15. On Some Power Series and their Sections, II.

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In this paper we shall prove the following theorem, which is an extension of a theorem given by Mr. Tsuji⁽¹⁾.

Theorem: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be regular for $|x| < 1$, and all its singularities $\omega_i (i=1, 2, \dots, k)$ on $|x| = 1$ be isolated and of the following nature;— $f(x)$ may be put in the form $(x - \omega_i)^{\lambda_i} \varphi_i(x)$ in $|x - \omega_i| < \delta$ and $\Re \frac{x}{\omega_i} \leq 1$ for a positive small δ , where λ_i are any real numbers and $\varphi_i(x)$ is limited in the above domain;—then, putting

$$(1) \quad f(x) = \sum_{n=0}^{\infty} a_n x^n = g_1\left(\frac{x}{\omega_1}\right) + g_2\left(\frac{x}{\omega_2}\right) + \dots + g_k\left(\frac{x}{\omega_k}\right) \\ = \sum_{n=0}^{\infty} b_{1,n} x^n + \sum_{n=0}^{\infty} b_{2,n} x^n + \dots + \sum_{n=0}^{\infty} b_{k,n} x^n,$$

where $g_1\left(\frac{x}{\omega_1}\right), \dots, g_k\left(\frac{x}{\omega_k}\right)$ are functions which are regular for $|x| \leq 1$ except only one point $x = \omega_i (i=1, \dots, k)$ respectively, and have $x = \omega_i$ as their isolated singularities of the above nature, we have for the n^{th} section

$$f_n(x) = \sum_{n=0}^n a_n x^n \quad \text{of} \quad f(x)$$

uniformly

(I) for $|x| \geq 1 + \varepsilon_1, \quad \varepsilon_1 > 0$

$$(2) \quad f_n(x) = x^{n+1} \left\{ \sum_{i=1}^k \omega_i^{-(n+1)} \left(\frac{b_{i,n}}{\frac{x}{\omega_i} - 1} - \frac{b_{i,n} - b_{i,n-1}}{\left(\frac{x}{\omega_i} - 1\right)^2} + \dots + (-1)^r \right) \right\}$$

(¹) Tsuji, this Journal **3** (1926) 96.

$$\times \frac{b_{i,n} - r b_{i,n-1} + \dots + (-1)^i \frac{r(r-1)\dots(r-(i-1))}{1 \cdot 2 \dots i} b_{i,n-i} + \dots + (-1)^r b_{i,n-r}}{\left(\frac{x}{\omega_i} - 1\right)^{r+1}} \Bigg) \\ + o(1) + O(\eta^n) \Bigg\}$$

(II) for $1 + \varepsilon_2 \geq |x| \geq 1$ and $|x - \omega_i| \geq \varepsilon_2$, $\varepsilon_2 > 0$

$$(3) \quad f_n(x) = f(x) + x^{n+1} \Bigg\{ \sum_{i=1}^k \omega_i^{-(n+1)} \left(\frac{b_{i,n}}{\frac{x}{\omega_i} - 1} - \frac{b_{i,n} - b_{i,n-1}}{\left(\frac{x}{\omega_i} - 1\right)^2} + \dots + (-1)^r \right. \\ \times \frac{b_{i,n} - r b_{i,n-1} + \dots + (-1)^i \frac{r(r-1)\dots(r-(i-1))}{1 \cdot 2 \dots i} b_{i,n-i} + \dots + (-1)^r b_{i,n-r}}{\left(\frac{x}{\omega_i} - 1\right)^{r+1}} \Bigg) \\ \left. + o(1) + O(\eta^n) \right\}$$

(III) for $|x| \geq 1$ and $|x - \omega_i| \geq \varepsilon_3$ we have (2) or (3).
where $\eta < 1$ and $b_{i,n}, b_{i,n-1}, \dots$ are independent of the decomposition of $f(x)$ into $g_1 + g_2 + \dots + g_k$ except a quantity $O(\eta^n)$.

