operatorname{Ker}(\mathfrak{G}(C^*)\to\mathfrak{G}(C_0^*))$ is a multiplicative subgroup of $\mathfrak{G}(C^*)$ for any $C\in W_k^{\mathrm{f}}$. On the other hand the subfunctor $\mathfrak{G}_{\mathfrak{e}}$[13] of $\mathfrak{G}$ is easily seen to be a k-monoid-subfunctor. Then the hyperalgebra $(\mathbf{T}_{\mathfrak{e}}(\mathfrak{G}),\mu,\eta)$ is clearly equal to the underlying bialgebra of $\mathfrak{G}_{\mathfrak{e}}$. The hyperalgebra $(\mathbf{S}_{\mathfrak{e}}(\mathfrak{G}),\mu,\eta)$ is called *the hyperalgebra of* $\mathfrak{G}$ and denoted by $\mathbf{hy}(\mathfrak{G})$.

If in particular $\mathfrak{G}$ is a k-*group*-functor, then the antipode S of $\mathbf{hy}(\mathfrak{G})$ is induced by the structure map $\mathfrak{i}:\mathfrak{G}\to\mathfrak{G}$, that is

$$S=\mathbf{S}_{\mathfrak{e}}(\mathfrak{i}):\mathbf{S}_{\mathfrak{e}}(\mathfrak{G})\to\mathbf{S}_{\mathfrak{e}}(\mathfrak{G}).$$

If a k-monoid-functor $\mathfrak{G}$ has the underlying bialgebra $\mathbf{T}(\mathfrak{G})$, then it follows directly from Proposition 2.1.4 (or from (1.3.4)) that the hyperal-

gebra $\mathbf{hy}(\mathfrak{G})$ exists and is equal to $\mathbf{T}(\mathfrak{G})^1$ the irreducible component of $\mathbf{T}(\mathfrak{G})$ containing 1, which is a subbialgebra of $\mathbf{T}(\mathfrak{G})$.

3.1.3 Let $\mathfrak{G}$ be a k-monoid-functor. Suppose that $\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})$ the tangent coalgebra to $\mathfrak{G}$ at the unit e in the strong sense exists (2.1.8). In view of (2.2.4) the structure maps $\mathfrak{p}$ and e induce coalgebra maps respectively

$$\mu=\mathbf{T}_{(\mathrm{e},\mathrm{e})}^{\mathrm{st}}(\mathfrak{p}):\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})\otimes\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})\simeq\mathbf{T}_{(\mathrm{e},\mathrm{e})}^{\mathrm{st}}(\mathfrak{G})\to\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G}) \quad \text{and}$$
$$\eta=\mathbf{T}_{1}^{\mathrm{st}}(\mathrm{e}):k\simeq\mathbf{T}_{1}^{\mathrm{st}}(\mathfrak{S}\mathfrak{p}\ k)\to\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})$$

which make $\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})$ into a hyperalgebra. Now consider the canonical isomorphism:

$$W_R(C,\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G}))\simeq\mathrm{Ker}\,(\mathfrak{G}(R\otimes C^*)\to\mathfrak{G}(R\otimes C_0^*))$$

for all $(R,C)\in W^{\mathrm{f}}$ (2.1.8). The right hand side is a submonoid of $\mathfrak{G}(R\otimes C^*)$. The hyperalgebra structure (μ,η) on $\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})$ determines a group structure on the left hand side (1.3.5). Then it is clear by the definition of (μ,η) that the above isomorphism becomes a monoid, in fact a group, isomorphism. In other words the hyperalgebra $(\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G}),\mu,\eta)$ *represents* the monoid-functor on W^{f}:

$$(R,C)\mapsto\mathrm{Ker}\,(\mathfrak{G}(R\otimes C^*)\to\mathfrak{G}(R\otimes C_0^*))$$

in the sense of (1.3.8). We call that hyperalgebra $(\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G}),\mu,\eta)$ the hyperalgebra of $\mathfrak{G}$ *in the strong sense* and denote by $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$.

If $\mathfrak{G}$ is a k-group-functor, then the antipode S of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ is clearly equal to $\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{i}):\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})\to\mathbf{T}_{\mathrm{e}}^{\mathrm{st}}(\mathfrak{G})$.

3.1.4 A k-monoid-functor (resp. k-group-functor) is called a *k-monoid* (resp. *k-group*) if its underlying k-functor is a *k-scheme* [7, II, § 1, 1.1].

Let $\mathfrak{G}$ be a k-monoid. It follows from Proposition 2.1.6 that the underlying bialgebra $\mathbf{T}(\mathfrak{G})$ exists and is equal to

$$\bigoplus_{x\in\mathfrak{G}}(\mathcal{O}_x)^0.$$

In particular the hyperalgebra $\mathbf{hy}(\mathfrak{G})$ exists and is equal to $(\mathcal{O}_{\mathrm{e}})^0$. The hyperalgebra structure on $(\mathcal{O}_{\mathrm{e}})^0$ is determined as follows: The direct product diagram in $\boldsymbol{M}_k\boldsymbol{E}$:

$$\mathfrak{G}\xleftarrow{\mathfrak{pr}_1}\mathfrak{G}\times\mathfrak{G}\xrightarrow{\mathfrak{pr}_2}\mathfrak{G}$$

induces a direct product diagram in W_k:

$$(\mathcal{O}_{\mathrm{e}})^0\leftarrow(\mathcal{O}_{(\mathrm{e},\mathrm{e})})^0\to(\mathcal{O}_{\mathrm{e}})^0$$

by (2.2.4). Then the multiplication $\mu: (\mathcal{O}_e)^0 \otimes (\mathcal{O}_e)^0 \to (\mathcal{O}_e)^0$ is the composite

$$(\mathcal{O}_e)^0 \otimes (\mathcal{O}_e)^0 \simeq (\mathcal{O}_{(e,e)})^0 \xrightarrow{(\mathfrak{p}_{(e,e)})^0} (\mathcal{O}_e)^0,$$

where $\mathfrak{p}_{(e,e)}: \mathcal{O}_e \to \mathcal{O}_{(e,e)}$ is the local homomorphism induced by the structure map $\mathfrak{p}: \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$. The unit of $(\mathcal{O}_e)^0$ is the unique group-like element.

If $\mathfrak{G}$ is a locally algebraic k-monoid, then it follows from Proposition 2.1.9 that $(\mathcal{O}_e)^0$ is the hyperalgebra in the strong sense.

3.1.5 Let $\mathfrak{f}: \mathfrak{G} \to \mathfrak{G}'$ be a map of k-monoid-functors. Suppose that both $\mathfrak{G}$ and $\mathfrak{G}'$ have the underlying bialgebra (resp. the hyperalgebra, resp. the hyperalgebra in the strong sense). Then the induced coalgebra map

$$\mathbf{T}(\mathfrak{f}): \mathbf{T}(\mathfrak{G}) \to \mathbf{T}(\mathfrak{G}')$$

(resp.

$$\mathbf{S}_e(\mathfrak{f}): \mathbf{hy}(\mathfrak{G}) \to \mathbf{hy}(\mathfrak{G}'),$$

resp.

$$\mathbf{T}_e^{st}(\mathfrak{f}): \mathbf{hy}^{st}(\mathfrak{G}) \to \mathbf{hy}^{st}(\mathfrak{G}))$$

is easily seen to be a bialgebra map. Thus the functor $\mathbf{T}(-)$ (resp. $\mathbf{hy}(-)$, resp. $\mathbf{hy}^{st}(-)$) can be considered as a covariant functor from some full subcategory of $\boldsymbol{Mon}_k$ to the category of cocommutative bialgebras (resp. hyperalgebras).

Let $\mathfrak{H}$ be the kernel of $\mathfrak{f}$. Lemmas 1.3.3 and 1.3.5 imply directly that the bialgebra kernel of $\mathbf{T}(\mathfrak{f})$ (resp. $\mathbf{S}_e(\mathfrak{f})$, resp. $\mathbf{T}_e^{st}(\mathfrak{f})$) is the underlying bialgebra (resp. the hyperalgebra, resp. the hyperalgebra in the strong sense) of $\mathfrak{H}$. In particular we have $\mathbf{hy}(\mathfrak{H}) = k$ (resp. $\mathbf{hy}^{st}(\mathfrak{H}) = k$) iff $\mathbf{T}_e(\mathfrak{f})$ (resp. $\mathbf{T}_e^{st}(\mathfrak{f})$) is injective by (1.3.6).

3.1.6 Let $\mathfrak{H}$ and $\mathfrak{G}$ be two k-group-functors. Let

$$\mathfrak{u}: \mathfrak{H} \times \mathfrak{G} \to \mathfrak{G}$$

be an action of $\mathfrak{H}$ on $\mathfrak{G}$ [7, II, § 1, 3.10], that is $\mathfrak{u}$ is an action in the sense of (2.1.10) such that $\mathfrak{H}(R)$ acts on $\mathfrak{G}(R)$ as group automorphisms for any $R \in \boldsymbol{M}_k$. Suppose that $\mathbf{hy}^{st}(\mathfrak{G})$ the hyperalgebra in the strong sense exists. Since $\mathfrak{e} \in \mathfrak{G}(k)$ is $\mathfrak{H}$-stable, there is a natural linear action of $\mathfrak{H}$ on $(\mathbf{hy}^{st}(\mathfrak{G}))_a$. Let $\mathfrak{H}$ act on $\mathfrak{G} \times \mathfrak{G}$ via the diagonal map: $\mathfrak{H} \to \mathfrak{H} \times \mathfrak{H}$. Then the structure map

$$\mathfrak{p}: \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$$

commutes with the action of $\mathfrak{H}$. This implies that the induced map

$$\mu: \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})\otimes\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})\rightarrow\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$$

is a k-$\mathfrak{H}$-module map, where the left hand side is viewed as a k-$\mathfrak{H}$-module via the diagonal map: $\mathfrak{H}\rightarrow\mathfrak{H}\times\mathfrak{H}$. Similarly if we let $\mathfrak{H}$ act on k trivially, then the structure map

$$\eta: k\rightarrow\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$$

is also seen to be a k-$\mathfrak{H}$-module map. Hence the subspace $(\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}))^{\mathfrak{H}}$ is a subalgebra of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ (but not necessarily a subcoalgebra). Let B be the largest subcoalgebra of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ which is contained in $(\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}))^{\mathfrak{H}}$. Then B is a subhyperalgebra of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ and is the hyperalgebra in the strong sense of $\mathfrak{G}^{\mathfrak{H}}$ by (2.1.10).

3.1.7 Let $K|k$ be a field extension. It is known [9, Proposition 3.2.3] that if C is a connected coalgebra then $K\otimes C$ is a connected K-coalgebra.

Let $\mathfrak{X}$ be a locally algebraic k-scheme and $e\in\mathfrak{X}(k)$. It follows from Proposition 2.2.8 that the canonical map

$$\xi: K\otimes\mathbf{T}(\mathfrak{X})\rightarrow\mathbf{T}^K(K\otimes\mathfrak{X})$$

induces an isomorphism:

$$K\otimes\mathbf{T}_e(\mathfrak{X})\xrightarrow{\simeq}\bigoplus_{\bar{x}|e}\mathbf{T}^K_{\bar{x}}(K\otimes\mathfrak{X}).$$

But since the left hand side is irreducible, we have

$$K\otimes\mathbf{T}_e(\mathfrak{X})\xrightarrow{\simeq}\mathbf{T}^K_{e_K}(K\otimes\mathfrak{X}),$$

where $e_K\in\mathfrak{X}(K)$ is viewed as a rational point of $K\otimes\mathfrak{X}$.

Let $\mathfrak{G}$ be a k-monoid-functor. Suppose that the underlying bialgebras $\mathbf{T}(\mathfrak{G})$ and $\mathbf{T}^K(K\otimes\mathfrak{G})$ exist. The naturality of the map ξ implies the commutativity of the diagrams:

$$\begin{tikzcd}
K\otimes\mathbf{T}(\mathfrak{G}\times\mathfrak{G}) \arrow[r, "\xi(\mathfrak{G}\times\mathfrak{G})"] \arrow[d, "K\otimes\mathbf{T}(\mathfrak{p})"] & \mathbf{T}^K(K\otimes(\mathfrak{G}\times\mathfrak{G})) \arrow[d, "\mathbf{T}^K(K\otimes\mathfrak{p})"] \\
K\otimes\mathbf{T}(\mathfrak{G}) \arrow[r, "\xi_{\mathfrak{G}}"] & \mathbf{T}^K(K\otimes\mathfrak{G})
\end{tikzcd}$$

$$\begin{tikzcd}
K\otimes\mathbf{T}(\mathfrak{Sp}\ k) \arrow[r, "\xi(\mathfrak{Sp}\ k)"] \arrow[d, "K\otimes\mathbf{T}(\mathfrak{e})"] & \mathbf{T}^K(\mathfrak{Sp}_K\ K) \arrow[d, "\mathbf{T}^K(\mathfrak{e}_K)"] \\
K\otimes\mathbf{T}(\mathfrak{G}) \arrow[r, "\xi_{\mathfrak{G}}"] & \mathbf{T}^K(\mathfrak{G})
\end{tikzcd} .$$

This means that the map

$$\xi: K\otimes\mathbf{T}(\mathfrak{G})\to\mathbf{T}^K(K\otimes\mathfrak{G})$$

is a bialgebra map. Hence we have proven:

PROROSITION. *Let $K|k$ be a field extension and $\mathfrak{G}$ a k-monoid-functor. If the underlying bialgebras $\mathbf{T}(\mathfrak{G})$ and $\mathbf{T}^K(K\otimes\mathfrak{G})$ exist then the canonical map $\xi: K\otimes\mathbf{T}(\mathfrak{G})\to\mathbf{T}^K(K\otimes\mathfrak{G})$ is a bialgebra map. If in particular $\mathfrak{G}$ is a locally algebraic k-monoid, then ξ induces an isomorphism of hyperalgecras: $K\otimes\mathbf{hy}(\mathfrak{G})\xrightarrow{\simeq}\mathbf{hy}^K(K\otimes\mathfrak{G})$.*

3.1.8 Let $k(\varepsilon)$ be the k-algebra on basis 1 and ε with $\varepsilon^2=0$. We define some algebra maps as follows:

$$v_\lambda: k(\varepsilon)\to k(\varepsilon),\quad \varepsilon\mapsto\lambda\varepsilon\quad \text{for } \lambda\in k$$
$$\iota: k(\varepsilon)\to k(\varepsilon)\otimes k(\varepsilon),\quad \varepsilon\mapsto\varepsilon\otimes\varepsilon$$
$$\iota_{i/n}: k(\varepsilon)\to\otimes^n k(\varepsilon),\quad \varepsilon\mapsto(\otimes^{i-1}1)\otimes\varepsilon\otimes(\otimes^{n-i}1)\quad \text{for } 1\leqq i\leqq n.$$

Let p be the characteristic exponent of k. The subalgebra of symmetric elements in $\otimes^p k(\varepsilon)$ is generated by $\sigma=\varepsilon_1+\cdots+\varepsilon_p$ and $\pi=\varepsilon_1\cdots\varepsilon_p$, where $\varepsilon_i=\iota_{i/p}(\varepsilon)$. Then there exists a unique algebra map

$$h: k(\sigma,\pi)\to k(\varepsilon)$$

such that $h(\pi)=\varepsilon$ and $h(\sigma)=\delta_{1,p}\varepsilon$.

Let $\mathfrak{G}$ be a k-group-functor. We put

$$\mathrm{Lie}\,(\mathfrak{G})=\mathrm{Ker}\,(\mathfrak{G}(k(\varepsilon))\to\mathfrak{G}(k))$$

where the right hand side is taken with respect to the canonical projection: $k(\varepsilon)\to k$, $\varepsilon\mapsto 0$.

PROPOSITION (cf. [7, II, §4 and §7]). *Let $\mathfrak{G}$ be a k-group-functor. Suppose that $\mathfrak{G}$ has the hyperalgebra* $\mathbf{hy}(\mathfrak{G})$.

(i) *The action:*

$$k\times\mathrm{Lie}\,(\mathfrak{G})\to\mathrm{Lie}\,(\mathfrak{G}),\qquad (\lambda,x)\mapsto\mathfrak{G}(v_\lambda)(x)$$

makes the group Lie $(\mathfrak{G})$ *into a vector space.*

(ii) *There exists a unique map*

$$\alpha: \mathrm{Lie}\,(\mathfrak{G})\times\mathrm{Lie}\,(\mathfrak{G})\to\mathrm{Lie}\,(\mathfrak{G})$$

such that

$$\mathfrak{G}(\iota)(\alpha(x,y))=\mathfrak{G}(\iota_{1/2})(x)\cdot\mathfrak{G}(\iota_{2/2})(y)\cdot\mathfrak{G}(\iota_{1/2})(x)^{-1}\cdot\mathfrak{G}(\iota_{2/2})(y)^{-1}$$

in the group $\mathfrak{G}(k(\varepsilon)\otimes k(\varepsilon))$ *for any* $x, y \in \mathrm{Lie}(\mathfrak{G})$. *Then the vector space* $\mathrm{Lie}(\mathfrak{G})$ *becomes a Lie algebra with* α *the bracket product.*

(iii) *The inclusion*: $k(\sigma, \pi) \hookrightarrow \otimes^p k(\varepsilon)$ *induces an injection*:

$$\mathrm{Ker}\,(\mathfrak{G}(k(\sigma, \pi)) \longrightarrow \mathfrak{G}(k)) \hookrightarrow \mathfrak{G}(\otimes^p k(\varepsilon))$$

where the left hand side is taken with respect to the canonical projection: $k(\sigma, \pi) \to k, \sigma \mapsto 0, \pi \mapsto 0$. *The map*:

$$\begin{aligned} &\mathfrak{G}(k(\varepsilon)) \to \mathfrak{G}(\otimes^p k(\varepsilon)) \\ &\qquad x \mapsto \mathfrak{G}(\iota_{1/p})(x) \cdots \mathfrak{G}(\iota_{p/p})(x) \end{aligned}$$

induces a map

$$f_p : \mathrm{Lie}\,(\mathfrak{G}) \to \mathrm{Ker}\,(\mathfrak{G}(k(\sigma, \pi)) \to \mathfrak{G}(k)).$$

Let P: $\mathrm{Lie}\,(\mathfrak{G}) \to \mathrm{Lie}\,(\mathfrak{G})$ *be the composite*:

$$\mathrm{Lie}\,(\mathfrak{G}) \xrightarrow{f_p} \mathrm{Ker}\,(\mathfrak{G}(k(\sigma, \pi)) \longrightarrow \mathfrak{G}(k)) \xrightarrow{\mathfrak{G}(h)} \mathrm{Lie}\,(\mathfrak{G}).$$

Then the map P *turns the Lie algebra* $\mathrm{Lie}\,(\mathfrak{G})$ *into a restricted Lie algebra.*

(iv) *The (restricted) Lie algebra* $\mathrm{Lie}\,(\mathfrak{G})$ *is naturally isomorphic to* $\mathrm{P}(\mathbf{hy}(\mathfrak{G}))$.