Remark: In a similar way we can obtain a perfectly similar theorem for the sections of a function which has only isolated singular points of the following nature;— $f(x)$ may be put in the form $(x - \omega_i)^{\lambda_i} \varphi_i(x) \times (\log_i(x - \omega_i))^k$ in $|x - \omega_i| < \delta$ and $\Re \frac{x}{\omega_i} \leq 1$.

Algebraic and algebrico-logarithmic singularities and some class of isolated essential singularities, for instance, $e^{\frac{1}{x-1}}$, $e^{\frac{1}{e^{x-1}}}$, $(x-1)^\lambda e^{\frac{1}{x-1}}$ etc. are implied in the above class of singularities.

The proof follows immediately by combining k functions $g_1, g_2, \dots, g_k$ of the following lemma:

Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be regular for $|x| \leq 1$ except $x=1$, and $x=1$ be an isolated singular point of $g(x)$ of the following nature;— $g(x)$ may be put in the following form $(x-1)^\lambda \varphi(x)$ in $|x-1| < \delta$ and $\Re x \leq 1$; then we have uniformly, for the n^{th} section $g_n(x) = \sum_{n=0}^n b_n x^n$ of $g(x)$;

(I) for $|x| \geq 1 + \varepsilon_1$

$$(4) \quad g_n(x) = x^{n+1} \left\{ \frac{b_n}{x-1} - \frac{b_n - b_{n-1}}{(x-1)^2} + \dots \right. \\ \left. \dots + (-1)^r \frac{b_n - r b_{n-1} + \dots + (-1)^i b_{n-i} \frac{r \dots (r-(i-1))}{1 \cdot 2 \dots i}}{(x-1)^{r+1}} + \dots + (-1)^r b_{n-r} \right\} + o(1)$$

(II) for $1 + \varepsilon_2 \geq |x| \geq 1$ and $|x-1| \geq \varepsilon_2$

$$(5) \quad g_n(x) = g(x) + x^{n+1} \left\{ \frac{b_n}{x-1} - \frac{b_n - b_{n-1}}{(x-1)^2} + \dots \right. \\ \left. \dots + (-1)^r \frac{b_n - r b_{n-1} + \dots + (-1)^i b_{n-i} \frac{r \dots (r-(i-1))}{1 \cdot 2 \dots i}}{(x-1)^{r+1}} + \dots + (-1)^r b_{n-r} \right\} + o(1)$$

and (III) for $|x| \geq 1$ and $|x-1| \geq \varepsilon_3$ we have (4) or (5).

Proof of the lemma:

(I) For $|x| \geq 1 + \varepsilon_1$ we have

$$g_n(x) = b_0 + b_1 x + \dots + b_n x^n = \frac{1}{2\pi i} \int_C \frac{g(z)}{z} \left[1 + \frac{x}{z} + \dots + \left(\frac{x}{z} \right)^n \right] dz \\ = \frac{x^{n+1}}{2\pi i} \int_C \frac{g(z) dz}{z^{n+1}(x-z)} - \frac{1}{2\pi i} \int_C \frac{g(z)}{x-z} dz$$

where C is any closed curve about the origin, lying in the domain of regularity of $g(x)$.

We take the path of integration C in the following way,

$$C = L + l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7$$

L : $z = (1 + \varepsilon)e^{i\theta}$, θ varying in the positive sense from 0 to 2π , where $0 < \varepsilon < \varepsilon_1$ is to be taken such that $g(x)$ has no singularity in $|x| \leq 1 + \varepsilon$ except on $|x| = 1$.

l_1 : $z = t$, t varying from $1 + \varepsilon$ to $1 + \varepsilon(n)$,

l_2 : $z = (1 + \varepsilon(n))e^{i\theta}$, θ varying in the negative sense from 0 to $-\theta_n$
where $\varepsilon(n)$ will be determined later and $\theta_n = \arccos \frac{1}{1 + \varepsilon(n)}$, $0 < \theta_n < \frac{\pi}{2}$,

$l_3: z=1+te^{-i\frac{\pi}{2}}, t$ varying
from $\sqrt{2\varepsilon(n)+\varepsilon^2(n)}$ to $\frac{1}{n}$,

$l_4: z=1+\frac{1}{n}e^{i\theta}, \theta$ varying
in the negative sense from
 $-\frac{\pi}{2}$ to $\frac{\pi}{2}$,

$l_5: z=1+te^{i\frac{\pi}{2}}, t$ varying
from $\frac{1}{n}$ to $\sqrt{2\varepsilon(n)+\varepsilon^2(n)}$,

$l_6: z=(1+\varepsilon(n))e^{i\theta}, \theta$ va-
rying in the negative sense from
 θ_n to 0,

$l_7: z=t, t$ varying from
 $1+\varepsilon(n)$ to $1+\varepsilon$.

Let $M\left(\frac{1}{v}\right)$ be the maximum
of $|g(x)|$ on $|x-1|=v$, then
we take $\varepsilon(n)$ so that $\varepsilon(n) \rightarrow 0$
and $M\left(\frac{1}{\varepsilon(n)}\right) e^{\frac{n}{\varepsilon(n)}} \rightarrow 0$ for $n \rightarrow \infty$.

Now

$$\frac{1}{2\pi i} \int_c \frac{g(z)}{x-z} dz = 0$$

and

$$\begin{aligned} \frac{1}{2\pi i} \int_c \frac{g(z) dz}{z^{n+1}(x-z)} &= \frac{1}{2\pi i} \int_c \frac{g(z)}{z^{n+1}(x-1)} \left[1 + \frac{z-1}{x-1} + \dots \right. \\ &\dots + \left(\frac{z-1}{x-1} \right)^r + \frac{\left(\frac{z-1}{x-1} \right)^{r+1}}{1 - \left(\frac{z-1}{x-1} \right)} \Big] dz = \frac{1}{2\pi i} \int_c \frac{g(z)}{z^{n+1}} \left[\frac{1}{x-1} + \frac{z-1}{(x-1)^2} + \dots \right. \\ &\dots + \frac{(z-1)^r}{(x-1)^{r+1}} \Big] dz + \frac{1}{2\pi i} \int_c \frac{g(z)}{z^{n+1}(x-z)} \left(\frac{z-1}{x-1} \right)^{r+1} dz \end{aligned}$$

where we take r so large that $g(z)(z-1)^{r+1} = (z-1)^l \varphi(z)$, $l > 0$, in
 $|z-1| < \delta$ and $\Re z \leq 1$, $\varphi(z)$ being limited there.

$$\frac{1}{2\pi i} \int_C \frac{g(z)}{z^{n+1}(x-z)} \left(\frac{z-1}{x-1} \right)^{r+1} dz = \frac{1}{2\pi i} \int_L + \frac{1}{2\pi i} \int_{l_1} + \dots + \frac{1}{2\pi i} \int_{l_7}.$$

Since, on L , $|g(z)(z-1)^{r+1}|$ is limited, we have

$$\left| \frac{1}{2\pi i} \int_L \right| < K \frac{1}{(1+\varepsilon)^n}$$

On $l_1 + l_7$, $|g(z)(z-1)^{r+1}| < M \left(\frac{1}{\varepsilon(n)} \right) K$

hence

$$\left| \frac{1}{2\pi i} \int_{l_1+l_7} \right| < M \left(\frac{1}{\varepsilon(n)} \right) \frac{K}{(1+\varepsilon(n))^n} = M \left(\frac{1}{\varepsilon(n)} \right) e^{-\frac{n}{\varepsilon(n)}} \cdot K$$