PROOF. Consider the natural isomorphisms:

$$\begin{aligned} \mathrm{P}(\mathbf{hy}\,(\mathfrak{G})) \simeq W_k(B_1, \mathbf{hy}(\mathfrak{G})) &\simeq \mathrm{Ker}\,(\mathfrak{G}(B_1^*) \to \mathfrak{G}(B_0^*)) \\ &\simeq \mathrm{Ker}\,(\mathfrak{G}(k(\varepsilon)) \to \mathfrak{G}(k)) = \mathrm{Lie}\,(\mathfrak{G}). \end{aligned}$$

Proposition then follows immediately from § 1.3b if one notices that the algebra maps

$$\begin{aligned} &v_\lambda : k(\varepsilon) \to k(\varepsilon) \\ &\iota : k(\varepsilon) \to k(\varepsilon) \otimes k(\varepsilon) \\ &\iota_{i/n} : k(\varepsilon) \to \otimes^n k(\varepsilon) \\ &\text{cano.} : k(\sigma, \pi) \hookrightarrow \otimes^p k(\varepsilon) \\ &h : k(\sigma, \pi) \to k(\varepsilon) \end{aligned}$$

are respectively the linear duals of the following coalgebra maps

$$\begin{aligned} &u_\lambda : B_1 \to B_1 \\ &\nu : B_1 \otimes B_1 \to B_1 \\ &pr_i : \otimes^n B_1 \to B_1 \\ &\text{cano.} : \otimes^p B_1 \to S^p B_1 \\ &g : B_1 \to S^p B_1. \end{aligned}$$

3.1.9 Let $\mathfrak{G}$ be a k-group-functor. For any $R \in \boldsymbol{M}_k$ we put

$$\mathfrak{Lie}\,(\mathfrak{G})(R) = \mathrm{Ker}\,(\mathfrak{G}(R(\varepsilon)) \to \mathfrak{G}(R))$$

where $R(\varepsilon) = R \otimes k(\varepsilon)$. In particular

$$\mathrm{Lie}\,(\mathfrak{G}) = \mathfrak{Lie}\,(\mathfrak{G})(k).$$

Suppose that $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ the hyperalgebra in the strong sense exists. Then we have

$$\begin{aligned} R \otimes \mathrm{P}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})) &\simeq W_R(B_1, \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})) \simeq \mathrm{Ker}\,(\mathfrak{G}(R \otimes B_1^*) \to \mathfrak{G}(R \otimes B_0^*)) \\ &\simeq \mathrm{Ker}\,(\mathfrak{G}(R(\varepsilon)) \to \mathfrak{G}(R)) = \mathfrak{Lie}\,(\mathfrak{G})(R). \end{aligned}$$

As a corollary to Proposition 1.3b.6 we have

PROPOSITION. *Let $\mathfrak{G}$ be a k-group-functor having $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ the hyperalgebra in the strong sense.*

(i) *The action:*

$$R \times \mathfrak{Lie}\,(\mathfrak{G})(R) \to \mathfrak{Lie}\,(\mathfrak{G})(R), \; (\lambda, x) \mapsto \mathfrak{G}(v_\lambda)(x)$$

makes the group $\mathfrak{Lie}\,(\mathfrak{G})(R)$ into an R-module, where v_λ is the R-algebra map: $R(\varepsilon) \to R(\varepsilon)$, $\varepsilon \mapsto \lambda\varepsilon$ for $\lambda \in R$.

(ii) *The maps:*

$$\begin{gathered} \mathfrak{G}(R(\varepsilon)) \times \mathfrak{G}(R(\varepsilon)) \to \mathfrak{G}(R(\varepsilon) \otimes_R R(\varepsilon)) \\ (x, y) \mapsto \mathfrak{G}(\iota_{1/2})(x) \cdot \mathfrak{G}(\iota_{2/2})(y) \cdot \mathfrak{G}(\iota_{1/2})(x)^{-1} \cdot \mathfrak{G}(\iota_{2/2})(y)^{-1} \\ \textit{and} \quad \mathfrak{G}(R(\varepsilon)) \to \mathfrak{G}(\otimes_R^p R(\varepsilon))\, x \mapsto \mathfrak{G}(\iota_{1/p})(x) \cdots \mathfrak{G}(\iota_{p/p})(x) \end{gathered}$$

induce respectively maps

$$\begin{aligned} &\alpha_R : \mathfrak{Lie}\,(\mathfrak{G})(R) \times \mathfrak{Lie}\,(\mathfrak{G})(R) \to \mathfrak{Lie}\,(\mathfrak{G})(R) \quad \textit{and} \\ &P_R : \mathfrak{Lie}\,(\mathfrak{G})(R) \to \mathfrak{Lie}\,(\mathfrak{G})(R) \end{aligned}$$

in the same way as in Proposition 3.1.8. *Then the R-module $\mathfrak{Lie}\,(\mathfrak{G})(R)$ becomes a (restricted) Lie algebra over R with α_R and P_R the structure maps.*

(iii) *The map:*

$$R \times \mathfrak{G}(k(\varepsilon)) \to \mathfrak{G}(R(\varepsilon)), \; (\lambda, x) \mapsto \mathfrak{G}(v_\lambda)(x)$$

induces an isomorphism of (restricted) Lie algebras over R:

$$R \otimes \mathrm{Lie}\,(\mathfrak{G}) \xrightarrow{\simeq} \mathfrak{Lie}\,(\mathfrak{G})(R).$$

3.2 Examples

3.2.1 Let V be a vector space. The commutative k-group-functors V_a and $\mathfrak{D}_a(V)$ are defined as follows [7, II, § 1, 2.1]:

$$V_a(R)=R\otimes V \quad \text{and}$$
$$\mathfrak{D}_a(V)(R)=\boldsymbol{Mod}_k(V,R)$$

for $R\in \boldsymbol{M}_k$.

PROPOSITION. $\mathbf{hy}^{\mathrm{st}}(V_a)\simeq \mathrm{B}_a(V)$ *and* $\mathbf{hy}(\mathfrak{D}_a(V))\simeq \mathrm{B}_a(V^*)$.

PROOF. Let $(R,C)\in \boldsymbol{W}^{\mathrm{f}}$. Then we have

$$\begin{aligned} \boldsymbol{W}_R(C,\mathrm{B}_a(V)) &\simeq \boldsymbol{Mod}_R(R\otimes(C/C_0),R\otimes V) \qquad (1.5.4)\\ &\simeq \boldsymbol{Mod}_k(C/C_0,R\otimes V)\\ &\simeq R\otimes(C/C_0)^*\otimes V\\ &\simeq \mathrm{Ker}\,(V_a(R\otimes C^*)\to V_a(R\otimes C_0^*))\end{aligned}$$

as groups. Since these isomorphisms are easily seen to be functorial with respect to $(R,C)\in \boldsymbol{W}^{\mathrm{f}}$, we have by definition

$$\mathbf{hy}^{\mathrm{st}}(V_a)\simeq \mathrm{B}_a(V).$$

Let $C\in \boldsymbol{W}_k^{\mathrm{f}}$. Since we have

$$\begin{aligned} \boldsymbol{W}_k(C,\mathrm{B}_a(V^*)) &\simeq \boldsymbol{Mod}_k(C/C_0,V^*)\\ &\simeq \boldsymbol{Mod}_k(V,(C/C_0)^*)\\ &\simeq \mathrm{Ker}\,(\mathfrak{D}_a(V)(C^*)\to \mathfrak{D}_a(V)(C_0^*))\end{aligned}$$

naturally as groups, it follows that

$$\mathbf{hy}(\mathfrak{D}_a(V))\simeq \mathrm{B}_a(V^*).$$

3.2.2 Let A be a (small) commutative bialgebra (resp. Hopf algebra). For any $R\in \boldsymbol{M}_k$, the set $\boldsymbol{M}_k(A,R)$ is easily seen to be a multiplicative submonoid (resp. subgroup) of the algebra $\boldsymbol{Mod}_k(A,R)$ [11, Theorem 4.0.5]. Hence the affine k-scheme

$$\mathfrak{Sp}\,A: R\mapsto \boldsymbol{M}_k(A,R)$$

can be considered as a k-monoid- (resp. k-group-) functor. On the other hand the dual coalgebra A^0 has a natural structure of (cocommutative) bialgebra (resp. Hopf algebra) [11, § 6.2]. Then the canonical isomorphism (2.1.5): $A^0\simeq \mathbf{T}(\mathfrak{Sp}\,A)$ is easily seen to be a bialgebra (resp. Hopf algebra) isomorphism. In particular $(A^0)^1$ the irreducible component of A^0 contain-

ing 1 can be canonically identified with the hyperalgebra of the k-monoid- (resp. k-group-) functor $\mathfrak{Sp}\, A$.

Let V be a vector space. We shall denote by $\mathbf{S}V$ the symmetric algebra on V. Then $\mathbf{S}V$ has a unique Hopf algebra structure (Δ, ε, S) determined by

$$\Delta(v)=v\otimes 1+1\otimes v,\ \varepsilon(v)=0 \quad \text{and} \quad S(v)=-v$$

for $v\in V$ [11, Proposition 3.2.3]. Then the k-group-functor $\mathfrak{D}_a(V)$ is clearly isomorphic to the k-group-functor $\mathfrak{Sp}\,\mathbf{S}V$ (cf. [7, II, § 1, 2.1]). Hence we have:

COROLLARY. *For any vector space V, there is a canonical isomorphism of hyperalgebras*:

$$\mathrm{B_a}(V^*)\simeq((\mathbf{S}V)^0)^1.$$

3.2.3 *The multiplicative group of an algebra.* Let A be an associative (not necessarily commutative) algebra. The k-group-functor $\boldsymbol{\mu}^A$ [7, II, § 1, 2.3] is defined by

$$\boldsymbol{\mu}^A(R)=\mathrm{U}(R\otimes A)$$

for $R\in \boldsymbol{M}_k$, where $\mathrm{U}(R\otimes A)$ denotes the unit group of the ring $R\otimes A$.

PROPOSITION. $\mathbf{hy}^{\mathrm{st}}(\boldsymbol{\mu}^A)\simeq \mathrm{B_m}(A)$.

PROOF. Let $(R, C)\in \boldsymbol{W}^{\mathrm{f}}$. We have functorially

$$\boldsymbol{W}_R(C, \mathrm{B_m}(A))\simeq \mathrm{T}_{(R\otimes A)}(C)$$

as groups (1.5.6). The canonical isomorphism of algebras:

$$\boldsymbol{Mod}_k(C, R\otimes A)\simeq R\otimes C^*\otimes A$$

clearly induces an isomorphism of multiplicative subgroups:

$$\mathrm{T}_{(R\otimes A)}(C)\simeq \mathrm{Ker}\,(\mathrm{U}(R\otimes C^*\otimes A)\to \mathrm{U}(R\otimes C_0^*\otimes A)).$$

From this the assertion follows immediately.

3.2.4 *Linear groups.* Let V be a vector space. The general linear group $\mathfrak{GL}(V)$ is the k-group-functor defined by

$$\mathfrak{GL}(V)(R)=\mathbf{GL}_R(R\otimes V)$$

for $R\in \boldsymbol{M}_k$ [7, II, § 1, 2.4]. If V is finite dimensional, then the special linear group $\mathfrak{SL}(V)$ is the k-group-subfunctor of $\mathfrak{GL}(V)$ defined by

$$\mathfrak{SL}(V)(R)=\mathbf{SL}_R(R\otimes V)$$

for $R \in \boldsymbol{M}_k$.

PROPOSITION. $\mathbf{hy}(\mathfrak{GL}(V)) \simeq \mathrm{B_m}(\mathrm{End}_k(V))$.

If V is finite dimensional, then

$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{GL}(V)) \simeq \mathrm{B_m}(\mathrm{End}_k(V)) \quad \textit{and}$$
$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{SL}(V)) \simeq \mathrm{Ker}_0(\mathbf{D}_V),$$

where $\mathbf{D}_V : \mathrm{B_m}(\mathrm{End}_k(V)) \to \mathrm{B_m}(k)$ is the hyperalgebra map define in (1.5.10).

PROOF. Let $C \in \boldsymbol{W}_k^{\mathrm{f}}$. Then the canonical isomorphism of algebras (1.5.9):

$$\mathrm{End}_{C^*}(C^* \otimes V) \simeq \boldsymbol{Mod}_k(C, \mathrm{End}_k(V))$$

induces an isomorphism of multiplicative subgroups:

$$\mathrm{Ker}\,(\mathbf{GL}_{C^*}(C^* \otimes V) \to \mathbf{GL}_{C_0^*}(C_0^* \otimes V)) \simeq \mathrm{T}_{\mathrm{End}_k(V)}(C).$$

Since we have

$$\boldsymbol{W}_k(C, \mathrm{B_m}(\mathrm{End}_k(V))) \simeq \mathrm{T}_{\mathrm{End}_k(V)}(C)$$

naturally as groups, it follows that

$$\mathbf{hy}(\mathfrak{GL}(V)) \simeq \mathrm{B_m}(\mathrm{End}_k(V)).$$

Suppose that V is finite dimensional. Since then $\mathfrak{GL}(V)$ is an algebraic k-group, it has the hyperalgebra in the strong sense which is equal to $\mathbf{hy}(\mathfrak{GL}(V))$. Now Proposition 1.5.10 means that the map of hyperalgebras

$$\mathbf{D}_V : \mathrm{B_m}(\mathrm{End}_k(V)) \to \mathrm{B_m}(k)$$

is obtained by applying the functor $\mathbf{hy}^{\mathrm{st}}(-)$ to the map of k-groups

$$\mathfrak{determinant} : \mathfrak{GL}(V) \to \mathfrak{GL}(k).$$

Hence $\mathfrak{SL}(V)$ the kernel of the determinant map has the Hopf kernel of $\mathbf{D}_V$ as its hyperalgebra in the strong sense by (3.1.5).

3.2.5 Let $\mathfrak{G}$ be a k-group-functor and V a vector space. Then the linear actions of $\mathfrak{G}$ on V_a (2.1.10) correspond bijectively to the maps of k-group-functors: $\mathfrak{G} \to \mathfrak{GL}(V)$ [7, II, § 2, 1.1].

Suppose that the hyperalgebra $\mathbf{hy}(\mathfrak{G})$ exists. Then any k-$\mathfrak{G}$-module (V, ρ), where $\rho : \mathfrak{G} \to \mathfrak{GL}(V)$, has a natural structure of left $\mathbf{hy}(\mathfrak{G})$-module via the composite

$$\mathbf{hy}(\mathfrak{G}) \xrightarrow{\mathbf{hy}(\rho)} \mathrm{B_m}(\mathrm{End}_k(V)) \xrightarrow{\pi'} \mathrm{End}_k(V),$$

where $\pi' = \pi'_{\mathrm{End}_k(V)}$ (1.5.5).

Let C be a finite dimensional subcoalgebra of $\mathbf{hy}(\mathfrak{G})$ with $i: C \hookrightarrow \mathbf{hy}(\mathfrak{G})$ the inclusion. Consider the map

$$V \to \boldsymbol{Mod}_k(C, V), v \mapsto (?)v,$$

where $(?)v: c \mapsto c \cdot v$. Let $\bar{i}$ be the element of $\mathfrak{G}(C^*)$ determined by i. Then we have:

LEMMA. *The above map*: $V \to \boldsymbol{Mod}_k(C, V)$ *factors as*

$$V \xrightarrow{\text{cano.}} C^* \otimes V \xrightarrow{\rho(\bar{i})} C^* \otimes V \underset{\simeq}{\xrightarrow{\text{cano.}}} \boldsymbol{Mod}_k(C, V).$$

PROOF. Under the canonical isomorphism:

$$\boldsymbol{Mod}_k(V, \boldsymbol{Mod}_k(C, V)) \simeq \boldsymbol{Mod}_k(C, \mathrm{End}_k(V))$$

the map: $v \mapsto (?)v$ corresponds to $\pi' \circ \mathbf{hy}(\rho) | C$ clearly. But via the canonical isomorphism of algebras:

$$\boldsymbol{Mod}_k(C, \mathrm{End}_k(V)) \simeq \mathrm{End}_{C^*}(C^* \otimes V)$$

the element $\pi' \circ \mathbf{hy}(\rho) | C$ corresponds to $\rho(\bar{i})$ by definition. This proves Lemma.

3.2.6 *The automorphism group of an algebra.* Let A be an associative (but not necessarily commutative) algebra. The k-group-functor $\mathfrak{Aut}(A)$ is defined by

$$\mathfrak{Aut}(A)(R) = \boldsymbol{Alg}_R\text{-aut}(R \otimes A)$$

for $R \in \boldsymbol{M}_k$, where the right hand side means the group of R-algebra automorphisms of $R \otimes A$ [7, II, § 1, 2.6].

PROPOSITION. $\mathbf{hy}(\mathfrak{Aut}(A)) \simeq \mathrm{M}_c(A, A)^1$.

PROOF. Let $C \in \boldsymbol{W}_k^f$. It follows from (1.5.11) that we have

$$\boldsymbol{W}_k(C, \mathrm{M}_c(A, A)) \simeq \boldsymbol{Alg}_{C^*}(C^* \otimes A, C^* \otimes A)$$

as monoids. Hence by (1.3.4) we have

$$\boldsymbol{W}_k(C, \mathrm{M}_c(A, A)^1) \simeq \mathrm{Ker}\,(\mathfrak{Aut}(A)(C^*) \to \mathfrak{Aut}(A)(C_0^*))$$

as groups. This means that $\mathbf{hy}(\mathfrak{Aut}(A)) \simeq \mathrm{M}_c(A, A)^1$.

3.2.7 Let V be a vector space and A an algebra. We defined in (1.5.1) *the additive C-Hopf-algebra* on V denoted by $\mathrm{C}_a(V)$. Then it is easy to see that

$$\mathbf{T}(V_a)\simeq \mathrm{C}_a(V) \quad \text{and} \quad \mathbf{T}(\mathfrak{D}_a(V))\simeq \mathrm{C}_a(V^*).$$

Consider the monoid-functor:

$$C\mapsto \boldsymbol{Mod}_k(C, A)^{\times}$$

on W_k, where $\boldsymbol{Mod}_k(C, A)^{\times}$ denotes the multiplicative monoid of the algebra $\boldsymbol{Mod}_k(C, A)$. Since this functor is represented by $\mathrm{C}(A)$, there exists a unique bialgebra structure (μ', η') on $\mathrm{C}(A)$ which makes the canonical isomorphism

$$W_k(C, \mathrm{C}(A))\xrightarrow{\simeq}\boldsymbol{Mod}_k(C, A)^{\times}, \sigma\mapsto\pi_A\circ\sigma$$

into a monoid isomorphism, where $\pi_A: \mathrm{C}(A)\to A$ is canonic. The bialgebra $(\mathrm{C}(A), \mu', \eta')$ is called *the multiplicative C-bialgebra* on V and denoted by $\mathrm{C}_{\mathrm{m}}(A)$. The bialgebra structure (μ', η') is easily seen to be unique such that the map

$$\pi_A: \mathrm{C}(A)\to A$$

turns into an algebra map.