On $l_2 + l_6$, $|g(z)(z-1)^{r+1}| < M \left(\frac{1}{\varepsilon(n)} \right) (2\varepsilon(n) + \varepsilon^2(n))^{\frac{r+1}{2}}$

then

$$\left| \frac{1}{2\pi i} \int_{l_2+l_6} \right| < M \left(\frac{1}{\varepsilon(n)} \right) \frac{K \varepsilon(n)^{\frac{r+1}{2}}}{(1+\varepsilon(n))^n} \int |dz| < M \left(\frac{1}{\varepsilon(n)} \right) e^{-\frac{n}{\varepsilon(n)}} \cdot K$$

On $l_3 + l_5$, $|g(z)(z-1)^{r+1}| < (z-1)^l K$

hence

$$\left| \frac{1}{2\pi i} \int_{l_3+l_5} \right| < K (2\varepsilon(n) + \varepsilon^2(n))^{\frac{l+1}{2}}$$

On l_4 , $|g(z)(z-1)^{r+1}| < (z-1)^l K$

hence

$$\left| \frac{1}{2\pi i} \int_{l_4} \right| < \left(\frac{1}{n} \right)^{l+1} \frac{K}{\left(1 - \frac{1}{n} \right)^n} = \left(\frac{1}{n} \right)^{l+1} K.$$

On the other hand $\int_C \frac{g(z)(z-1)^\lambda}{z^{n+1}} dz$ is the coefficient of z^n in the expansion of $g(z)(z-1)^\lambda$ in the power series of z , that is

$$g(z)(z-1)^\lambda = (-1)^\lambda \left\{ b_0 + (b_1 - \lambda b_0)z + \dots + \left(b_n - b_{n-1}\lambda + b_{n-2} \frac{\lambda(\lambda-1)}{1 \cdot 2} + \dots \right. \right. \\ \left. \left. \dots (-1)^l b_{n-l} \frac{\lambda(\lambda-1) \dots (\lambda-(l-1))}{1 \cdot 2 \dots l} + \dots (-1)^{\lambda-1} b_{n-\lambda+1} \lambda + (-1)^\lambda b_{n-\lambda} \right\} z^n + \dots \right\}.$$

Hence we have (4).

(II) For $1+\varepsilon_2 \geq |x| \geq 1$, $|x-1| \geq \varepsilon_2$ we have, by taking $\varepsilon > \varepsilon_2$,

$$\frac{1}{2\pi i} \int_C \frac{g(z)}{x-z} = -g(x).$$

Similarly as in (I) we have (5).

(III) Combining the two cases (I) and (II), (6) follows at once by suitable choice of $\varepsilon_1, \varepsilon_2, \varepsilon$.

Under various assumptions on the coefficients of $\sum a_n x^n$ we will obtain several simple properties of the sections of the power series.

Example I: Let all the singularities ω_i ($i=1, \dots, k$) on $|x|=1$ of $h(x)$ be algebraic, so that $h(x) = c_i^0 \left(1 + \frac{x}{\omega_i}\right)^{-\lambda_i} + \dots$ and $-\lambda = \min(-\lambda_i)$, then by taking the path of integration C in the proof of the lemma such as

$L: z = (1+\varepsilon)e^{i\theta}$, θ varying in the positive sense from 0 to 2π ,

$l_1: z = t$, t varying from $1+\varepsilon$ to $1 + \frac{1}{n^{1-\varepsilon}}$,

$l_2: z = t$, t varying from $1 + \frac{1}{n^{1-\varepsilon}}$ to $1 + \frac{1}{n}$,

$l_3: z = 1 + \frac{1}{n}e^{i\theta}$, θ varying from 0 to -2π ,

$l_4: z = t$, t varying from $1 + \frac{1}{n}$ to $1 + \frac{1}{n^{1-\varepsilon}}$,