Let $(A_a)^{\times}$ be the k-monoid-functor defined by

$$\boldsymbol{M}_k\to\boldsymbol{Mon}, \qquad R\mapsto(R\otimes A)^{\times}.$$

Since then

$$W_k(C, \mathrm{C}_{\mathrm{m}}(A))\simeq\boldsymbol{Mod}_k(C, A)^{\times}\simeq(C^*\otimes A)^{\times}$$

for any $C\in W_k^{\mathrm{f}}$, it follows that

$$\mathbf{T}((A_a)^{\times})\simeq\mathrm{C}_{\mathrm{m}}(A).$$

Let $\mathfrak{End}\,(V)$ and $\mathfrak{End}_{\mathfrak{alg}}\,(A)$ be the k-monoid-functors defined by

$$\mathfrak{End}\,(V): R\mapsto(\mathrm{End}_R\,(R\otimes V))^{\times}$$
$$\mathfrak{End}_{\mathfrak{alg}}\,(A): R\mapsto\boldsymbol{Alg}_R(R\otimes A, R\otimes A)$$

(cf. [7, II, § 1, 2.4 and 2.6]). Since then

$$W_k(C, \mathrm{C}_{\mathrm{m}}(\mathfrak{End}\,(V)))\simeq\boldsymbol{Mod}_k(C, \mathrm{End}_k\,(V))^{\times}\simeq\mathrm{End}_{C^*}\,(C^*\otimes V)^{\times}$$

and

$$W_k(C, \mathrm{M}_{\mathrm{c}}(A, A))\simeq\boldsymbol{Alg}_{C^*}(C^*\otimes A)$$

for any $C\in W_k^{\mathrm{f}}$ (1.5.9 and 1.5.11), it follows that

$$\mathbf{T}(\mathfrak{End}\,(V))\simeq\mathrm{C}_{\mathrm{m}}(\mathrm{End}_k\,(V)) \quad \text{and} \quad \mathbf{T}(\mathfrak{End}_{\mathfrak{alg}}\,(A))\simeq\mathrm{M}_{\mathrm{c}}(A, A).$$

Summarizing we have:

PROPOSITION.
$$\begin{aligned}
&\mathbf{T}(V_a) &&\simeq \mathrm{C}_a(V)\\
&\mathbf{T}(\mathfrak{D}_a(V)) &&\simeq \mathrm{C}_a(V^*)\\
&\mathbf{T}((A_a)^\times) &&\simeq \mathrm{C}_m(A)\\
&\mathbf{T}(\mathfrak{End}(V)) &&\simeq \mathrm{C}_m(\mathrm{End}_k(V))\\
&\mathbf{T}(\mathfrak{End}_{alg}(A)) &&\simeq \mathrm{M}_c(A, A).
\end{aligned}$$

3.2.8 Let $\mathfrak{G}$ be a k-monoid-functor. We define a k-group-functor $\mathfrak{U}(\mathfrak{G})$ by

$$\mathfrak{U}(\mathfrak{G})(R) = \mathrm{U}(\mathfrak{G}(R))$$

for any $R \in M_k$, where $\mathrm{U}(\mathfrak{G}(R))$ is the group of invertible elements in $\mathfrak{G}(R)$ (1.3a.1) (cf. [7, II, § 1, 1.4]). Suppose that $\mathfrak{G}$ has the underlying bialgebra $\mathbf{T}(\mathfrak{G})$. Let $\mathbf{T}(\mathfrak{G})'$ be the largest sub-Hopf algebra of $\mathbf{T}(\mathfrak{G})$ (1.3a.5). Then we have

$$W_k(C, \mathbf{T}(\mathfrak{G})') = \mathrm{U}(W_k(C, \mathbf{T}(\mathfrak{G}))) \simeq \mathrm{U}(\mathfrak{G}(C^*)) = \mathfrak{U}(\mathfrak{G})(C^*)$$

for any $C \in W_k^f$. Hence the Hopf algebra $\mathbf{T}(\mathfrak{G})'$ is the underlying Hopf algebra of $\mathfrak{U}(\mathfrak{G})$. In particular we have

$$\mathbf{hy}(\mathfrak{G}) = \mathbf{T}(\mathfrak{G})^1 = \mathbf{T}(\mathfrak{G})'^1 = \mathbf{hy}(\mathfrak{U}(\mathfrak{G})).$$

Let V be a vector space and A an algebra. Since then we have

$$\boldsymbol{\mu}^A = \mathfrak{U}((A_a)^\times), \mathfrak{GL}(V) = \mathfrak{U}(\mathfrak{End}(V)) \quad \text{and} \quad \mathfrak{Aut}(A) = \mathfrak{U}(\mathfrak{End}_{alg}(A)),$$

it follows that the underlying Hopf algebras

$$\mathbf{T}(\boldsymbol{\mu}^A), \mathbf{T}(\mathfrak{GL}(V)) \quad \text{and} \quad \mathbf{T}(\mathfrak{Aut}(A))$$

are respectively the largest sub-Hopf algebras of the bialgebras

$$\mathrm{C}_m(A), \mathrm{C}_m(\mathrm{End}_k(V)) \quad \text{and} \quad \mathrm{M}_c(A, A).$$

3.3 The hyperalgebra of an algebraic k-group

3.3.1 Let $\mathfrak{G}$ be a *locally algebraic k-group*. It follows from (2.1.11) that the hyperalgebra $\mathbf{hy}(\mathfrak{G})$ of $\mathfrak{G}$, which is in the strong sense (2.1.9), is of *finite type* as a coalgebra (that is $\mathrm{Lie}(\mathfrak{G}) = \mathrm{P}(\mathbf{hy}(\mathfrak{G}))$ is finite dimensional) and that

$$\dim \mathfrak{G} = \dim_e \mathfrak{G} = K \dim \mathbf{hy}(\mathfrak{G}).$$

Let $\mathfrak{G}^0$ be the *connected component* of e in $\mathfrak{G}$ [7, II, § 5, 1.1]. $\mathfrak{G}^0$ is an *open*

subgroup of $\mathfrak{G}$ and algebraic. Since the structure sheaf $\mathcal{O}_{\mathfrak{G}^0}$ has the same stalk over $\mathfrak{e}$ as $\mathcal{O}_{\mathfrak{G}}$, we have

$$\mathbf{hy}(\mathfrak{G}^0)=\mathbf{hy}(\mathfrak{G}).$$

3.3.2 Let $\mathfrak{G}$ be a k-group. A *subgroup* (resp. an open subgroup, resp. a closed subgroup) of $\mathfrak{G}$ means a k-group-subfunctor of $\mathfrak{G}$ which is a subscheme (resp. an open subscheme, resp. a closed subscheme). If $\mathfrak{G}$ is algebraic then *any subgroup* of $\mathfrak{G}$ is *closed* by [7, II, § 5, 5.1 (b)].

Let $\mathfrak{f}: \mathfrak{G}\to\mathfrak{H}$ be a map of algebraic k-groups. It follows from [7, III, § 3, 5.2 and 2.6] that there exists a unique subgroup $\mathfrak{H}'$ of $\mathfrak{H}$ containing the image $\mathfrak{f}(\mathfrak{G})$ such that the induced map

$$\mathfrak{f}: \mathfrak{G}\to\mathfrak{H}'$$

is *faithfully flat*. In the following we shall write

$$\mathfrak{H}'=\widetilde{\mathfrak{f}(\mathfrak{G})}$$

and call it the *image-subgroup* of $\mathfrak{f}$ (which coincides with the sheaf-image of $\mathfrak{f}$ defined in [7, III, § 1, 2.3]). Since now the induced map:

$$\mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\widetilde{\mathfrak{f}(\mathfrak{G})})$$

(resp.

$$\mathbf{hy}(\widetilde{\mathfrak{f}(\mathfrak{G})})\to\mathbf{hy}(\mathfrak{H}))$$

is surjective (resp. injective) by Proposition 2.3.1 (resp. by Proposition 2.2.5), we have:

PROPOSITION. *Let $\mathfrak{f}: \mathfrak{G}\to\mathfrak{H}$ be a map of algebraic k-groups. Then* $\mathbf{hy}(\widetilde{\mathfrak{f}(\mathfrak{G})})$ *the hyperalgebra of the image-subgroup of $\mathfrak{f}$ is the image of the induced map of hyperalgebras*

$$\mathbf{hy}(\mathfrak{f}): \mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{H}).$$ [14)]

3.3.3 A locally algebraic k-scheme $\mathfrak{X}$ is said to be étale (resp. smooth) at a point x if the canonical projection: $\mathfrak{X}\to\mathfrak{Sp}\,k$ is étale (resp. smooth) at x.

PROPOSITION. *Let $\mathfrak{f}: \mathfrak{G}\to\mathfrak{H}$ be a map of locally algebraic k-groups. The following are equivalent:*

(i) *$\mathfrak{f}$ is non-ramified at $\mathfrak{e}$.*

(ii) *$\mathfrak{Ker}(\mathfrak{f})$ is étale at $\mathfrak{e}$.*

(iii) *$\mathbf{hy}(\mathfrak{f}): \mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{H})$ is injective.*

In particular $\mathfrak{G}$ is étale at $\mathfrak{e}$ iff $\mathbf{hy}(\mathfrak{G})=k$.

PROOF. It follows from Proposition 2.3.5 that

$$\mathfrak{f} \text{ is non-ramified at } \mathfrak{e} \Leftrightarrow \mathbf{hy}(\mathfrak{f}): \mathbf{hy}(\mathfrak{G}) \to \mathbf{hy}(\mathfrak{H}) \text{ is injective}$$

and that

$$\mathfrak{G} \text{ is étale at } \mathfrak{e} \Leftrightarrow \mathbf{hy}(\mathfrak{G}) = k.$$

Since $\mathbf{hy}(\mathfrak{Ker}\,(\mathfrak{f}))$ is the Hopf kernel of $\mathbf{hy}(\mathfrak{f})$ (3.1.5), we have

$$\mathbf{hy}(\mathfrak{f}) \text{ is injective} \Leftrightarrow \mathbf{hy}(\mathfrak{Ker}\,(\mathfrak{f})) = k$$

by (1.3.6). This proves Proposition.

REMARK. A locally algebraic k-scheme $\mathfrak{X}$ is said to be étale (resp. smooth) if it is étale (resp. smooth) at its every point. It is known that a locally algebraic k-group $\mathfrak{G}$ is étale (resp. smooth) iff it is at $\mathfrak{e}$ [7, II, § 5, 1.4 and 2.1].

3.3.4 PROPOSITION. *Let* $\mathfrak{f}: \mathfrak{G} \to \mathfrak{H}$ *be a map of locally algebraic k-groups. The following are equivalent:*

(i) $\mathfrak{f}$ *is flat at* $\mathfrak{e}$.

(ii) *The induced map* $\mathbf{hy}(\mathfrak{f}): \mathbf{hy}(\mathfrak{G}) \to \mathbf{hy}(\mathfrak{H})$ *is surjective.*

PROOF. (i)$\Rightarrow$(ii) is contained in Proposition 2.3.1.

(ii)$\Rightarrow$(i). It is enough to show that the induced map

$$\mathfrak{f}: \mathfrak{G}^0 \to \mathfrak{H}^0$$

is flat at $\mathfrak{e}$. Since $\mathbf{hy}(\mathfrak{G}^0) = \mathbf{hy}(\mathfrak{G})$ and $\mathbf{hy}(\mathfrak{H}^0) = \mathbf{hy}(\mathfrak{H})$, we can assume that $\mathfrak{G}$ and $\mathfrak{H}$ are algebraic. Decompose $\mathfrak{f}$ as follows:

$$\mathfrak{f}: \mathfrak{G} \longrightarrow \widetilde{\mathfrak{f}(\mathfrak{G})} \hookrightarrow \mathfrak{H}.$$

Since $\mathbf{hy}(\mathfrak{f})$ is surjective, it follows from (3.3.2) that

$$\mathbf{hy}(\widetilde{\mathfrak{f}(\mathfrak{G})}) = \mathbf{hy}(\mathfrak{H}).$$

Hence the inclusion: $\widetilde{\mathfrak{f}(\mathfrak{G})} \hookrightarrow \mathfrak{H}$ is étale at $\mathfrak{e}$, and in particular flat at $\mathfrak{e}$, by Proposition 2.3.5. Since the induced map: $\mathfrak{G} \to \widetilde{\mathfrak{f}(\mathfrak{G})}$ is faithfully flat, the assertion follows.

3.3.5 PROPOSITION. *Let p be the characteristic exponent of k. Let* $\mathfrak{G}$ *be a locally algebraic k-group. The following are equivalent:*

(i) $\mathfrak{G}$ *is smooth at* $\mathfrak{e}$.

(ii) $\mathbf{hy}(\mathfrak{G}) \simeq \mathrm{B}(U)$ *as a coalgebra for some vector space U.*

(iii) $\dim \mathfrak{G} = [\mathrm{Lie}\,(\mathfrak{G}) : k]$.

(iv) $\mathfrak{F}_{\mathfrak{G}}: \mathfrak{G}\to\mathfrak{G}^{(p)}$ *the Frobenius map of* $\mathfrak{G}$ *is flat at* $\mathfrak{e}$.
(v) $\boldsymbol{Mod}_k(\mathbf{hy}(\mathfrak{G}), k^{1/p})$ *is a reduced algebra.*
In particular if $p=1$ *then any locally algebraic k-group is smooth at* $\mathfrak{e}$.

PROOF. It follows from Propositions 3.3.4 and 2.2.9 that

$$\text{(iv)}\Leftrightarrow \Upsilon_{\mathbf{hy}(\mathfrak{G})}: \mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{G})^{(p)} \text{ is surjective.}$$

Since $\dim \mathfrak{G}=K\dim \mathbf{hy}(\mathfrak{G})$ and $\text{Lie}\,(\mathfrak{G})=\text{P}(\mathbf{hy}(\mathfrak{G}))$, we have

$$\text{(iii)}\Leftrightarrow K\dim \mathbf{hy}(\mathfrak{G})=\dim_k \text{P}(\mathbf{hy}(\mathfrak{G})).$$

Therefore the assertion results from Proposition 2.3.5, Theorem 1.8.1, Proposition 1.6.1 and Theorem 1.9.4.

REMARK. As is said before the condition (i) implies that $\mathfrak{G}$ is smooth.[15]

3.3.6 PROPOSITION. *Let* $\mathfrak{G}$ *be a locally algebraic k-group and* $\mathfrak{H}$ *a subgroup of* $\mathfrak{G}$. *Then* $\mathfrak{H}$ *is an open subgroup of* $\mathfrak{G}$ *iff* $\mathbf{hy}(\mathfrak{H})=\mathbf{hy}(\mathfrak{G})$.

PROOF. It is enough to prove the "if" part. Suppose that $\mathbf{hy}(\mathfrak{H})=\mathbf{hy}(\mathfrak{G})$. Let $\bar{k}$ be the algebraic closure of k. Then $\bar{k}\otimes\mathfrak{H}$ is a subgroup of the locally algebraic $\bar{k}$-group $\bar{k}\otimes\mathfrak{G}$ and satisfies

$$\mathbf{hy}^{\bar{k}}(\bar{k}\otimes\mathfrak{H})=\mathbf{hy}^{\bar{k}}(\bar{k}\otimes\mathfrak{G})$$

by (3.1.7). Since $\mathfrak{H}$ is open in $\mathfrak{G}$ iff $\bar{k}\otimes\mathfrak{H}$ is open in $\bar{k}\otimes\mathfrak{G}$ by [7, I, § 2, 5.3], we can assume that $k=\bar{k}$.

Now let $\mathcal{O}_{\mathfrak{H},\mathfrak{e}}$ and $\mathcal{O}_{\mathfrak{G},\mathfrak{e}}$ be the fibres over $\mathfrak{e}$ of the structure sheaves $\mathcal{O}_{\mathfrak{H}}$ and $\mathcal{O}_{\mathfrak{G}}$ respectively. Since $\mathfrak{H}$ is a subscheme of $\mathfrak{G}$, the induced map:

$$\mathcal{O}_{\mathfrak{G},\mathfrak{e}}\to\mathcal{O}_{\mathfrak{H},\mathfrak{e}}$$

is surjective. But since $(\mathcal{O}_{\mathfrak{G},\mathfrak{e}})^0=\mathbf{hy}(\mathfrak{G})$ is dense in $(\mathcal{O}_{\mathfrak{G},\mathfrak{e}})^*$ by (1.1.1), it follows from the assumption $\mathbf{hy}(\mathfrak{H})=\mathbf{hy}(\mathfrak{G})$ that the above map is injective too. Thus we have

$$\mathcal{O}_{\mathfrak{G},\mathfrak{e}}\xrightarrow{\simeq}\mathcal{O}_{\mathfrak{H},\mathfrak{e}}.$$

This means that $\mathfrak{H}$ contains an open subscheme $\mathfrak{U}$ of $\mathfrak{G}$ such that $\mathfrak{e}\in\mathfrak{U}$ in view of [7, I, § 3, 4.2]. Let $\mathfrak{V}$ be the smallest open subscheme of $\mathfrak{G}$ which contains all $\mathfrak{U}x$ for $x\in\mathfrak{H}(k)$. Then $\mathfrak{V}$ is an open subscheme of $\mathfrak{H}$ and satisfies $\mathfrak{V}(k)=\mathfrak{H}(k)$. Hence $\mathfrak{H}=\mathfrak{V}$ by [7, I, § 3, 6.8]. Therefore $\mathfrak{H}$ is an open subgroup of $\mathfrak{G}$. (That $\mathfrak{V}$ is contained in $\mathfrak{H}$ follows from [7, I, § 1, 4.13]).

3.3.7 COROLLARY. *Let* $\mathfrak{f}: \mathfrak{G}\to\mathfrak{H}$ *be a map of algebraic k-groups. Then the image-subgroup* $\widetilde{\mathfrak{f}(\mathfrak{G})}$ *is an open subgroup of* $\mathfrak{H}$ *iff the induced map* $\mathbf{hy}(\mathfrak{f})$:

$\mathbf{hy}(\mathfrak{G}) \to \mathbf{hy}(\mathfrak{H})$ *is surjective. (The latter condition is equivalent to saying that* $\mathfrak{f}$ *is flat at* $\mathfrak{e}$*).*[16]

3.3.8 COROLLARY. *Let* $\mathfrak{G}$ *be a locally algebraic k-group and* $\mathfrak{H}$ *a subgroup of* $\mathfrak{G}$. *Then*

$$\mathfrak{H}^0 = \mathfrak{G}^0 \Leftrightarrow \mathbf{hy}(\mathfrak{G}) = \mathbf{hy}(\mathfrak{H})$$

PROOF. If $\mathfrak{H}^0 = \mathfrak{G}^0$ then $\mathbf{hy}(\mathfrak{H}) = \mathbf{hy}(\mathfrak{H}^0) = \mathbf{hy}(\mathfrak{G}^0) = \mathbf{hy}(\mathfrak{G})$. If $\mathbf{hy}(\mathfrak{H}) = \mathbf{hy}(\mathfrak{G})$ then $\mathfrak{H}$ is an open subgroup of $\mathfrak{G}$. Hence $\mathfrak{H}^0$ is open in $\mathfrak{G}^0$. Since $\mathfrak{G}^0$ is algebraic $\mathfrak{H}^0$ is closed too. Since $\mathfrak{G}^0$ is connected $\mathfrak{H}^0 = \mathfrak{G}^0$.

3.3.9 Let $\mathfrak{G}$ be a locally algebraic k-group. Then for any subgroup $\mathfrak{H}$ of $\mathfrak{G}$ we have $\mathbf{hy}(\mathfrak{H}) = \mathbf{T}(\mathfrak{H}) \cap \mathbf{hy}(\mathfrak{G})$, since $\mathbf{hy}(\mathfrak{H}) = \mathbf{T}(\mathfrak{H})^1$. In particular if $\mathfrak{K}$ is another subgroup of $\mathfrak{G}$, we have

$$\mathbf{hy}(\mathfrak{H} \cap \mathfrak{K}) = \mathbf{hy}(\mathfrak{H}) \cap \mathbf{hy}(\mathfrak{K})$$

since $\mathbf{T}(\mathfrak{H} \cap \mathfrak{K}) = \mathbf{T}(\mathfrak{H}) \cap \mathbf{T}(\mathfrak{K})$ by Proposition 2.2.1.

COROLLARY. *Let* $\mathfrak{G}$ *be a locally algebraic k-group and* $\mathfrak{H}, \mathfrak{K}$ *two subgroups of* $\mathfrak{G}$. *Then we have*

$$\mathfrak{H}^0 \subset \mathfrak{K} \Leftrightarrow \mathbf{hy}(\mathfrak{H}) \subset \mathbf{hy}(\mathfrak{K}).$$

PROOF. $\mathbf{hy}(\mathfrak{H}) \subset \mathbf{hy}(\mathfrak{K}) \Leftrightarrow \mathbf{hy}(\mathfrak{H}) = \mathbf{hy}(\mathfrak{H} \cap \mathfrak{K}) \Leftrightarrow \mathfrak{H}^0 = (\mathfrak{H} \cap \mathfrak{K})^0 \Leftrightarrow \mathfrak{H}^0 \subset \mathfrak{K}$.