$l_5: z = t$, t varying from $1 + \frac{1}{n^{1-\varepsilon}}$ to $1+\varepsilon$,

we obtain

$$h_n(x) = x^{n+1} \left\{ \frac{\lambda(\lambda+1)\dots(\lambda+n-1)}{n!} \left[\sum_{i=1}^k \frac{c_i^0 \omega_i^{-(n+1)}}{x\omega_i^{-1}-1} + \sum_{i=1}^k \left(\frac{\varepsilon_1^{(i)}}{x\omega_i^{-1}-1} + \dots \right. \right. \right. \\ \left. \left. \left. \dots + \frac{\varepsilon_r^{(i)}}{(x\omega_i^{-1}-1)^r} \right) \right] + O(n^{\lambda-r-1+\varepsilon}) \right\}$$

uniformly for $|x| \geq 1 + \varepsilon_1$.

Similarly for $1+\varepsilon_2 \geq |x| \geq 1$, $|x-\omega_i| \geq \varepsilon_2$; and for $|x| \geq 1$, $|x-\omega_i| \geq \varepsilon_3$.

Example II: Let

$$f(x) = \sum_{i=1}^k c_i^0 \left(\frac{x}{\omega_i} - 1 \right)^{-\lambda} + \sum_{i=1}^k d_i e^{\frac{1}{\frac{x}{\omega_i} - 1}} + \varphi(x)$$

where $\varphi(x)$ is regular for $|x| \leq 1$ and $\lambda \geq 1$

First we consider $g(x) = d_0 e^{\frac{1}{x-1}} + \varphi(x) = \sum_{n=0}^{\infty} a_n x^n + \varphi(x)$, then by taking the path of integration C in the proof of the lemma such as

$$\begin{aligned} L: & z = (1+\varepsilon)e^{i\theta}, \quad \theta \text{ varying in the positive sense from } 0 \text{ to } 2\pi, \\ l_1: & z = t, \quad t \text{ varying from } 1+\varepsilon \text{ to } 1+\frac{1}{n^{\frac{1}{2}-\varepsilon}}, \\ l_2: & z = \left(1+\frac{1}{n^{\frac{1}{2}-\varepsilon}}\right)e^{i\theta}, \quad \theta \text{ varying in the negative sense from } 0 \text{ to } -\theta_n, \\ l_3: & z = 1+te^{-i\frac{\pi}{2}}, \quad t \text{ varying from } \sqrt{2n^{-1+2\varepsilon}+n^{-2+\varepsilon}} \text{ to } \frac{1}{n}, \\ l_4: & z = 1+\frac{1}{n}e^{i\theta}, \quad \theta \text{ varying in the negative sense from } -\frac{\pi}{2} \text{ to } \frac{\pi}{2}, \\ l_5: & z = 1+te^{i\frac{\pi}{2}}, \quad t \text{ varying from } \frac{1}{n} \text{ to } \sqrt{2n^{-1+2\varepsilon}+n^{-2+\varepsilon}}, \\ l_6: & z = \left(1+\frac{1}{n^{\frac{1}{2}-\varepsilon}}\right)e^{i\theta}, \quad \theta \text{ varying in the negative sense from } \theta_n \text{ to } 0, \\ l_7: & z = t, \quad t \text{ varying from } 1+\frac{1}{n^{\frac{1}{2}-\varepsilon}} \text{ to } 1+\varepsilon. \end{aligned}$$

and by choosing a positive integer s , we obtain

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_L \frac{g(z)(z-1)^{s+1}}{z^{n+1}(x-z)} dz \right| &< K \frac{1}{(1+\varepsilon)^n}, \quad \left| \frac{1}{2\pi i} \int_{l_1+l_7} \right| < K e^{-n^{\frac{1}{2}+\varepsilon}}, \\ \left| \frac{1}{2\pi i} \int_{l_2+l_6} \right| &< K e^{-n^{\frac{1}{2}+\varepsilon}}, \quad \left| \frac{1}{2\pi i} \int_{l_3+l_5} \right| < K n^{-\frac{s+2}{4}+\frac{\varepsilon s}{2}} \end{aligned}$$

$$\text{and } \left| \frac{1}{2\pi i} \int_{l_4} \right| < K n^{-(s+2)}.$$

Hence we have for a small positive ε

$$\begin{aligned} g_n(x) = x^n &\left\{ \frac{a_n}{x-1} - \frac{a_n - a_{n-1}}{(x-1)^2} + \dots \right. \\ &+ (-1)^s \frac{a_n - a_{n-1}s + \dots + (-1)^{s-1}s a_{n-s+1} + (-1)^s a_{n-s}}{(x-1)^{s+1}} \\ &\left. + O\left(n^{-\frac{s+1}{4}-1+\varepsilon}\right) + O(\eta^n) \right\} \end{aligned}$$

uniformly for $|x| \geq 1+\varepsilon_1$

and similarly for $1 + \varepsilon_2 \geq |x| \geq 1$, $|x-1| \geq \varepsilon_2$ and for $|x-1| \geq \varepsilon_3$, $|x| \geq 1$.

$$\text{On the other hand} \quad a_n = \int_C \frac{g(z)}{z^{n+1}} dz$$

where the path of integration C is the same as above, that is,

$$C = L + l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7,$$

and we have

$$a_n = O\left(n^{-\frac{1}{4} + \varepsilon}\right).$$

Similarly

$$\begin{aligned} a_n - a_{n-1} &= \int_C \frac{g(z)(z-1)}{z^{n+1}} dz = O\left(n^{-\frac{3}{4} + \varepsilon}\right) \\ &\dots\dots\dots \\ &\dots\dots\dots \\ a_n - sa_{n-1} + \dots + (-1)^s a_{n-s} &= \int \frac{g(z)(z-1)^{s+1}}{z^{n+1}} dz = O\left(n^{-\frac{s+2}{4} + \varepsilon}\right). \end{aligned}$$

Now consider

$$f(x) = \sum_{i=1}^k c_i^0 \left(\frac{x}{\omega_i} - 1\right)^{-\lambda} + \sum_{i=1}^k d_i^0 e^{\frac{x}{\omega_i} - 1} + \varphi(x).$$

This can be decomposed into

$$\begin{aligned} f(x) &= h_1\left(\frac{x}{\omega_1}\right) + h_2\left(\frac{x}{\omega_2}\right) + \dots + h_k\left(\frac{x}{\omega_k}\right) \\ &\quad + g_1\left(\frac{x}{\omega_1}\right) + g_2\left(\frac{x}{\omega_2}\right) + \dots + g_k\left(\frac{x}{\omega_k}\right) + \varphi(x). \end{aligned}$$

By the above

$$\begin{aligned} h_1\left(\frac{x}{\omega_1}\right) + \dots + h_k\left(\frac{x}{\omega_k}\right) &= x^{n+1} \binom{\lambda + n - 1}{n} \left\{ \sum_{i=1}^k \left(\frac{\omega_i^{-(n+1)}}{\frac{x}{\omega_i} - 1} \right. \right. \\ &\quad \left. \left. + \frac{\varepsilon_1}{\frac{x}{\omega_i} - 1} + \dots + \frac{\varepsilon_r}{\left(\frac{x}{\omega_i} - 1\right)^{r+1}} \right) + O\left(n^{\lambda - r - 1}\right) \right\} \end{aligned}$$

$$g_1\left(\frac{x}{\omega_1}\right) + \dots + g_k\left(\frac{x}{\omega_k}\right) = x^{n+1} \left\{ \sum_{i=1}^k \omega_i^{-(n+1)} \left(\frac{a_n}{\frac{x}{\omega_i} - 1} \right. \right. \\ \left. \left. + \dots + (-1)^s \frac{a_n - s a_{n-1} + \dots + (-1)^s a_{n-s}}{\left(\frac{x}{\omega_i} - 1 \right)^{s+1}} \right) + O\left(n^{-\frac{s+2}{4} + \varepsilon}\right) \right\}.$$