3.3.10 COROLLARY. *Let* $\mathfrak{f}_1$ *and* $\mathfrak{f}_2$ *be maps of locally algebraic k-groups*: $\mathfrak{G} \rightrightarrows \mathfrak{G}'$. *If* $\mathfrak{G}$ *is connected then*

$$\mathfrak{f}_1 = \mathfrak{f}_2 \Leftrightarrow \mathbf{hy}(\mathfrak{f}_1) = \mathbf{hy}(\mathfrak{f}_2) : \mathbf{hy}(\mathfrak{G}) \rightrightarrows \mathbf{hy}(\mathfrak{G}').$$ [17]

PROOF. The equalizer $\mathfrak{K} = \mathfrak{Ker}\,(\mathfrak{f}_1, \mathfrak{f}_2)$ is a subgroup of $\mathfrak{G}$ by [7, I, § 2, 5.6]. Since $\mathbf{T}(\mathfrak{K})$ is the equalizer Ker $(\mathbf{T}(\mathfrak{f}_1), \mathrm{T}(\mathfrak{f}_2))$ by Proposition 2.2.1, it follows easily that

$$\mathbf{hy}(\mathfrak{K}) = \mathrm{Ker}\ (\mathbf{hy}(\mathfrak{f}_1), \mathbf{hy}(\mathfrak{f}_2))$$

(cf. (3.3.9)). Hence we have

$$\begin{aligned} \mathbf{hy}(\mathfrak{f}_1) = \mathbf{hy}(\mathfrak{f}_2) &\Leftrightarrow \mathbf{hy}(\mathfrak{K}) = \mathbf{hy}(\mathfrak{G}) \\ &\Leftrightarrow \mathfrak{K}^0 = \mathfrak{G}^0 = \mathfrak{G} \\ &\Leftrightarrow \mathfrak{G} = \mathfrak{K}. \end{aligned}$$

3.3.11 For a k-scheme $\mathfrak{X}$ we denote by $\mathfrak{X}_{\mathrm{red}}$ the *reduced part* of $\mathfrak{X}$ which is the smallest closed subscheme of $\mathfrak{X}$ such that $|\mathfrak{X}| = |\mathfrak{X}_{\mathrm{red}}|$ [7, I, § 2, 4.10].

Let $\mathfrak{G}$ be a locally algebraic k-group. Suppose that k is *perfect*. Then

it is known that $\mathfrak{G}_{\mathrm{red}}$ is a smooth subgroup of $\mathfrak{G}$ [7, II, § 5, 2.3]. Hence $\mathfrak{G}_{\mathrm{red}}$ is the largest smooth subgroup of $\mathfrak{G}$.

PROPOSITION. *Suppose that k is perfect. Let $\mathfrak{G}$ be a locally algebraic k-group. Then* $\mathbf{hy}(\mathfrak{G}_{\mathrm{red}})$ *the hyperalgebra of the reduced part of $\mathfrak{G}$ is* $\mathbf{hy}(\mathfrak{G})_{\mathrm{red}}$ *the reduced part of* $\mathbf{hy}(\mathfrak{G})$ *in the sense of* (1.9.5).

PROOF. Since $\mathbf{hy}(\mathfrak{G}_{\mathrm{red}})$ is reduced in the sense of (1.9.5), we have $\mathbf{hy}(\mathfrak{G}_{\mathrm{red}})\subset\mathbf{hy}(\mathfrak{G})_{\mathrm{red}}$. In order to show $\mathbf{hy}(\mathfrak{G})_{\mathrm{red}}\subset\mathbf{hy}(\mathfrak{G}_{\mathrm{red}})$ we can assume that $\mathfrak{G}=\mathfrak{G}^0$, since $(\mathfrak{G}^0)_{\mathrm{red}}=(\mathfrak{G}_{\mathrm{red}})^0$. The image-subgroup of the Frobenius map $\mathfrak{F}_{\mathfrak{G}}:\mathfrak{G}\to\mathfrak{G}^{(p)}$, which is well-defined because $\mathfrak{G}$ is algebraic, is of the form $\mathfrak{G}'^{(p)}$ for some uniquely determined subgroup $\mathfrak{G}'$ of $\mathfrak{G}$, since k is perfect. If we put inductively

$$\mathfrak{G}_{(0)}=\mathfrak{G},\ \mathfrak{G}_{(n+1)}=(\mathfrak{G}_{(n)})'$$

then $\{\mathfrak{G}_{(n)}\}$ form a descending chain of closed subgroups of $\mathfrak{G}$ and satisfy $\mathbf{hy}(\mathfrak{G}_{(n)})=\mathbf{hy}(\mathfrak{G})_{(n)}$ in the same notation as in Proof 1.9.5. Since $\mathfrak{G}$ is algebraic there exists an integer N such that $\mathfrak{G}_{(N)}=\mathfrak{G}_{(N+1)}$. Then $\mathfrak{G}_{(N)}$ is smooth by Proposition 3.3.5 (iv) and hence contained in $\mathfrak{G}_{\mathrm{red}}$. Therefore we have by Proof 1.9.5

$$\mathbf{hy}(\mathfrak{G})_{\mathrm{red}}\subset\mathbf{hy}(\mathfrak{G})_{(N)}=\mathbf{hy}(\mathfrak{G}_{(N)})\subset\mathbf{hy}(\mathfrak{G}_{\mathrm{red}}).$$

3.3.12 In view of Theorem 1.9.6 we have:

COROLLARY. *Suppose that k is perfect. Then for any locally algebraic k-group $\mathfrak{G}$*

$$\mathrm{Lie}(\mathfrak{G}_{\mathrm{red}})=\{l\in\mathrm{Lie}(\mathfrak{G})\mid \textit{There is an } \infty\textit{-sequence of divided powers in } \mathbf{hy}(\mathfrak{G}) \textit{ lying over } l\}.$$

3.4 $\mathbf{hy}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H}))$ and $\mathbf{hy}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H}))$

Let $R\in\boldsymbol{M}_k$. We shall denote by $\boldsymbol{M}_R$ the category of (small) commutative R-algebras, which can be naturally identified with $\boldsymbol{M}_k/R$ the category of pairs (S,ϕ) with $S\in\boldsymbol{M}_k$ and $\phi\in\boldsymbol{M}_k(R,S)$.

Let $\mathfrak{G}$ be a k-group-functor having the hyperalgebra $\mathbf{hy}(\mathfrak{G})$. The canonical isomorphism:

$$W_k(C,\mathbf{hy}(\mathfrak{G}))\xrightarrow{\simeq}\mathrm{Ker}\,(\mathfrak{G}(C^*)\longrightarrow\mathfrak{G}(C_0^*))$$

will be denoted by

$$\sigma\mapsto\exp\,(\sigma,C)$$

where $C \in W_k^f$. If in particular $\mathbf{hy}(\mathfrak{G})$ is in the strong sense, then

$$\exp(\sigma, C) = \exp(\sigma, k, C)$$

in the notation of (2.1.8).

3.4.1 Let $\mathfrak{G}$ be a k-group-functor. A *k-$\mathfrak{G}$-module hyperalgebra* is a pair (I, ρ) where I is a hyperalgebra and

$$\rho : \mathfrak{G} \to \mathfrak{G}\mathfrak{L}(I)$$

is a linear representation of $\mathfrak{G}$ on I such that $\mathfrak{G}(R)$ acts on $R \otimes I$ as R-Hopf algebra automorphisms for any $R \in \boldsymbol{M}_k$.

Let (I, ρ) be a k-$\mathfrak{G}$-module hyperalgebra. We shall write

$$a \rightharpoonup x = \rho(a)(x) \quad \text{and} \quad [a, x] = \sum (a \rightharpoonup x_{(1)}) S_I(x_{(2)})$$

for $R \in \boldsymbol{M}_k$, $a \in \mathfrak{G}(R)$ and $x \in R \otimes I$, where S_I denotes the antipode of I. A simple calculation shows that

$$\begin{aligned}
[ab, x] &= \sum [a, b \rightharpoonup x_{(1)}][b, x_{(2)}] \\
[a, xy] &= \sum (a \rightharpoonup x_{(1)})[a, y] S_I(x_{(2)}) \quad \text{and} \\
a \rightharpoonup x &= \sum [a, x_{(1)}] x_{(2)}
\end{aligned}$$

for $a, b \in \mathfrak{G}(R)$ and $x, y \in R \otimes I$ where $R \in \boldsymbol{M}_k$.

Let A be a subalgebra and C a subcoalgebra of I such that

$$A \cdot C \subset C.$$

We define a sub-functor $\mathfrak{G}_{A,C}$ of $\mathfrak{G}$ as follows: For $R \in \boldsymbol{M}_k$, $\mathfrak{G}_{A,C}(R)$ is the set of $a \in \mathfrak{G}(R)$ such that

$$[a, R \otimes C] \subset R \otimes A.$$

We claim that $\mathfrak{G}_{A,C}$ is a sub-monoid-functor of $\mathfrak{G}$ such that C is $\mathfrak{G}_{A,C}$-stable. Indeed let $a, b \in \mathfrak{G}_{A,C}(R)$. Then

$$\begin{aligned}
&a \rightharpoonup (R \otimes C) \subset [a, R \otimes C](R \otimes C) \subset R \otimes AC \subset R \otimes C. \quad \text{and} \\
&[ab, R \otimes C] \subset [a, b \rightharpoonup (R \otimes C)][b, R \otimes C] \\
&\qquad \subset [a, R \otimes C][b, R \otimes C] \subset R \otimes AA \subset R \otimes A.
\end{aligned}$$

Hence C is $\mathfrak{G}_{A,C}$-stable and $ab \in \mathfrak{G}_{A,C}(R)$. Since $1 \in \mathfrak{G}_{A,C}(R)$ clearly, it follows that $\mathfrak{G}_{A,C}$ is a sub-monoid-functor of $\mathfrak{G}$.

Let J be a sub-hyperalgebra of I which is $\mathfrak{G}$-stable. We put

$$V = \{x \in I \mid [\mathfrak{G}(R), x] \subset R \otimes J \text{ for any } R \in \boldsymbol{M}_k\}.$$

Since

$$[\mathfrak{G}(R), JV] \subset (\mathfrak{G}(R) \rightharpoonup J)[\mathfrak{G}(R), V]J \subset R\otimes J,$$

we have $JV \subset V$. Let

$$\mathbf{X}_I(\mathfrak{G}, J)$$

denote the largest subcoalgebra of I contained in V. It follows that

$$J\cdot \mathbf{X}_I(\mathfrak{G}, J) \subset \mathbf{X}_I(\mathfrak{G}, J) \quad \text{and} \quad J \subset \mathbf{X}_I(\mathfrak{G}, J).$$

Now Lemma 2.4.5 implies that for any $R \in \boldsymbol{M}_k$, $R\otimes V$ is the set of $x \in R\otimes I$ such that

$$[\mathfrak{G}(S), x] \subset S\otimes J$$

for any $S \in \boldsymbol{M}_R$. Let $(R, D) \in \boldsymbol{W}$ and $\sigma \in \boldsymbol{W}_R(D, I)$. It follows from Lemma 1.2.5 that

$$\sigma(R\otimes D) \subset R\otimes V \Leftrightarrow \sigma(R\otimes D) \subset R\otimes \mathbf{X}_I(\mathfrak{G}, J).$$

Therefore the set $\boldsymbol{W}_R(D, \mathbf{X}_I(\mathfrak{G}, J))$ can be canonically identified with the set of $\sigma \in \boldsymbol{W}_R(D, I)$ such that

$$(\rho(g)\circ \boldsymbol{W}_\phi(\sigma)) * (\boldsymbol{W}_\eta(S_I)\circ \boldsymbol{W}_\phi(\sigma)) \in \boldsymbol{W}_S(D, J)$$

for any $S \in \boldsymbol{M}_k$, $\phi \in \boldsymbol{M}_k(R, S)$ and $g \in \mathfrak{G}(S)$.

3.4.2 LEMMA. *Let $\mathfrak{X}$ be a k-functor and V, W vector spaces. Let*

$$\mathrm{u}: \mathfrak{X}\times V_a \rightarrow W_a$$

be a map of k-functors such that for any $R \in \boldsymbol{M}_k$ and $a \in \mathfrak{X}(R)$, the map:

$$R\otimes V \rightarrow R\otimes W, \ x \mapsto \mathrm{u}(a, x)$$

is R-linear. Let W' be a subspace of W. Let $\mathfrak{Y}$ be the sub-functor of V_a defined by

$$\mathfrak{Y}(R) = \{x \in R\otimes V \mid \mathrm{u}(\mathfrak{X}(S), x) \subset S\otimes W' \textit{ for any } S \in \boldsymbol{M}_R\}.$$

Then we have $\mathfrak{Y} = V'_a$, where

$$V' = \mathfrak{Y}(k).$$

PROOF. $\mathfrak{Y}$ clearly contains V'_a. Let $R \in \boldsymbol{M}_k$ and $x \in \mathfrak{Y}(R)$. Write

$$x = \sum r_i \otimes x_i$$

with $r_i \in R$ linearly independent and $x_i \in V$. Let $S \in \boldsymbol{M}_k$ and $a \in \mathfrak{X}(S)$. Then we have

$$\sum r_i \otimes \mathfrak{u}(a, x_i) \in R \otimes S \otimes W'$$

since $R \otimes S \in \boldsymbol{M}_R$. This means that each x_i belongs to V'. Hence $Y = V'_a$.

3.4.3 Let $\mathfrak{G}$ and $\mathfrak{J}$ be k-group-functors and

$$\mathfrak{u} : \mathfrak{G} \times \mathfrak{J} \to \mathfrak{J}$$

an action of $\mathfrak{G}$ on $\mathfrak{J}$ (3.1.6). For $a \in \mathfrak{G}(R)$ and $x \in \mathfrak{J}(R)$, where $R \in \boldsymbol{M}_k$, we shall write

$$a \rightharpoonup x = \mathfrak{u}(a, x) \quad \text{and} \quad [a, x] = (a \rightharpoonup x)x^{-1}.$$

Let $\mathfrak{K}$ be a sub-group-functor of $\mathfrak{J}$ which is $\mathfrak{G}$-stable. A sub-functor $\mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K})$ of $\mathfrak{J}$ is defined as follows: For any $R \in \boldsymbol{M}_k$, $\mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K})(R)$ is the set of $x \in \mathfrak{J}(R)$ such that

$$[\mathfrak{G}(S), x_S] \subset \mathfrak{K}(S)$$

for any $S \in \boldsymbol{M}_R$. It is easy to show that $\mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K})$ contains $\mathfrak{K}$ and that

$$\mathfrak{K} \cdot \mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K}) \subset \mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K}).$$

Suppose that $\mathfrak{J}$ has the hyperalgebra in the strong sense $\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$ (3.1.3). We have seen in (3.1.6) that the action $\mathfrak{u}$ induces a natural linear action:

$$\mathfrak{G} \times \mathbf{hy}^{\mathrm{st}}(\mathfrak{J})_a \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{J})_a$$

such that $\mathfrak{G}(R)$ acts on $R \otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$ as R-Hopf algebra automorphisms. Thus the hyperalgebra $\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$ turns into a k-$\mathfrak{G}$-module hyperalgebra. Let $\mathfrak{K}$ be a $\mathfrak{G}$-stable sub-group-functor of $\mathfrak{J}$. Suppose that $\mathfrak{K}$ also has the hyperalgebra in the strong sense. Then the induced map:

$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{K}) \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$$

is injective by Proposition 2.2.5. Hence $\mathbf{hy}^{\mathrm{st}}(\mathfrak{K})$ can be identified with a $\mathfrak{G}$-stable sub-hyperalgebra of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$.

PROPOSITION. $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K})) = \mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})}(\mathfrak{G}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{K}))$.

PROOF. Let $(R, C) \in W^{\mathrm{f}}$. Consider the canonical injection:

$$W_R(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{J})) \hookrightarrow \mathfrak{J}(R \otimes C^*), \sigma \longmapsto \exp(\sigma, R, C).$$

We have shown in (3.4.1) that

$$\begin{aligned} &\sigma \in W_R(C, \mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})}(\mathfrak{G}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{K}))) \\ &\Leftrightarrow (\rho(g) \circ W_\phi(\sigma)) * (W_\eta(S_{\mathfrak{J}}) \circ W_\phi(\sigma)) \in W_S(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{K})) \\ &\qquad\qquad \text{for any } S \in \boldsymbol{M}_k, \phi \in \boldsymbol{M}_k(R, S) \text{ and } g \in \mathfrak{G}(S) \end{aligned}$$

where $S_{\mathfrak{Z}}$ denotes the antipode of the hyperalgebra $\mathbf{hy}^{\mathrm{st}}(\mathfrak{Z})$ and ρ the induced linear representation of $\mathfrak{G}$ on $\mathbf{hy}^{\mathrm{st}}(\mathfrak{Z})$. If we notice that

$$\begin{aligned}&\exp\,((\rho(g)\circ W_{\phi}(\sigma))*(W_{\eta}(S_{\mathfrak{Z}})\circ W_{\phi}(\sigma)), S, C)\\&=(g_{S\otimes C^*}\rightharpoonup\exp(\sigma, R, C)_{S\otimes C^*})\,\exp\,(\sigma, R, C)^{-1}_{S\otimes C^*}\end{aligned}$$

then it follows from Lemma below that

$$\begin{aligned}&\sigma\in W_R(C, \mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{Z})}(\mathfrak{G}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{K})))\\&\Leftrightarrow(g_{S\otimes C^*}\rightharpoonup\exp\,(\sigma, R, C)_{S\otimes C^*})\,\exp\,(\sigma, R, C)^{-1}_{S\otimes C^*}\in\mathfrak{K}(S\otimes C^*)\\&\qquad\qquad\qquad\text{for any } S\in \boldsymbol{M}_R \text{ and } g\in\mathfrak{G}(S)\\&\Leftrightarrow\exp\,(\sigma, R, C)\in\mathfrak{X}_{\mathfrak{Z}}(\mathfrak{G}, \mathfrak{K})(R\otimes C^*).\end{aligned}$$

This means $\mathbf{T}^{\mathrm{st}}_{e}(\mathfrak{X}_{\mathfrak{Z}}(\mathfrak{G}, \mathfrak{K}))=\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{Z})}(\mathfrak{G}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{K}))$.

3.4.4 LEMMA. *Let $\mathfrak{G}, \mathfrak{Z}$ and $\mathfrak{K}$ be as above. Let $R, T\in \boldsymbol{M}_k$ and $x\in \mathfrak{Z}(R\otimes T)$. Then $x\in\mathfrak{X}_{\mathfrak{Z}}(\mathfrak{G}, \mathfrak{K})(R\otimes T)$ iff $[a_{S\otimes T}, x_{S\otimes T}]\in\mathfrak{K}(S\otimes T)$ for any $S\in \boldsymbol{M}_R$ and $a\in\mathfrak{G}(S)$.*

PROOF. Exercise (cf. [7, II, § 1, 3.5]).

3.4.5 Let $\mathfrak{G}$ be a k-group-functor. Suppose that $\mathbf{hy}(\mathfrak{G})$ the hyperalgebra of $\mathfrak{G}$ exists. A pair (V, ρ) with V a vector space and $\rho: \mathfrak{G}\to\mathfrak{G}\mathfrak{L}(V)$ a map of k-group-functors is called a k-$\mathfrak{G}$-module. Such a pair determines a structure of $\mathbf{hy}(\mathfrak{G})$-module on V as is seen in (3.2.5). Let V and W be k-$\mathfrak{G}$-modules. If we view $V\otimes W$ as a k-$\mathfrak{G}$-module by pull back along the diagonal map: $\mathfrak{G}\to\mathfrak{G}\times\mathfrak{G}$, then the induced $\mathbf{hy}(\mathfrak{G})$-module $V\otimes W$ is obtained from the $(\mathbf{hy}(\mathfrak{G})\otimes\mathbf{hy}(\mathfrak{G}))$-module $V\otimes W$ by pull back along the structure map $\Delta: \mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{G})\otimes\mathbf{hy}(\mathfrak{G})$. If $f: V\to W$ is a map of k-$\mathfrak{G}$-modules, then f is also $\mathbf{hy}(\mathfrak{G})$-linear.

Now let I be a k-$\mathfrak{G}$-module hyperalgebra. Then by definition, the structure maps

$$\begin{aligned}&\Delta: I\to I\otimes I,\quad \mu: I\otimes I\to I,\quad S_I: I\to I,\\&\eta: k\to I\quad\text{and}\quad\varepsilon: I\to k\end{aligned}$$

are k-$\mathfrak{G}$-module maps. Hence they are $\mathbf{hy}(\mathfrak{G})$-linear also. This means that I becomes a $\mathbf{hy}(\mathfrak{G})$-module hyperalgebra in the sense of (1.10.1).

Let A be a subalgebra and C a subcoalgebra of I such that

$$A\cdot C\subset C.$$

We have defined a sub-hyperalgebra $\mathbf{hy}(\mathfrak{G})_{A,C}$ of $\mathbf{hy}(\mathfrak{G})$ in (1.10.4).

PROPOSITION. $\mathbf{hy}(\mathfrak{G}_{A,C})=\mathbf{hy}(\mathfrak{G})_{A,C}$. *If $\mathbf{hy}(\mathfrak{G})$ is in the strong sense, then*

$\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}_{A,C})=\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})_{A,C}$.

PROOF. Let $D\in W_k^{\mathrm{f}}$ and $\sigma\in W_k(D,\mathbf{hy}(\mathfrak{G}))$. Then we have

$$\begin{aligned}\exp(\sigma,D)\in\mathfrak{G}_{A,C}(D^*)&\Leftrightarrow[\exp(\sigma,D),D^*\otimes C]\subset D^*\otimes A\\&\Leftrightarrow\sum(\exp(\sigma,D)\rightharpoonup c_{(1)})S_I(c_{(2)})\in D^*\otimes A\qquad\text{for any } c\in C.\end{aligned}$$

If we identify $D^*\otimes I$ with $\boldsymbol{Mod}_k(D,I)$ naturally, Lemma 3.2.5 means that

$$(\exp(\sigma,D)\rightharpoonup c)(d)=\sigma(d)\rightharpoonup c$$

for $c\in C$ and $d\in D$. Hence we have

$$\begin{aligned}&\exp(\sigma,D)\in\mathfrak{G}_{A,C}(D^*)\\&\Leftrightarrow\sum(\sigma(d)\rightharpoonup c_{(1)})S_I(c_{(2)})\in A\quad\text{for any } d\in D\quad\text{and}\quad c\in C\\&\Leftrightarrow[\sigma(D),C]\subset A\\&\Leftrightarrow\sigma(D)\subset\mathbf{hy}(\mathfrak{G})_{A,C}.\end{aligned}$$

This implies that $\mathbf{hy}(\mathfrak{G}_{A,C})=\mathbf{hy}(\mathfrak{G})_{A,C}$.

Suppose next that $\mathbf{hy}(\mathfrak{G})$ is in the strong sense. Let $(R,D)\in W^{\mathrm{f}}$ and $\sigma\in W_R(D,\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}))$. Then we have

$$\begin{aligned}&\exp(\sigma,R,D)\in\mathfrak{G}_{A,C}(R\otimes D^*)\\&\Leftrightarrow\sum(\exp(\sigma,R,D)\rightharpoonup c_{(1)})S_I(c_{(2)})\in R\otimes D^*\otimes A\qquad\text{for any } c\in C.\end{aligned}$$

If we identify $R\otimes D^*\otimes I$ with $\boldsymbol{Mod}_R(R\otimes D,R\otimes I)$ naturally, then it follows from Lemma below that

$$(\exp(\sigma,R,D)\rightharpoonup c)(d)=\sigma(d)\rightharpoonup c$$

for $c\in R\otimes I$ and $d\in R\otimes D$. Hence we have

$$\exp(\sigma,R,D)\in\mathfrak{G}_{A,C}(R\otimes D^*)\Leftrightarrow[\sigma(R\otimes D),R\otimes C]\subset R\otimes A.$$

Put

$$V=\{x\in\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})\mid[x,C]\subset A\}.$$

Then it is easy to show that

$$R\otimes V=\{x\in R\otimes\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})\mid[x,R\otimes C]\subset R\otimes A\}.$$

Since $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})_{A,C}$ is the largest subcoalgebra of V, it follows from Lemma 1.2.5 that

$$\begin{aligned}[\sigma(R\otimes D),R\otimes C]\subset R\otimes A&\Leftrightarrow\sigma(R\otimes D)\subset R\otimes V\\&\Leftrightarrow\sigma(R\otimes D)\subset R\otimes\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})_{A,C}.\end{aligned}$$

Therefore we have

$$\exp(\sigma, R, D) \in \mathfrak{G}_{A,C}(R\otimes D^*) \Leftrightarrow \sigma(R\otimes D) \subset R\otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})_{A,C}.$$

This means that $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}_{A,C}) = \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})_{A,C}$.

3.4.6 LEMMA. *Let $\mathfrak{G}$ be a k-group-functor which has the hyperalebra in the strong sense $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$. Let (V, ρ) be a k-$\mathfrak{G}$-module. (Hence V becomes a $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$-module naturally). Let $(R, C) \in W^{\mathrm{f}}$ and $\sigma \in W_R(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{G}))$. If we identify $R\otimes C^*\otimes V$ with $\boldsymbol{Mod}_R(R\otimes C, R\otimes V)$ canonically, then we have*

$$\sigma(c) \rightharpoonup x = (\rho(\exp(\sigma, R, C))(x))(c)$$

foy $c \in R\otimes C$ and $x \in R\otimes V$, where '$\rightharpoonup$' denotes the induced action of $R\otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ on $R\otimes V$.

PROOF. Since C is finite dimensional, there exist a finite dimensional subcoalgebra D of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ and $\sigma' \in W_R(C, D)$ such that

$$\sigma = W_\eta(i) \circ \sigma'$$

where $i: D \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ is the canonical injection. Lemma 3.2.5 implies that

$$(\rho(\exp(i, k, D))(x))(d) = d \rightharpoonup x$$

for $d \in D$ and $x \in V$, under the identification:

$$D^*\otimes V \simeq \boldsymbol{Mod}_k(D, V).$$

Since we have

$$\exp(W_\eta(i), R, D) = \exp(i, k, D)_{R\otimes D^*},$$

it follows that

$$(\rho(\exp(W_\eta(i), R, D))(x))(d) = d \rightharpoonup x \in R\otimes V$$

for $d \in R\otimes D$ and $x \in R\otimes V$. Since $\exp(\sigma, R, C)$ is equal to $\exp(W_\eta(i), R, D)_{R\otimes C^*}$ taken with respect to the R-algebra map

$$\boldsymbol{Mod}_R(\sigma, R): R\otimes D^* \to R\otimes C^*,$$

it follows that

$$(\rho(\exp(\sigma, R, C))(x))(c) = \sigma(c) \rightharpoonup x$$

for $c \in R\otimes C$ and $x \in R\otimes V$.

3.4.7 Let G be a hyperalgebra and I a G-module hyperalgebra (1.10.1).

Let J be a G-stable sub-hyperalgebra of I. Put

$$V=\{x\in I \mid [G, x]\subset J\}$$

where $[a, x]=\sum(a\rightharpoonup x_{(1)})S_I(x_{(2)})$ for $a\in G$ and $x\in I$ (1.10.4). We denote by $\mathbf{X}_I(G, J)$ the largest subcoalgebra of I contained in V. Then we have

$$J\subset\mathbf{X}_I(G, J) \quad \text{and} \quad J\cdot\mathbf{X}_I(G, J)\subset\mathbf{X}_I(G, J).$$

PROPOSITION. *Let $\mathfrak{G}$ be a connected algebraic k-group and I a k-$\mathfrak{G}$-module hyperalgebra. (Then I is a* $\mathbf{hy}(\mathfrak{G})$*-module hyperalgebra). Let J be a $\mathfrak{G}$-stable sub-hyperalgebra of I. Then we have*

$$\mathbf{X}_I(\mathfrak{G}, J)=\mathbf{X}_I(\mathbf{hy}(\mathfrak{G}), J).$$

PROOF. Let Φ (resp. Ψ) be the set of subcoalgebras C of I such that $J\subset C$, $J\cdot C\subset C$ and $\mathfrak{G}=\mathfrak{G}_{J,C}$ (resp. $\mathbf{hy}(\mathfrak{G})=\mathbf{hy}(\mathfrak{G})_{J,C}$). Then by definition, $\mathbf{X}_I(\mathfrak{G}, J)$ (resp. $\mathbf{X}_I(\mathbf{hy}(\mathfrak{G}), J)$) is the largest element of Φ (resp. Ψ). Hence we have only to prove $\Phi=\Psi$. The inclusion $\Phi\subset\Psi$ always holds by Proposition 3.4.5. But $\mathfrak{G}_{J,C}$ is a closed subgroup of $\mathfrak{G}$ by Lemmas below. Hence if $\mathbf{hy}(\mathfrak{G})=\mathbf{hy}(\mathfrak{G}_{J,C})$, then $\mathfrak{G}_{J,C}=\mathfrak{G}$ by Corollary 3.3.8, since $\mathfrak{G}$ is connected. This proves Proposition.

3.4.8 LEMMA. *Let $\mathfrak{G}$ be a k-group-functor and I a k-$\mathfrak{G}$-module hyperalgebra. For a subalgebra A and a subcoalgebra C of I such that $A\cdot C\subset C$, the sub-monoid-functor $\mathfrak{G}_{A,C}$ of $\mathfrak{G}$ is a closed submonoid.*

PROOF. Let $R\in \boldsymbol{M}_k$ and $g\in\mathfrak{G}(R)$. Form a pullback diagram as follows:

$$\begin{array}{ccc} \mathfrak{G}_{A,C} & \hookrightarrow & \mathfrak{G} \\ \uparrow & & \uparrow g \\ \mathfrak{Z} & \hookrightarrow & \mathfrak{Sp}\, R. \end{array}$$

It is enough to show thot $\mathfrak{Z}=\mathfrak{Sp}\,(R/m)$ for some ideal m of R. Let A' be a subspace of I such that $I=A\oplus A'$. Take a basis $\{e_\lambda\}$ for A'. There exists a linear map $f_\lambda\colon C\to R$ for each λ such that

$$[g, c]\equiv\sum_\lambda f_\lambda(c)\otimes e_\lambda \bmod R\otimes A$$

for all $c\in C$. Put

$$m=\sum_{\lambda,c} R\cdot f_\lambda(c).$$

Let $\phi\in \boldsymbol{M}_k(R, S)$. Then we have

$$\begin{aligned}\phi \in \mathfrak{Z}(S) &\Leftrightarrow g_S \text{ (taken with respect to } \phi) \in \mathfrak{G}_{A,C}(S)\\ &\Leftrightarrow [g_S, C] \subset S \otimes A \Leftrightarrow \phi(f_\lambda(c)) = 0 \qquad \text{for any } \lambda, c.\end{aligned}$$

Hence we have $\mathfrak{Z} = \mathfrak{Sp}\,(R/m)$.

3.4.9 LEMMA. *Let $\mathfrak{G}$ be an algebraic k-group. Then any closed sub-monoid of $\mathfrak{G}$ is a sub-group.*

PROOF. Let $\mathfrak{H}$ be a closed sub-monoid of $\mathfrak{G}$ and $\mathfrak{H}'$ the largest subgroup-functor of $\mathfrak{H}$. Then we have a pullback diagram:

$$\begin{array}{ccc}\mathfrak{H} & \xrightarrow{\mathfrak{i}} & \mathfrak{G}\\ \cup & & \cup\\ \mathfrak{H}' & \longrightarrow & \mathfrak{H}\end{array}$$

where $\mathfrak{i} = \mathfrak{i}_{\mathfrak{G}} | \mathfrak{H}$ (3.1.1). Therefore $\mathfrak{H}'$ is a closed subgroup of $\mathfrak{G}$. Since $\mathbf{hy}(\mathfrak{H}) = \mathbf{hy}(\mathfrak{H}')$, the same method as in Proposition 3.3.6 shows that $\mathfrak{H}'$ is open in $\mathfrak{H}$. (In fact Proposition 3.3.6 holds for a locally algebraic *k-monoid* $\mathfrak{G}$ and its sub-*group* $\mathfrak{H}$). In order to say $\mathfrak{H} = \mathfrak{H}'$ we can assume that k is algebraically closed, since $\bar{k} \otimes \mathfrak{H} = \bar{k} \otimes \mathfrak{H}'$ implies $\mathfrak{H} = \mathfrak{H}'$. But then we have only to prove $\mathfrak{H}(k) = \mathfrak{H}'(k)$ in view of [7, I, § 3, 6.8]. Let $x \in \mathfrak{H}(k)$. Then $\{x^n \mathfrak{H}\}_n$ form a descending chain of closed subschemes of $\mathfrak{G}$. Since $\mathfrak{G}$ is algebraic, this must be stationary. This means that x is invertible in $\mathfrak{H}(k)$. Hence $\mathfrak{H}(k) = \mathfrak{H}'(k)$. This proves Lemma.

3.4.10 Let $\mathfrak{G}, \mathfrak{J}$ be k-group-functors and

$$\mathfrak{u} : \mathfrak{G} \times \mathfrak{J} \to \mathfrak{J}$$

an action of $\mathfrak{G}$ on $\mathfrak{J}$. Let $\mathfrak{K}$ be a $\mathfrak{G}$-stable sub-group-functor of $\mathfrak{J}$.

COROLLARY. *Suppose that the hyperalgebras in the strong sense $\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$ and $\mathbf{hy}^{\mathrm{st}}(\mathfrak{K})$ exist and that $\mathfrak{G}$ is a connected algebraic k-group. Then we have*

$$\mathbf{T}^{\mathrm{st}}_{\mathfrak{e}}(\mathfrak{X}_{\mathfrak{J}}(\mathfrak{G}, \mathfrak{K})) = \mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})}(\mathfrak{G}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{K})) = \mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}), \mathbf{hy}^{\mathrm{st}}(\mathfrak{K})).$$

3.4.11 Let $\mathfrak{G}, \mathfrak{J}$ be k-group-functors and

$$\mathfrak{u} : \mathfrak{G} \times \mathfrak{J} \to \mathfrak{J}$$

an action of $\mathfrak{G}$ on $\mathfrak{J}$. Suppose that $\mathbf{hy}(\mathfrak{G})$ and $\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$ exist. Then $\mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$ becomes a k-$\mathfrak{G}$-module hyperalgebra by (3.4.3) and so a $\mathbf{hy}(\mathfrak{G})$-module hyperalgebra by (3.4.5).

LEMMA. *The induced coalgebra map*

$$\mathbf{S}_{(\mathfrak{e},\mathfrak{e})}(\mathfrak{u}) : \mathbf{hy}(\mathfrak{G}) \otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{J}) \simeq \mathbf{S}_{(\mathfrak{e},\mathfrak{e})}(\mathfrak{G} \times \mathfrak{J}) \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{J})$$

coincides with the structure map of the $\mathbf{hy}(\mathfrak{G})$*-module hyperalgebra* $\mathbf{hy}^{\mathrm{st}}(\mathfrak{F})$.

PROOF. Let $\rho: \mathfrak{G} \to \mathfrak{G}\mathfrak{L}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{F}))$ be the structure map of the k-$\mathfrak{G}$-module hyperalgebra $\mathbf{hy}^{\mathrm{st}}(\mathfrak{F})$. Let $(R, C) \in W^{\mathrm{f}}$, $\sigma \in W_k(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{F}))$ and $g \in \mathfrak{G}(R)$. Then we have by definition

$$\mathfrak{u}(g_{R\otimes C^*}, \exp(\sigma, C)_{R\otimes C^*}) = \exp(\rho(g) \circ W_\eta(\sigma), R, C).$$

Let $D \in W_k^{\mathrm{f}}$ and $\tau \in W_k(D, \mathbf{hy}(\mathfrak{G}))$. Put

$$R = D^* \quad \text{and} \quad g = \exp(\tau, D).$$

Then we have

$$\begin{aligned} &\mathfrak{u}(\exp(\tau, D)_{D^*\otimes C^*}, \exp(\sigma, C)_{D^*\otimes C^*}) \\ &\quad = \exp(\rho(\exp(\tau, D) \circ W_\eta(\sigma), D^*, C). \end{aligned}$$

Put $\nu = \rho(\exp(\tau, D)) \circ W_\eta(\sigma) \in W_{D^*}(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{F}))$. Let

$$\nu' : D\otimes C \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{F})$$

be the composite:

$$\begin{array}{ccc} D\otimes C \longrightarrow D\otimes D^* \otimes C & \xrightarrow{1\otimes\nu} D\otimes D^*\otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{F}) \longrightarrow \mathbf{hy}^{\mathrm{st}}(\mathfrak{F}) \\ d\otimes c \longmapsto d\otimes 1\otimes c & d\otimes X\otimes y \longmapsto \langle d, X\rangle y. \end{array}$$

Then Lemma below means that ν' is a coalgebra map and

$$\exp(\nu, D^*, C) = \exp(\nu', k, D\otimes C).$$

On the other hand Lemma 3.2.5 means that

$$\nu' = p \circ (\tau\otimes\sigma)$$

where

$$p : \mathbf{hy}(\mathfrak{G}) \otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{F}) \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{F})$$

denotes the structure map of the $\mathbf{hy}(\mathfrak{G})$-module hyperalgebra $\mathbf{hy}^{\mathrm{st}}(\mathfrak{F})$. Hence we have

$$\begin{aligned} &\exp(p \circ (\tau\otimes\sigma), D\otimes C) \\ &= \mathfrak{u}(\exp(\tau, D)_{D^*\otimes C^*}, \exp(\sigma, C)_{D^*\otimes C^*}) \\ &= \mathfrak{u}(\exp(\tau\otimes\sigma, D\otimes C)) \\ &= \exp(\mathbf{S}_{(e,e)}(\mathfrak{u}) \circ (\tau\otimes\sigma), D\otimes C). \end{aligned}$$

Therefore we have

$$p\circ(\tau\otimes\sigma)=\mathbf{S}_{(e,e)}(\mathrm{u})\circ(\tau\otimes\sigma).$$

This means that

$$p=\mathbf{S}_{(e,e)}(\mathrm{u}).$$

3.4.12 Let $D, C\in W_k^f$ and $H\in W_k$. The composite of natural isomorphisms:

$$\begin{aligned}\boldsymbol{Mod}_k(D\otimes C, H)\simeq\boldsymbol{Mod}_k(C, \boldsymbol{Mod}_k(D, H))&\simeq\boldsymbol{Mod}_k(C, D^*\otimes H)\\ &\simeq\boldsymbol{Mod}_{D^*}(D^*\otimes C, D^*\otimes H)\end{aligned}$$

induces a bijection:

$$W_k(D\otimes C, H)\simeq\boldsymbol{Coalg}_{D^*}(D^*\otimes C, D^*\otimes H),$$

as is verified by simple calculation. We shall denote by ν' the element of $W_k(D\otimes C, H)$ which corresponds to $\nu\in W_{D^*}(C, H)$. Then ν' is the composite:

$$\begin{array}{ccccc} D\otimes C\longrightarrow D\otimes D^*\otimes C & \xrightarrow{1\otimes\nu} & D\otimes D^*\otimes H\longrightarrow H\\ d\otimes c\longmapsto d\otimes 1\otimes c & & d\otimes X\otimes y\longmapsto\langle d, X\rangle y.\end{array}$$

Let $\mathfrak{G}$ be a k-group-functor having the hyperalgebra in the strong sense $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$. Let $\nu\in W_{D^*}(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{G}))$. Then there exist a finite dimensional sub-coalgebra E of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ and $\upsilon\in W_{D^*}(C, E)$ such that

$$\nu=W_\eta(i)\circ\upsilon$$

where $i: E\to\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ is the canonical injection. Then we have

$$\nu'=(W_\eta(i)\circ\upsilon)'=i\circ\upsilon'.$$

Since the algebra map

$${}^t(\upsilon'): E^*\to D^*\otimes C^*$$

factors as

$$E^*\xrightarrow{\eta\otimes 1}D^*\otimes E^*\xrightarrow{\boldsymbol{Mod}_{D^*}(\upsilon, D^*)}D^*\otimes C^*,$$

it follows that

$$\begin{aligned}\exp(\nu, D^*, C)=\exp(W_\eta(i), D^*, E)_{D^*\otimes C^*}&=\exp(i, k, E)_{D^*\otimes C^*}\\ &=\exp(\nu', k, D\otimes C).\end{aligned}$$

Thus we have proven:

LEMMA. *Let $\mathfrak{G}$ be a k-group-functor such that $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ exists. Let*

$D, C \in W_k^{\mathrm{f}}$ and $\nu \in W_{D^*}(C, \mathbf{hy}^{\mathrm{st}}(\mathfrak{J}))$. Then the composite

$$\begin{array}{llll} \nu' : D\otimes C \longrightarrow D\otimes D^*\otimes C \xrightarrow{1\otimes\nu} D\otimes D^*\otimes \mathbf{hy}^{\mathrm{st}}(\mathfrak{J}) \longrightarrow \mathbf{hy}^{\mathrm{st}}(\mathfrak{J}) \\ \qquad d\otimes c \longmapsto d\otimes 1\otimes c \qquad\qquad\qquad d\otimes X\otimes y \longmapsto \langle d, X\rangle y \end{array}$$

is a coalgebra map and

$$\exp(\nu, D^*, C) = \exp(\nu', k, D\otimes C).$$

3.4.13 Let $\mathfrak{G}$ be a k-group-functor. Let

$$\mathfrak{u} : \mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$$

be the action of $\mathfrak{G}$ on $\mathfrak{G}$ by *inner automorphisms*, that is

$$\mathfrak{u}(a, x) = axa^{-1}$$

for a, $x \in \mathfrak{G}(R)$, where $R \in \boldsymbol{M}_k$. Suppose that $\mathfrak{G}$ has the hyperalgebra in the strong sense $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$. Then the action $\mathfrak{u}$ makes $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ into a k-$\mathfrak{G}$-module hyperalgebra, whose structure map will be denoted by

$$\mathfrak{Ad} : \mathfrak{G} \to \mathfrak{G}\mathfrak{L}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}))$$

and called the *adjoint representation* of $\mathfrak{G}$. This representation induces a structure of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$-module hyperalgebra on $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$, whose structure map coincides with

$$\mathbf{T}^{\mathrm{st}}_{(e,e)}(\mathfrak{u}) : \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})\otimes\mathbf{hy}^{\mathrm{st}}(\mathfrak{G}) \to \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$$

by Lemma 3.4.11. But we have

$$\mathbf{T}^{\mathrm{st}}_{(e,e)}(\mathfrak{u})(a\otimes x) = \sum a_{(1)}\cdot x\cdot S_{\mathfrak{G}}(a_{(2)})$$

for $a, x \in \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$, where $S_{\mathfrak{G}}$ denotes the antipode of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$, by simple calculation. Therefore the adjoint representation of $\mathfrak{G}$ induces the adjoint action of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ on $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ (1.10.3,1)).

Let $\mathfrak{H}$ be a sub-group-functor of $\mathfrak{G}$. The *normalizer* $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})$ (resp. the *centralizer* $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})$) of $\mathfrak{H}$ in $\mathfrak{G}$ is the sub-group-functor of $\mathfrak{G}$ defined by

$$\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})(R) = \{g \in \mathfrak{G}(R) \mid g_S\mathfrak{H}(S)g_S^{-1} = \mathfrak{H}(S) \text{ for any } S \in \boldsymbol{M}_R\}$$

(resp. by

$$\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})(R) = \{g \in \mathfrak{G}(R) \mid g_Sxg_S^{-1} = x \text{ for any } S \in \boldsymbol{M}_R \text{ and } x \in \mathfrak{H}(S)\})$$

for $R \in \boldsymbol{M}_k$. Consider the sub-functors $\mathfrak{X}_{\mathfrak{G}}(\mathfrak{H}, \mathfrak{H})$ and $\mathfrak{X}_{\mathfrak{G}}(\mathfrak{H}, \mathfrak{e})$ taken with respect to the inner-automorphism action:

$$\mathfrak{H}\times\mathfrak{G}\to\mathfrak{G},\ (a, x)\mapsto axa^{-1}.$$

Since we have

$$\mathfrak{X}_{\mathfrak{G}}(\mathfrak{H}, \mathfrak{H})(R)=\{g\in\mathfrak{G}(R)\mid g_S\mathfrak{H}(S)g_S^{-1}\subset\mathfrak{H}(S) \text{ for any } S\in \boldsymbol{M}_R\},$$

it follows that $\mathfrak{X}_{\mathfrak{G}}(\mathfrak{H}, \mathfrak{H})$ is a sub-monoid-functor of $\mathfrak{G}$ and has $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})$ as its largest sub-group-functor. Therefore if both $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ and $\mathbf{hy}^{\mathrm{st}}(\mathfrak{H})$ exist, then we have

$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H}))=\mathbf{hy}^{\mathrm{st}}(\mathfrak{X}_{\mathfrak{G}}(\mathfrak{H}, \mathfrak{H}))=\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathfrak{H}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{H}))$$

where $\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathfrak{H}, \mathbf{hy}^{\mathrm{st}}(\mathfrak{H}))$ is taken with respect to the adjoint representation

$$\mathfrak{Ad}: \mathfrak{H}\to\mathfrak{GL}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})).$$

Similarly we have

$$\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})=\mathfrak{X}_{\mathfrak{G}}(\mathfrak{H}, \mathfrak{e}) \quad\text{and}\quad \mathbf{hy}^{\mathrm{st}}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H}))=\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathfrak{H}, k).$$

In other words we have:

PROPOSITION. *Let $\mathfrak{G}$ be a k-group-functor having the hyperalgebra in the strong sense $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$. Let $\mathfrak{H}$ be a sub-group-functor of $\mathfrak{G}$.*

(i) *$\mathbf{hy}^{\mathrm{st}}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H}))$ is the largest element of the set of subcoalgebras C of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ such that*

$$\sum \mathfrak{Ad}(h)(c_{(1)})S_{\mathfrak{G}}(c_{(2)})\in R\otimes k$$

for any $R\in \boldsymbol{M}_k$, $h\in\mathfrak{H}(R)$ and $c\in C$, where $S_{\mathfrak{G}}$ is the antipode of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$.

(ii) *Suppose that $\mathbf{hy}^{\mathrm{st}}(\mathfrak{H})$ exists. Then $\mathbf{hy}^{\mathrm{st}}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H}))$ is the largest element of the set of subcoalgebras D of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ such that*

$$\sum \mathfrak{Ad}(h)(d_{(1)})S_{\mathfrak{G}}(d_{(2)})\in R\otimes\mathbf{hy}^{\mathrm{st}}(\mathfrak{H})$$

for any $R\in \boldsymbol{M}_k$, $h\in\mathfrak{H}(R)$ and $d\in D$.

3.4.14 Let $\mathfrak{G}$ be a locally algebraic k-group and $\mathfrak{H}$ a subgroup of $\mathfrak{G}$. If $\mathfrak{H}$ is connected, then it follows from Corollary 3.4.10 that

$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H}))=\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}), \mathbf{hy}^{\mathrm{st}}(\mathfrak{H})) \quad\text{and}$$
$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H}))=\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}), k)$$

where the right hande sides are taken with respect to the adjoint action. (Notice that $\mathfrak{H}$ is algebraic, since it is connected). But since $\mathbf{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}), \mathbf{hy}^{\mathrm{st}}(\mathfrak{H}))$ is the largest element of the set of sub-coalgebras C of $\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$ such that

$$[\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}), C] \subset \mathbf{hy}^{\mathrm{st}}(\mathfrak{H}),$$

it follows that

$$\mathrm{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}), \mathbf{hy}^{\mathrm{st}}(\mathfrak{H}) = \mathrm{N}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}))$$

in the notation of (1.10.5). (The reader should notice that

$$S_{\mathfrak{G}}([a, x]) = [x, a]$$

for any $a, x \in \mathbf{hy}^{\mathrm{st}}(\mathfrak{G})$). Since we have similarly

$$\mathrm{X}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H}), k) = \mathrm{C}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H})),$$

follows the following:

PROPOSITION. *Let $\mathfrak{G}$ be a locally algebraic k-group and $\mathfrak{H}$ a connected subgroup of $\mathfrak{G}$. Then we have*

$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})) = \mathrm{N}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H})) \quad \text{and}$$
$$\mathbf{hy}^{\mathrm{st}}(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})) = \mathrm{C}_{\mathbf{hy}^{\mathrm{st}}(\mathfrak{G})}(\mathbf{hy}^{\mathrm{st}}(\mathfrak{H})).$$

3.4.15 COROLLARY. *Let $\mathfrak{G}$ be a connected algebraic k-group and $\mathfrak{H}$ a closed subgroup of $\mathfrak{G}$. Then we have*

$$\begin{aligned} &\mathfrak{H} \textit{ is normal (resp. central) in } \mathfrak{G} \\ &\Leftrightarrow \sum \mathfrak{Ad}\,(h)(c_{(1)}) S_{\mathfrak{G}}(c_{(2)}) \in R \otimes \mathbf{hy}(\mathfrak{H}) \ (\textit{resp. } \in R \otimes k) \\ &\qquad \textit{for any } R \in \boldsymbol{M}_k,\ h \in \mathfrak{H}(R) \textit{ and } c \in \mathbf{hy}(\mathfrak{G}) \end{aligned}$$

where $S_{\mathfrak{G}}$ denotes the antipode of $\mathbf{hy}(\mathfrak{G})$. Suppose further that $\mathfrak{H}$ is connected. Then we have

$$\begin{aligned} &\mathfrak{H} \textit{ is normal (resp. central) in } \mathfrak{G} \\ &\Leftrightarrow [\mathbf{hy}(\mathfrak{G}), \mathbf{hy}(\mathfrak{H})] \subset \mathbf{hy}(\mathfrak{H}) \ (\textit{resp. } \subset k). \end{aligned}$$

PROOF. Since $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{H})$ and $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{H})$ are closed subgroups of $\mathfrak{G}$, our assertion follows immediately from Propositions 3.4.13 and 3.4.14 in view of Corollary 3.3.8.

3.4.16 Let $\mathfrak{G}$ be a k-group-functor and

$$\rho : \mathfrak{G} \to \mathfrak{G}\mathfrak{L}(V)$$

a linear representation of $\mathfrak{G}$ on a vector space V. For subspaces W, W' of V such that $W' \subset W$, the sub-monoid-functor $\mathfrak{G}_{W', W}$ [7, II, § 2, 1.4] is defined by

$$\mathfrak{G}_{W',W}(R)=\{g\in\mathfrak{G}(R)\mid g\cdot x-x\in R\otimes W' \text{ for any } x\in W\}$$

for $R\in \boldsymbol{M}_k$. Suppose that $\mathbf{hy}(\mathfrak{G})$ exists. Then V is a $\mathbf{hy}(\mathfrak{G})$-module by (3.2.5).

PROPOSITION. *$\mathbf{hy}(\mathfrak{G}_{W',W})$ is the largest element of the set of subcoalgebras C of $\mathbf{hy}(\mathfrak{G})$ such that*

$$(c\rightharpoonup x)-\varepsilon(c)x\in W'$$

for any $c\in C$ and $x\in W$, where '$\rightharpoonup$' denotes the action of $\mathbf{hy}(\mathfrak{G})$ on V.

PROOF. Let $D\in W_k^f$ and $\sigma\in W_k(D,\mathbf{hy}(\mathfrak{G}))$. Then we have

$$\exp(\sigma,D)\in\mathfrak{G}_{W',W}(D^*)\Leftrightarrow\exp(\sigma,D)\cdot x-x\in D^*\otimes W' \qquad \text{for any } x\in W.$$

If we identify $D^*\otimes V$ with $\boldsymbol{Mod}_k(D,V)$ canonically, then we have

$$(\exp(\sigma,D)\cdot x)(d)=\sigma(d)\rightharpoonup x$$

for $d\in D$. Hence we have

$$\exp(\sigma,D)\in\mathfrak{G}_{W',W}(D^*)\Leftrightarrow(\sigma(d)\rightharpoonup x)-\varepsilon(d)x\in W' \qquad \text{for any } d\in D \text{ and } x\in W.$$

Since there exists the largest subcoalgebra C of $\mathbf{hy}(\mathfrak{G})$ such that

$$(c-x)\rightharpoonup\varepsilon(c)x\in W'$$

for any $c\in C$ and $x\in W$, it follows that $\mathbf{hy}(\mathfrak{G}_{W',W})=C$.

3.4.17 It is easy to see that $\mathfrak{G}_{W',W}$ is a closed submonoid of $\mathfrak{G}$ (cf. (3.4.8)). In particular if $\mathfrak{G}$ is an algebraic k-group, then $\mathfrak{G}_{W',W}$ is a closed subgroup of $\mathfrak{G}$ by (3.4.9).

COROLLARY. *Let $\mathfrak{G}$ be a connected algebraic k-group and*

$$\rho:\mathfrak{G}\rightarrow\mathfrak{GL}(V)$$

a linear representation of $\mathfrak{G}$ on a vector space V.

(i) *A subspace W of V is $\mathfrak{G}$-stable iff $\mathbf{hy}(\mathfrak{G})$-stable.*

(ii) *$V^{\mathfrak{G}}=V^{\mathbf{hy}(\mathfrak{G})}=\{x\in V\mid c\rightharpoonup x=\varepsilon(c)x$ for any $c\in\mathbf{hy}(\mathfrak{G})\}$.*

PROOF. (i) W is $\mathfrak{G}$-stable$\Leftrightarrow\mathfrak{G}=\mathfrak{G}_{W,W}\Leftrightarrow\mathbf{hy}(\mathfrak{G})=\mathbf{hy}(\mathfrak{G}_{W,W})$
$\Leftrightarrow W$ is $\mathbf{hy}(\mathfrak{G})$-stable.

(ii) Let W be a subspace of V. Then

$$W\subset V^{\mathfrak{G}}\Leftrightarrow\mathfrak{G}=\mathfrak{G}_{0,W}\Leftrightarrow\mathbf{hy}(\mathfrak{G})=\mathbf{hy}(\mathfrak{G}_{0,W})\Leftrightarrow W\subset V^{\mathbf{hy}(\mathfrak{G})}.$$

3.4a $\mathbf{hy}(\mathfrak{N}_{\mathfrak{H}}^{-1}(\mathfrak{G}))$ **and** $\mathbf{hy}(\mathfrak{C}_{\mathfrak{H}}^{-1}(\mathfrak{G}))$

3.4a.1 Let $\mathfrak{G}$ be a k-group-functor, $\mathfrak{X}$ a k-functor and

$$\mathfrak{u}: \mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$$

an action of $\mathfrak{G}$ on $\mathfrak{X}$. Let $\mathfrak{Y}$ be a subfunctor of $\mathfrak{X}$. We define a subfunctor $\mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$ (resp. $\mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$) of $\mathfrak{Y}$ as follows: Let $R\in \boldsymbol{M}_k$. Then $\mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})(R)$ (resp. $\mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})(R)$) is the set of $y\in\mathfrak{Y}(R)$ such that

$$g\rightharpoonup y_S\in\mathfrak{Y}(S) \qquad (\text{resp. } g\rightharpoonup y_S=y_S)$$

for any $S\in \boldsymbol{M}_R$ and $g\in\mathfrak{G}(S)$, where we put as usual

$$g\rightharpoonup y_S=\mathfrak{u}(g, y_S).$$

The subfunctor $\mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$ (resp. $\mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$) of $\mathfrak{Y}$ is the largest which is $\mathfrak{G}$-stable (resp. on which $\mathfrak{G}$ acts trivially).

PROPOSITION. *Suppose that $\mathfrak{G}$ is a k-group (-scheme). If $\mathfrak{Y}$ is a closed subfunctor of $\mathfrak{X}$, then $\mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$ is also a closed subfuntor of $\mathfrak{X}$. If $\mathfrak{X}$ is a separated k-functor* [7, I, § 2, 5.4], *then $\mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$ is a closed subfunctor of $\mathfrak{Y}$.*

PROOF. Recall that given a map of k-functors

$$\mathfrak{p}: \mathfrak{U}\times\mathfrak{V}\to\mathfrak{Z}$$

and a subfunctor $\mathfrak{Z}'$ of $\mathfrak{Z}$, the transporter of $\mathfrak{U}$ in $\mathfrak{Z}'$, written $\mathfrak{Transp}_{\mathfrak{p}}(\mathfrak{U}, \mathfrak{Z}')$, is the largest subfunctor of $\mathfrak{V}$ such that

$$\mathfrak{p}(\mathfrak{U}\times\mathfrak{Transp}_{\mathfrak{p}}(\mathfrak{U}, \mathfrak{Z}'))\subset\mathfrak{Z}'$$

[7, I, § 2, 7.4]. Since we have

$$\mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})=\mathfrak{Transp}_{\mathfrak{u}}(\mathfrak{G}, \mathfrak{Y}),$$

the first part follows from [7, I, § 2, 7.7]. Let

$$\mathfrak{v}: \mathfrak{G}\times\mathfrak{X}\to\mathfrak{X}$$

be the trivial action. Let

$$\mathfrak{u}', \mathfrak{v}': \mathfrak{X}\rightrightarrows\mathfrak{Hom}_k(\mathfrak{G}, \mathfrak{X})$$

be the maps canonically associated with $\mathfrak{u}$ and $\mathfrak{v}$ respectively. Then $\mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})$ is the equalizer of the diagram

$$\mathfrak{u}', \mathfrak{v}': \mathfrak{Y}\rightrightarrows\mathfrak{Hom}_k(\mathfrak{G}, \mathfrak{X}).$$

Since $\mathfrak{Hom}_k(\mathfrak{G}, \mathfrak{X})$ is separated if $\mathfrak{X}$ is separated by [7, I, § 2, 7.8], the latter

part follows.

3.4a.2 LEMMA. *Let $y \in \mathfrak{Y}(R \otimes T)$ with $R, T \in \boldsymbol{M}_k$. Then we have*

$$y \in \mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})(R\otimes T)(\mathit{resp.}\ y \in \mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})(R\otimes T))$$
$$\Leftrightarrow g_{S\otimes T} \rightharpoonup y_{S\otimes T} \in \mathfrak{Y}(S\otimes T)(\mathit{resp.}\ g_{S\otimes T} \rightharpoonup y_{S\otimes T} = y_{S\otimes T})$$
$$\textit{for any } S \in \boldsymbol{M}_R \textit{ and } g \in \mathfrak{G}(S).$$

PROOF. Exercise.

3.4a.3 Let $\mathfrak{G}$ be a k-group-functor and

$$\rho : \mathfrak{G} \to \mathfrak{G}\mathfrak{L}(V)$$

a linear representation of $\mathfrak{G}$ on a vector space V. For a subspace W of V, we put

$$\mathrm{N}_W^{-1}(\mathfrak{G}) = \{w \in W \mid \rho(g)(w) \in R \otimes W \text{ for any } R \in \boldsymbol{M}_k \text{ and } g \in \mathfrak{G}(R)\}$$
$$\mathrm{C}_W^{-1}(\mathfrak{G}) = \{w \in W \mid \rho(g)(w) = 1 \otimes w \text{ for any } R \in \boldsymbol{M}_k \text{ and } g \in \mathfrak{G}(R)\}.$$

LEMMA. *If we view ρ as an action of $\mathfrak{G}$ on V_a, then we have*

$$(\mathrm{N}_W^{-1}(\mathfrak{G}))_a = \mathfrak{N}_{W_a}^{-1}(\mathfrak{G}) \quad \text{and} \quad (\mathrm{C}_W^{-1}(\mathfrak{G}))_a = \mathfrak{C}_{W_a}^{-1}(\mathfrak{G}).$$

PROOF. This follows from Lemma 2.4.5.

3.4a.4 Let $\mathfrak{u} : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of a k-group-functor $\mathfrak{G}$ on a k-functor $\mathfrak{X}$ and $\mathfrak{Y}$ a subfunctor of $\mathfrak{X}$. Let $e \in \mathfrak{Y}(k) \cap \mathfrak{X}^{\mathfrak{G}}(k)$ and suppose that both $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$ and $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y})$ exist. We know that $\mathfrak{u}$ induces a natural linear representation of $\mathfrak{G}$ on $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{X})$, which we shall write ρ.

PROPOSITION. $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G}))$ (*resp.* $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G}))$) *is the largest subcoalgebra of $\mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y})$ contained in* $\mathrm{N}_{\mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y})}^{-1}(\mathfrak{G})$ (*resp.* $\mathrm{C}_{\mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y})}^{-1}(\mathfrak{G})$).

PROOF. Let $(R, C) \in \boldsymbol{W}^{\mathrm{f}}$ and $\sigma \in \boldsymbol{W}_R(C, \mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y}))$. Then we have

$$\exp(\sigma, R, C) \in \mathfrak{N}_{\mathfrak{Y}}^{-1}(\mathfrak{G})(R\otimes C^*)$$
$$\Leftrightarrow g_{S\otimes C^*} \rightharpoonup \exp(\boldsymbol{W}_\phi(\sigma), S, C) \in \mathfrak{Y}(S\otimes C^*)$$
$$\text{for any } S \in \boldsymbol{M}_k, \phi \in \boldsymbol{M}_k(R, S) \text{ and } g \in \mathfrak{G}(S)$$
$$\Leftrightarrow \rho(g)\circ \boldsymbol{W}_\phi(\sigma) \in \boldsymbol{W}_S(C, \mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y}))$$
$$\text{for any } S \in \boldsymbol{M}_k, \phi \in \boldsymbol{M}_k(R, S) \text{ and } g \in \mathfrak{G}(S)$$
$$\Leftrightarrow \sigma(R\otimes C) \subset \mathfrak{N}_{W_a}^{-1}(\mathfrak{G})(R) = R \otimes \mathrm{N}_W^{-1}(\mathfrak{G})$$
$$(\text{where we put } W = \mathbf{T}_e^{\mathrm{st}}(\mathfrak{Y})).$$

Similarly we have

$$\exp(\sigma, R, C) \in \mathfrak{C}_{\mathfrak{Y}}^{-1}(\mathfrak{G})(R\otimes C^*) \Leftrightarrow \sigma(R\otimes C) \subset R\otimes \mathrm{C}_W^{-1}(\mathfrak{G}).$$

Hence our assertion follows from Lemma 1.2.5.

3.4a.5 Let $\mathfrak{G}$ be a k-group-functor and $\mathfrak{H}$ a sub-group-functor of $\mathfrak{G}$. The subfunctor $\mathfrak{N}_{\mathfrak{H}}^{-1}(\mathfrak{G})$ (resp. $\mathfrak{C}_{\mathfrak{H}}^{-1}(\mathfrak{G})$) of $\mathfrak{H}$ taken with respect to the inner-automorphism action

$$\mathfrak{u}: \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}, \qquad (g, x)\mapsto gxg^{-1}$$

is clearly a sub-group-functor of $\mathfrak{H}$ and the largest normal (resp. central) sub-group-functor of $\mathfrak{G}$ contained in $\mathfrak{H}$. If $\mathfrak{G}$ is a k-scheme and $\mathfrak{H}$ is a closed subgroup of $\mathfrak{G}$, then $\mathfrak{N}_{\mathfrak{H}}^{-1}(\mathfrak{G})$ and $\mathfrak{C}_{\mathfrak{H}}^{-1}(\mathfrak{G})$ are closed subgroups of $\mathfrak{H}$ by Proposition 3.4a. 1 (, since any k-group is separated).

COROLLARY. *Let $\mathfrak{G}$ be a locally algebraic k-group and $\mathfrak{H}$ a connected closed subgroup of $\mathfrak{G}$. Then we have*

$\mathfrak{H}$ is normal (resp. central) in $\mathfrak{G}$

$\Leftrightarrow$**hy**($\mathfrak{H}$) *is $\mathfrak{G}$-stable (resp. $\mathfrak{G}$ acts trivially on* **hy**($\mathfrak{H}$)) *under the adjoint representation of $\mathfrak{G}$.*

PROOF. Since $\mathfrak{N}_{\mathfrak{H}}^{-1}(\mathfrak{G})$ and $\mathfrak{C}_{\mathfrak{H}}^{-1}(\mathfrak{G})$ are closed subgroups of $\mathfrak{H}$, it follows from Corollary 3.3.8 that

$\mathfrak{H}$ is normal (resp. central) in $\mathfrak{G}$

$\Leftrightarrow \mathfrak{H}=\mathfrak{N}_{\mathfrak{H}}^{-1}(\mathfrak{G})$ (resp. $\mathfrak{C}_{\mathfrak{H}}^{-1}(\mathfrak{G})$)

$\Leftrightarrow$**hy**($\mathfrak{H}$)$=$**hy**($\mathfrak{N}_{\mathfrak{H}}^{-1}(\mathfrak{G})$) (resp. **hy**($\mathfrak{H}$)$=$**hy**($\mathfrak{C}_{\mathfrak{H}}^{-1}(\mathfrak{G})$))

$\Leftrightarrow$**hy**($\mathfrak{H}$)$=\mathrm{N}_{\mathbf{hy}(\mathfrak{H})}^{-1}(\mathfrak{G})$)

$\Leftrightarrow$**hy**($\mathfrak{H}$) is $\mathfrak{G}$-stable (resp. $\mathfrak{G}$ acts trivially on **hy**($\mathfrak{H}$)) under the adjoint representation of $\mathfrak{G}$.

3.5 hy($\mathscr{D}(\mathfrak{G})$)

Let $\mathfrak{G}$ be an algebraic k-group which is smooth. Then the *derived group* $\mathscr{D}(\mathfrak{G})$ of $\mathfrak{G}$ is defined as follows [7, II, § 5, 4.8 and III, § 3, 3.7d)]: For $R\in \boldsymbol{M}_k$, $\mathscr{D}(\mathfrak{G})(R)$ is the set of $g\in\mathfrak{G}(R)$ such that there exists an $S\in\boldsymbol{M}_R$ which is faithfully flat and finitely presented over R such that g_S belongs to $[\mathfrak{G}(S), \mathfrak{G}(S)]$ the commutator subgroup of $\mathfrak{G}(S)$. It is known that $\mathscr{D}(\mathfrak{G})$ is a closed smooth subgroup of $\mathfrak{G}$. The purpose of this section is to compute **hy**($\mathscr{D}(\mathfrak{G})$) the hyperalgebra of $\mathscr{D}(\mathfrak{G})$.

In the following $\mathfrak{G}$ denotes an algebraic smooth k-group. The hyperalgebra of $\mathfrak{G}$ is simply denoted by **hy**($\mathfrak{G}$) (, although it is in the strong sense).

$S_{\mathfrak{G}}$ denotes the antipode of $\mathbf{hy}(\mathfrak{G})$. Let $R \in \boldsymbol{M}_k$, $a, x \in \mathfrak{G}(R)$ and $u, v \in R \otimes \mathbf{hy}(\mathfrak{G})$. We shall write

$$\begin{aligned} &a \rightharpoonup x = axa^{-1}, &&[a, x] = (a \rightharpoonup x)x^{-1} \\ &a \rightharpoonup v = \mathfrak{Ad}(a)(v), &&[a, v] = \sum (a \rightharpoonup v_{(1)}) S_{\mathfrak{G}}(v_{(2)}) \\ &u \rightharpoonup v = \sum u_{(1)} v S_{\mathfrak{G}}(u_{(2)}), &&[u, v] = \sum (u \rightharpoonup v_{(1)}) S_{\mathfrak{G}}(v_{(2)}) \end{aligned}$$

(cf. § 3.4 and § 1.10).

3.5.1 Let Φ be the set of subalgebras A of $\mathbf{hy}(\mathfrak{G})$ such that

$$\mathfrak{G} = \mathfrak{G}_{A, \mathbf{hy}(\mathfrak{G})}$$

with respect to the adjoint representation of $\mathfrak{G}$, that is

$$[\mathfrak{G}(R), R \otimes \mathbf{hy}(\mathfrak{G})] \subset R \otimes A$$

for any $R \in \boldsymbol{M}_k$. The reader may easily verify that Φ contains the *smallest* element, which we shall denote by $\mathbf{d}(\mathfrak{G})$.

LEMMA. *Let $K | k$ be an extension of fields. Then*

$$\mathbf{d}(K \otimes \mathfrak{G}) = K \otimes \mathbf{d}(\mathfrak{G}).$$

PROOF. Notice that $\mathbf{hy}^K(K \otimes \mathfrak{G}) = K \otimes \mathbf{hy}(\mathfrak{G})$ (3.1.7). The inclusion $\mathbf{d}(K \otimes \mathfrak{G}) \subset K \otimes \mathbf{d}(\mathfrak{G})$ is clear. Let $R \in \boldsymbol{M}_k$. Since $R \otimes K \in \boldsymbol{M}_k$ and $R \otimes \mathbf{d}(K \otimes \mathfrak{G}) = (R \otimes K) \otimes_K \mathbf{d}(K \otimes \mathfrak{G})$, it follows that

$$[\mathfrak{G}(R), R \otimes \mathbf{hy}(\mathfrak{G})] \subset (R \otimes k \otimes \mathbf{hy}(\mathfrak{G})) \cap (R \otimes \mathbf{d}(K \otimes \mathfrak{G})) = R \otimes A$$

where $A = (k \otimes \mathbf{hy}(\mathfrak{G})) \cap \mathbf{d}(K \otimes \mathfrak{G})$. This means that $A \in \Phi$. Hence

$$\mathbf{d}(K \otimes \mathfrak{G}) \supset K \otimes A \supset \mathbf{d}(\mathfrak{G}).$$

3.5.2 LEMMA. $\mathbf{hy}(\mathscr{D}(\mathfrak{G})) \supset \mathbf{d}(\mathfrak{G})$.

PROOF. Let C be a finite dimensional subcoalgebra of $\mathbf{hy}(\mathfrak{G})$. For any $R \in \boldsymbol{M}_k$ and $g \in \mathfrak{G}(R)$ we have

$$[g_{R \otimes C^*}, \exp(i, C)_{R \otimes C^*}] = \exp(\sigma, R, C) \in \mathscr{D}(\mathfrak{G})(R \otimes C^*)$$

where $i \colon C \rightarrow \mathbf{hy}(\mathfrak{G})$ is the canonical injection and

$$\sigma = (\mathfrak{Ad}(g) \circ W_\eta(i)) * (W_\eta(S_{\mathfrak{G}}) \circ W_\eta(i)).$$

Hence $\sigma \in W_R(C, \mathbf{hy}(\mathscr{D}(\mathfrak{G})))$. This means that

$$\sigma(R \otimes C) = [g, R \otimes C] \subset R \otimes \mathbf{hy}(\mathscr{D}(\mathfrak{G})).$$

Since $\mathbf{hy}(\mathfrak{G})$ is a directed union of finite dimensional subcoalgebras, it follows that $\mathbf{hy}(\mathscr{D}(\mathfrak{G})) \in \Phi$.

3.5.3 LEMMA. (i) $[\mathbf{hy}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})] \subset \mathbf{d}(\mathfrak{G})$.

(ii) *For any $R \in \boldsymbol{M}_k$, $a, b \in \mathfrak{G}(R)$ and $x, y \in R \otimes \mathbf{hy}(\mathfrak{G})$, we have*

$$\sum (a \rightharpoonup x_{(1)})(b \rightharpoonup y_{(1)}) S_{\mathfrak{G}}(x_{(2)}) S_{\mathfrak{G}}(y_{(2)}) \in R \otimes \mathbf{d}(\mathfrak{G}).$$

PROOF. (i) Since $\mathfrak{G} = \mathfrak{G}_{\mathbf{d}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})}$, it follows that

$$\mathbf{hy}(\mathfrak{G}) = \mathbf{hy}(\mathfrak{G})_{\mathbf{d}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})}.$$

This means that $[\mathbf{hy}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})] \subset \mathbf{d}(\mathfrak{G})$.

(ii) This is clear since

$$\sum (a \rightharpoonup x_{(1)})(b \rightharpoonup y_{(1)}) S_{\mathfrak{G}}(x_{(2)}) S_{\mathfrak{G}}(y_{(2)}) = \sum [a, x_{(1)}][x_{(2)}, b \rightharpoonup y_{(1)}][b, y_{(2)}].$$

3.5.4 It follows from Proof of [7, II, § 5, 4.8] that for some integer $n > 0$, the map

$$\mathfrak{f} : \mathfrak{G}^n \times \mathfrak{G}^n \to \mathscr{D}(\mathfrak{G}) \qquad (x, y) \mapsto [x_1, y_1] \cdots [x_n, y_n]$$

is flat on some open dense subscheme $\mathfrak{U}$ of $\mathfrak{G}^n \times \mathfrak{G}^n$. Suppose that k is algebraically closed. Then $\mathfrak{U}(k) \neq \emptyset$ by [7, I, § 3, 7.8]. Hence $\mathfrak{f}$ is flat at some rational point (u, v) of $\mathfrak{G}^n \times \mathfrak{G}^n$. Then the map

$$\begin{aligned} \psi : \ & \mathfrak{G}^n \times \mathfrak{G}^n \to \mathscr{D}(\mathfrak{G}) \\ & (x, y) \mapsto [u_1 x_1, v_1(u_1 \rightharpoonup y_1)] \cdots [u_n x_n, v_n(u_n \rightharpoonup y_n)] \end{aligned}$$

is flat at (e, e). Since

$$\begin{aligned} & [u_1 x_1, v_1(u_1 \rightharpoonup y_1)] \cdots [u_n x_n, v_n(u_n \rightharpoonup y_n)] \\ & \quad = u_1 v_1 (v_1^{-1} \rightharpoonup x_1)(u_1 \rightharpoonup y_1) x_1^{-1} y_1^{-1} u_1^{-1} v_1^{-1} u_2 v_2 (v_2^{-1} \rightharpoonup x_2)(u_2 \rightharpoonup y_2) \cdots \end{aligned}$$

we can find out elements $a, b, c, d \in \mathfrak{G}^n(k)$ such that

$$\begin{aligned} & \psi(c \rightharpoonup x, d \rightharpoonup y) \\ & \quad = (a_1 \rightharpoonup x_1)(b_1 \rightharpoonup y_1) x_1^{-1} y_1^{-1} \cdots (a_n \rightharpoonup x_n)(b_n \rightharpoonup y_n) x_n^{-1} y_n^{-1} \psi(e, e) \end{aligned}$$

for any $R \in \boldsymbol{M}_k$ and $x, y \in \mathfrak{G}^n(R)$, where we put

$$c \rightharpoonup x = (c_1 \rightharpoonup x_1, \cdots, c_n \rightharpoonup x_n).$$

This prove the following:

LEMMA. *If k is algebraically closed, then there exist an integer $n > 0$ and elements $a_1, \cdots, a_n, b_1, \cdots, b_n$ of $\mathfrak{G}(k)$ such that the map*

$$\omega : \mathfrak{G}^n \times \mathfrak{G}^n \to \mathscr{D}(\mathfrak{G})$$

defined by

$$\omega(x, y) = (a_1 \rightharpoonup x_1)(b_1 \rightharpoonup y_1)x_1^{-1}y_1^{-1} \cdots (a_n \rightharpoonup x_n)(b_n \rightharpoonup y_n)x_n^{-1}y_n^{-1}$$

is flat at $(\mathfrak{e}, \mathfrak{e})$.

3.5.5 THEOREM. $\mathbf{hy}(\mathscr{D}(\mathfrak{G})) = \mathbf{d}(\mathfrak{G})$.[18]

PROOF. Let $K|k$ be an extension of fields. Then we have

$$\mathscr{D}(K \otimes \mathscr{D}) = K \otimes \mathscr{D}(\mathfrak{G}).$$

Hence $K \otimes \mathbf{hy}(\mathscr{D}(\mathfrak{G})) = \mathbf{hy}_K(\mathscr{D}(K \otimes \mathfrak{G}))$. In view of Lemma 3.5.1, the equality

$$\mathbf{hy}^K(\mathscr{D}(K \otimes \mathfrak{G})) = \mathbf{d}(K \otimes \mathfrak{G})$$

means the desired equality. Thus we can assume that k is algebraically closed. Take an integer n and elements a_i, $b_i \in \mathfrak{G}(k)$ as in Lemma 3.5.4. A simple calculation shows that

$$\begin{aligned} &\mathbf{T}_{(\mathfrak{e},\mathfrak{e})}(\omega)({}^1u \otimes \cdots \otimes {}^n u \otimes {}^1 v \otimes \cdots \otimes {}^n v) \\ &= \sum (a_1 \rightharpoonup {}^1u_{(1)})(b_1 \rightharpoonup {}^1v_{(1)}) S_{\mathfrak{G}}({}^1u_{(2)}) S_{\mathfrak{G}}({}^1v_{(2)}) \cdots \\ &\qquad \cdots (a_n \rightharpoonup {}^n u_{(1)})(b_n \rightharpoonup {}^n v_{(1)}) S_{\mathfrak{G}}({}^n u_{(2)}) S_{\mathfrak{G}}({}^n v_{(2)}) \end{aligned}$$

for ${}^i u, {}^i v \in \mathbf{hy}(\mathfrak{G}), i = 1, \cdots, n$. Since the map

$$\mathbf{T}_{(\mathfrak{e},\mathfrak{e})}(\omega) : \otimes^{2n} \mathbf{hy}(\mathfrak{G}) \to \mathbf{hy}(\mathscr{D}(\mathfrak{G}))$$

is surjective by Proposition 2.3.1, it follows from Lemma 3.5.3 that $\mathbf{hy}(\mathscr{D}(\mathfrak{G})) \subset \mathbf{d}(\mathfrak{G})$. This proves Theorem in view of Lemma 3.5.2.

3.5.6 COROLLARY. *If* $\mathfrak{G}$ *is connected, then* $\mathbf{hy}(\mathscr{D}(\mathfrak{G}))$ *is the subalgebra of* $\mathbf{hy}(\mathfrak{G})$ *generated by* $[\mathbf{hy}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})]$.

PROOF. Let A be a subalgebra of $\mathbf{hy}(\mathfrak{G})$. Since $\mathfrak{G}_{A, \mathbf{hy}(\mathfrak{G})}$ is a closed subgroup of $\mathfrak{G}$, we have

$$\begin{aligned} A \supset \mathbf{d}(\mathfrak{G}) &\Leftrightarrow \mathfrak{G} = \mathfrak{G}_{A,\mathbf{hy}(\mathfrak{G})} \Leftrightarrow \mathbf{hy}(\mathfrak{G}) = \mathbf{hy}(\mathfrak{G}_{A,\mathbf{hy}(\mathfrak{G})}) = \mathbf{hy}(\mathfrak{G})_{A,\mathbf{hy}(\mathfrak{G})} \\ &\Leftrightarrow [\mathbf{hy}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})] \subset A. \end{aligned}$$

3.6 Algebraic sub-hyperalgebra

Let $\mathfrak{G}$ be a locally algebraic k-group. A sub-hyperalgebra of $\mathbf{hy}(\mathfrak{G})$ is said to be *algebraic* if it is of the form $\mathbf{hy}(\mathfrak{H})$ for some *closed subgroup* $\mathfrak{H}$ of $\mathfrak{G}$. Let $\mathfrak{H}$ be a (not necessarily closed) subgroup of $\mathfrak{G}$. Since $\mathfrak{G}^0$ is algebraic, $\mathfrak{H}^0$ is a closed subgroup of $\mathfrak{G}^0$ (3.3.2). But $\mathfrak{G}^0$ is a closed (and open) subgroup

of $\mathfrak{G}$ [7, II, § 5, 1.8]. Hence $\mathfrak{H}^0$ is a closed subgroup of $\mathfrak{G}$. Therefore $\mathbf{hy}(\mathfrak{H})$ $=\mathbf{hy}(\mathfrak{H}^0)$ is an algebraic sub-hyperalgebra of $\mathbf{hy}(\mathfrak{G})$. Corollary 3.3.9 implies that the map: $\mathfrak{H}\mapsto\mathbf{hy}(\mathfrak{H})$ is a bijection from the set of connected subgroups of $\mathfrak{G}$ onto the set of algebraic subhyperalgebras of $\mathbf{hy}(\mathfrak{G})$.

Let $\{\mathfrak{H}_\lambda\}$ be a family of closed subgroups of $\mathfrak{G}$. Then $\cap\mathfrak{H}_\lambda$ is also a closed subgroup of $\mathfrak{G}$ and we have

$$\mathbf{hy}(\cap\mathfrak{H}_\lambda)=\cap\mathbf{hy}(\mathfrak{H}_\lambda).$$

This is clear, since we have

$$W_k(C, \cap\mathbf{hy}(\mathfrak{H}_\lambda))=\cap W_k(C, \mathfrak{H}_\lambda)\simeq\cap\operatorname{Ker}(\mathfrak{H}_\lambda(C^*)\to\mathfrak{H}_\lambda(C_0^*))$$

for any $C\in W_k^{\mathrm{f}}$. Let J be a sub-hyperalgebra of $\mathbf{hy}(\mathfrak{G})$. Let $\{\mathfrak{H}_\lambda\}$ be the set of closed subgroups of $\mathfrak{G}$ such that $\mathbf{hy}(\mathfrak{H}_\lambda)\supset J$. Then $\mathbf{hy}(\cap\mathfrak{H}_\lambda)$ is the *smallest* algebraic sub-hyperalgebra of $\mathbf{hy}(\mathfrak{G})$ containing J, which we shall denote by $\mathbf{A}(J)$ and call the *algebraic envelope* of J.

In the following $\mathfrak{G}$ will denote a locally algebraic k-group.

3.6.1 LEMMA. *Let I be a k-$\mathfrak{G}$-module hyperalgebra* (3.4.1). *Let A be a subalgebra and C a subcoalgebra of I such that $A\cdot C\subset C$. Then* $\mathbf{hy}(\mathfrak{G})_{A,C}$ *is an algebraic sub-hyperalgebra of* $\mathbf{hy}(\mathfrak{G})$. (*Notice that I is a* $\mathbf{hy}(\mathfrak{G})$*-module hyperalgebra* (3.4.5)).

PROOF. We can assume that $\mathfrak{G}$ is connected and hence algebraic. Then $\mathfrak{G}_{A,C}$ is a closed subgroup of $\mathfrak{G}$ (3.4.8 and 3.4.9) and $\mathbf{hy}(\mathfrak{G})_{A,C}=\mathbf{hy}(\mathfrak{G}_{A,C})$ (3.4.5).

3.6.2 For $x, y\in\mathbf{hy}(\mathfrak{G})$, we put

$$[x, y]=\sum x_{(1)}y_{(1)}S_{\mathfrak{G}}(x_{(2)})S_{\mathfrak{G}}(y_{(2)})$$

where $S_{\mathfrak{G}}$ denotes the antipode of $\mathbf{hy}(\mathfrak{G})$.

COROLLARY. *Let J and K be sub-hyperalgebras of* $\mathbf{hy}(\mathfrak{G})$ *such that $K\subset J$. Let A be a subalgebra and C a subcoalgebra of* $\mathbf{hy}(\mathfrak{G})$ *such that $A\cdot C\subset C$.*

(i) *If $[J, C]\subset A$, then $[\mathbf{A}(J), C]\subset A$.*

(ii) $\mathrm{N}_{\mathbf{hy}(\mathfrak{G})}(J)$ *and* $\mathrm{C}_{\mathbf{hy}(\mathfrak{G})}(J)$ (1.10.5) *are algebraic sub-hyperalgebras.*

(iii) *If K is normal* (*resp. central*) *in J, then so is in $\mathbf{A}(J)$.*

(iv) *If $[J, J]\subset K$, then $[\mathbf{A}(J), \mathbf{A}(J)]\subset K$.*

PROOF. (i), (ii) and (iii) follow immediately since the adjoint action of $\mathbf{hy}(\mathfrak{G})$ on $\mathbf{hy}(\mathfrak{G})$ is induced by the adjoint representation of $\mathfrak{G}$.

(iv) $[J, J]\subset K\Rightarrow[\mathbf{A}(J), J]\subset K\Rightarrow[J, \mathbf{A}(J)]=S_{\mathfrak{G}}([\mathbf{A}(J), J])\subset K\Rightarrow[\mathbf{A}(J), \mathbf{A}(J)]\subset K$.

3.6.3 PROPOSITION. *Let J is a sub-hyperalgebra of* $\mathbf{hy}(\mathfrak{G})$. *Then the subalgebra of* $\mathbf{hy}(\mathfrak{G})$ *generated by* $[J, J]$ *contains* $[\mathbf{A}(J), \mathbf{A}(J)]$. *If k is perfect and J is reduced* (1.9.5), *then the subalgebra of* $\mathbf{hy}(\mathfrak{G})$ *generated by* $[J, J]$ *is an algebraic sub-hyperalgebra of* $\mathbf{hy}(\mathfrak{G})$.[19]

PROOF. Since $[J, J]$ is a subcoalgebra of J, the subalgebra of J generated by $[J, J]$ is a sub-hyperalgebra. Hence the first part follows from Corollary 3.6.2 (iv). Suppose that k is perfect and J is reduced. Let $\mathfrak{H}$ be a connected subgroup of $\mathfrak{G}$ such that $\mathbf{hy}(\mathfrak{H})=\mathbf{A}(J)$. Then the image-subgroup (3.3.2) of the Frobenius map

$$\mathfrak{F}: \mathfrak{H}\to\mathfrak{H}^{(p)}$$

is of the form $\mathfrak{H}'^{(p)}$ for some closed subgroup $\mathfrak{H}'$ of $\mathfrak{H}$. We have

$$\mathbf{hy}(\mathfrak{H}')^{(p)}=\mathrm{Im}\,(\mathbf{hy}(\mathfrak{F})=\Upsilon_{\mathbf{hy}(\mathfrak{H})}: \mathbf{hy}(\mathfrak{H})\to\mathbf{hy}(\mathfrak{H})^{(p)})\supset J^{(p)}$$

since the map $\Upsilon_J: J\to J^{(p)}$ is surjective (1.9.4). Since $\mathfrak{H}$ is connected, it follows that $\mathfrak{H}=\mathfrak{H}'$. This implies that $\mathfrak{H}$ is smooth by (3.3.5 (iv)). Therefore the subalgebra generated by $[J, J]$, which is the same as the one generated by $[\mathbf{A}(J), \mathbf{A}(J)]$, is equal to $\mathbf{hy}(\mathscr{D}(\mathfrak{H}))$ by (3.5.8). This proves Proposition.

3.6.4 COROLLARY. Let J be a sub-hyperalgebra of $\mathbf{hy}(\mathfrak{G})$. Suppose that k is perfect. If J is reduced and generated by $[J, J]$ as an algebra, then J is algebraic.

3.6.5 PROPOSITION. *Let $\mathfrak{f}: \mathfrak{G}\to\mathfrak{H}$ be a map of connected algebraic k-groups. Then the induced map of hyperalgebras*

$$\mathbf{hy}(\mathfrak{f}): \mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{H})$$

is bijective iff $\mathfrak{f}$ is faithfully flat and $\mathfrak{Ker}\,(\mathfrak{f})$ is finite [7, I, § 5, 1.1] *and étale.*

PROOF. Since $\mathrm{Im}\,(\mathbf{hy}(\mathfrak{f}))=\mathbf{hy}(\widetilde{\mathfrak{f}(\mathfrak{G})})$ (3.3.2), it follows that $\mathbf{hy}(\mathfrak{f})$ is surjective iff $\widetilde{\mathfrak{f}(\mathfrak{G})}=\mathfrak{H}$, or equivalently $\mathfrak{f}$ is faithfully flat. On the other hand $\mathbf{hy}(\mathfrak{f})$ is injective iff $\mathfrak{Ker}\,(\mathfrak{f})$ is étale (3.3.3). But an algebraic k-group is finite if it is étale (cf. [7, II, § 5, 1.10]).

3.6.6 PROPOSITION. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be connected algebraic k-groups. Suppose that k is perfect, $\mathfrak{G}$ is* smooth *and* $\mathfrak{G}=\mathscr{D}(\mathfrak{G})$. *Then for any hyperalgebra map* $\phi: \mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{H})$, *there exist a connected algebraic k-group $\mathfrak{G}'$ and maps of k-groups*

$$\mathfrak{p}: \mathfrak{G}'\to\mathfrak{G} \quad \textit{and} \quad \omega: \mathfrak{G}'\to\mathfrak{H}$$

such that $\mathbf{hy}(\omega)=\phi\circ\mathbf{hy}(\mathfrak{p})$ *and* $\mathbf{hy}(\mathfrak{p})$ *is bijective.*

PROOF. We have

$$\mathbf{hy}(\mathfrak{G}\times\mathfrak{H})=\mathbf{hy}(\mathfrak{G})\otimes\mathbf{hy}(\mathfrak{H}).$$

Let $\psi:\mathbf{hy}(\mathfrak{G})\to\mathbf{hy}(\mathfrak{G}\times\mathfrak{H})$ be the composite:

$$\mathbf{hy}(\mathfrak{G})\xrightarrow{\Delta}\mathbf{hy}(\mathfrak{G})\otimes\mathbf{hy}(\mathfrak{G})\xrightarrow{1\otimes\phi}\mathbf{hy}(\mathfrak{G})\otimes\mathbf{hy}(\mathfrak{H}).$$

Then ψ is an injective hyperalgebra map, since $(1\otimes\varepsilon)\circ\psi=1$. Let J be the image of ψ. Since $\mathfrak{G}$ is smooth, J is reduced. Since $\mathfrak{G}=\mathscr{D}(\mathfrak{G})$, J is generated by $[J,J]$ as an algebra. Hence J is an algebraic sub-hyperalgebra of $\mathbf{hy}(\mathfrak{G}\times\mathfrak{H})$. Let $\mathfrak{G}'$ be a connected subgroup of $\mathfrak{G}\times\mathfrak{H}$ such that $\mathbf{hy}(\mathfrak{G}')=J$. If we put

$$\mathfrak{p}=\mathfrak{pr}_1\,|\,\mathfrak{G}':\mathfrak{G}'\to\mathfrak{G}\quad\text{and}$$
$$\omega=\mathfrak{pr}_2\,|\,\mathfrak{G}':\mathfrak{G}'\to\mathfrak{H}$$

then the assertion holds.

3.6.7 COROLLARY. *Let $\mathfrak{G}$ be a connected algebraic smooth k-group. Suppose that $\mathfrak{G}=\mathscr{D}(\mathfrak{G})$ and k is perfect. Then the condition:*

(*SC*) *Any map of connected algebraic k-groups $\mathfrak{p}:\mathfrak{G}'\to\mathfrak{G}$ which is faithfully flat with the kernel $\mathfrak{Ker}\,(\mathfrak{p})$ finite and etale is an isomorphism.*

implies that the map:

$$\boldsymbol{Gr}_k(\mathfrak{G},\mathfrak{H})\to\boldsymbol{Hopf}_k\,(\mathbf{hy}(\mathfrak{G}),\mathbf{hy}(\mathfrak{H})),\ \mathfrak{f}\mapsto\mathbf{hy}(\mathfrak{f})$$

is bijective for any locally algebraic k-group $\mathfrak{H}$.[20)]

PROOF. This is injective by (3.3.10). Since $\mathbf{hy}(\mathfrak{H})=\mathbf{hy}(\mathfrak{H}^0)$, the surjectivity follows from (3.6.6).

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Foot Notes

P. 5, 1) A^0 denotes the dual k-coalgebra [11, §6.0] of a k-algebra A.

P. 8, 2) I.e., the set of finite dimensional subcoalgebras of C forms a directed family whose union equals C.

P. 13, 3) This abuse of notation (see 1.2.8) may lead no confusion.

P. 27, 4) It will be shown that $K\dim C$ is *finite* [15, 4.3.4.3].

P. 29, 5) We call $(\mathrm{C}(V), \mu, \eta, S)$ the *additive C-Hopf algebra on* V and denote it by $\mathrm{C_a}(V)$.

P. 49, 6) The case of cocomutative coalgebras of *finite type* (but not necessarily connected) will be treated in Part II [15, 4.4.5.8].

P. 63, 7) It is to generalize the results here to the case of *p-Lie algebras.*

P. 70, 8) We view $\|\mathfrak{X}\|$ as a subset of $|\mathfrak{X}|$ via ω, since the map ω is injective, e.g., by [7, I, §2, 2.7].

P. 72, 9) For each $\alpha \in \|\mathfrak{X}\|$, let $\mathfrak{X}_\alpha \subset \mathfrak{X}$ be the following subfunctor: $\mathfrak{X}_\alpha(R)$ $(R \in \boldsymbol{M}_k)$ is the set of $\rho \in \mathfrak{X}(R)$ with $\langle \mathfrak{X}(\phi)(\rho)\rangle = \alpha$ for all $\phi: R \to K$ in $\boldsymbol{M}_k$ with $K \in \boldsymbol{Fld}_k^{\mathrm{f}}$. If $\mathbf{T}(\mathfrak{X}_\alpha)$ exists, we define $\mathrm{S}_\alpha(\mathfrak{X}) = \mathbf{T}(\mathfrak{X}_\alpha)$. Then the set $\boldsymbol{W}_k(C, \mathrm{S}_\alpha(\mathfrak{X}))$ is naturally isomorphic to

$$\{f \in \mathfrak{X}(C^*) \mid \langle f_{D^*}\rangle = \alpha \text{ for all } D \subset C \text{ simple}\}$$

for $C \in \boldsymbol{W}_k^{\mathrm{f}}$.

P. 72, 10) Note that $\boldsymbol{W}_k(C_0, \mathrm{S}_e(\mathfrak{X}))$ consists of ohly one element for $C \in \boldsymbol{W}_k^{\mathrm{f}}$.

P. 94, 11) A *subscheme* $\mathfrak{Y}$ of a locally algebraic k-scheme $\mathfrak{X}$ is clearly locally algebraic with the inclusion $\mathfrak{i}: \mathfrak{Y} \to \mathfrak{X}$ *quasi-compact.* Hence $\mathfrak{Y} = \mathfrak{X}$ iff $\mathbf{T}(\mathfrak{Y}) = \mathbf{T}(\mathfrak{X})$. More generally let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{Y}$ be a map of locally algebraic k-schemes. If $\mathfrak{f}$ is *faithfully flat,* then $\mathbf{T}(\mathfrak{X})$ is an *injective cogenerasor* in the category of $\mathbf{T}(\mathfrak{Y})$-comodules, or equivalently $\mathbf{T}(\mathfrak{X})$ is an injective $\mathbf{T}(\mathfrak{Y})$-commodule with $\mathbf{T}(\mathfrak{f})$ *surjective.* The converse holds true, if $\mathfrak{f}$ is *quasi-compact.*

P. 96, 12) The classes $\mathscr{P}_n, \mathscr{P}_e$ and $\mathscr{P}_e$ are explicity described in [15, 4.4.3.3, 4.4.5.1].

P. 101, 13) This is in the sence of the footnote 9).

P. 114, 14) Similarly $\mathbf{T}(\widetilde{\mathfrak{f}(\mathfrak{G})})$ equals the image of $\mathbf{T}(\mathfrak{f}): \mathbf{T}(\mathfrak{G}) \to \mathbf{T}(\mathfrak{H})$.

P. 116, 15) Since the Frobenius map $\mathfrak{F}_{\mathfrak{X}}: \mathfrak{X} \to \mathfrak{X}^{(p)}$ is always *surjective,* the condition (iv) implies that $\mathfrak{F}_{\mathfrak{G}}$ is *faithfully flat.*

P. 117, 16) On the other hand, $\widetilde{f(\mathfrak{G})} = \mathfrak{H}$ if and only if $\mathbf{T}(\mathfrak{f}): \mathbf{T}(\mathfrak{G}) \to \mathbf{T}(\mathfrak{H})$ is *surjective.*

P. 117, 17) In general $\mathfrak{f}_1 = \mathfrak{f}_2$ if and only if $\mathbf{T}(\mathfrak{f}_1) = \mathbf{T}(\mathfrak{f}_2)$ for $\mathfrak{G}$ arbitrary.

P. 138, 18) More generally let $\mathfrak{G}$ be an algebraic k-group and $\mathfrak{H}, \mathfrak{K}$ be *smooth* closed subgroups of $\mathfrak{G}$ satisfying at least *one of* the following conditions: (a) One of $\mathfrak{H}$ and $\mathfrak{K}$ is connected or (b) One of $\mathfrak{H}$ and $\mathfrak{K}$ normalizes the other. The commutator subgroup $[\mathfrak{H}, \mathfrak{K}]$ which is smooth closed, can be defined (see [7, II, §5, 4.9] in case (a) and [Borel, Linear algebraic groups, pp. 108, 111] in case (b)). We can prove that $\mathbf{T}([\mathfrak{H}, \mathfrak{K}])$ is the *subalgebra* of $\mathbf{T}(\mathfrak{G})$ generated by $[\mathbf{T}(\mathfrak{H}), \mathbf{T}(\mathfrak{K})]$ with the notation of 3.5. If $\mathfrak{H}$ is *connected,* $\mathbf{hy}([\mathfrak{H}, \mathfrak{K}])$ is the subalgebra of $\mathbf{hy}(\mathfrak{G})$ generated by $[\mathbf{hy}(\mathfrak{H}), \mathbf{T}(\mathfrak{K})]$. If both $\mathfrak{H}$ and $\mathfrak{K}$ are connected, $\mathbf{hy}([\mathfrak{H}, \mathfrak{K}])$ is generated by $[\mathbf{hy}(\mathfrak{H}), \mathbf{hy}(\mathfrak{K})]$ as an algebra. On the other hand if both $\mathfrak{H}$ and $\mathfrak{K}$ normalize the other, then $\mathbf{hy}([\mathfrak{H}, \mathfrak{K}])$ is the subalgebra gederated by $[\mathbf{T}(\mathfrak{H}), \mathbf{hy}(\mathfrak{K})] + [\mathbf{hy}(\mathfrak{H}), \mathbf{T}(\mathfrak{K})]$. In particular $\mathbf{hy}(\mathscr{D}(\mathfrak{G}))$ is generated by $[\mathbf{T}(\mathfrak{G}), \mathbf{hy}(\mathfrak{G})]$ as an algebra.

Further the above results can be generalized to the case of *not necessarily smooth* subgroups $\mathfrak{H}$ and $\mathfrak{K}$ (but satisfying one of (a) and (b)).

P. 140, 19) It will be shown elsewhere that the assumptions on perfectness of k and reducedness of J can be dropped.

P. 141, 20) In fact the assumptions of smoothness and perfectness are unnecessarily. The assumption $\mathfrak{G} = \mathscr{D}(\mathfrak{G})$ can be weakened to that $\mathfrak{G}/\mathscr{D}(\mathfrak{G})$ is *finite* [16, Lem. 1.8].