Hence, putting $s=4r$,

we have

$$f_n(x) = x^{n+1} \binom{\lambda + n - 1}{n} \left\{ \sum_{i=1}^k \frac{\omega_i^{-(n+1)}}{\left(\frac{x}{\omega_i} - 1 \right)} \right. \\ \left. + \frac{\varepsilon_1'}{\left(\frac{x}{\omega_i} - 1 \right)} + \dots + \frac{\varepsilon_s'}{\left(\frac{x}{\omega_i} - 1 \right)^{s+1}} + O\left(n^{\lambda-r-1}\right) \right\}.$$

For these functions in the example II we can obtain also the following all properties of the sections which Mr. Tsuji has obtained for the sections of the functions in the example I.

Under the conditions in the examples I or II we have

I $\lim_{n \rightarrow \infty} [|f_n(x_1)| + |f_n(x_2)| + \dots + |f_n(x_k)|] = \infty,$

where $x_1, x_2, \dots, x_k$ are any k points outside the unit circle.

II $\lim_{n \rightarrow \infty} [|f_n(x)| + |f_{n+1}(x)| + \dots + |f_{n+k-1}(x)|] = \infty,$

uniformly for $|x| \geq 1 + \varepsilon_1$.

III There exists a domain outside $|x|=1$ near a certain singular point on $|x|=1$ in which $\lim_{n \rightarrow \infty} f_n(x) = \infty$ uniformly.

IV $\lim_{n \rightarrow \infty} f_n(x) = \infty$ almost every where, outside $|x|=1$ i.e. with exception of at most a point set of linear measure zero.

V The number of the roots of $f_n(x)=0$ which lie outside $|x| = 1 + \varepsilon_1 > 1$ is at most $k-1$.

VI The derived set of the roots of $f_n(x)=0, f_n'(x)=0, f_n''(x)=0, \dots, \dots, \int_0^x f(x)dx=0, \dots$ are identically equal outside $|x|=1$.

VII The derived set of the roots of $f_n(x)=0$ outside $|x|=1$

(i) has no point when $k=1$.

(ii) forms a circle which cuts the unit circle orthogonally when $k=2$,

(iii) forms certain domains which are bounded by algebraic curves when $k \geq 3$ except certain cases.

VIII Let the number of the roots of $f_n(x)=0$ in the domain

$$1 + \varepsilon_2 \geq |x| \geq 1 - \varepsilon_1 \quad \theta_0 + \theta \geq \arg x \geq \theta_0$$

be denoted by $N(\theta)$, then $\lim \frac{N(\theta)}{n} = \frac{\theta}{2\pi}$.

The same relation holds when we take $\varepsilon_1 = \frac{\omega_1(n)}{n}$, $\varepsilon_2 = \frac{\omega_2(n)}{n}$, where

$$\frac{\omega_1(n)}{n} \rightarrow 0, \quad \frac{\omega_2(n)}{n} \rightarrow 0, \quad \frac{\omega_1(n)}{\log n} \rightarrow \infty, \quad \frac{\omega_2(n)}{\log n} \rightarrow \infty.$$

Let the number of the roots of $f_n(x)=0$ in the domain

$$1 - \varepsilon_1 \leq |x| \leq 1, \quad \theta_0 \leq \arg x \leq \theta_0 + \theta$$

be denoted by $N_i(\theta)$, then under certain conditions

$$\lim \frac{N_i(\theta)}{n} = \frac{\theta}{2\pi}.$$

IX If every amplitude of all the singularities is a rational multiple of 2π then $f_n(x) \rightarrow \infty$ uniformly for points outside the unit circle except certain points.

The above results and my previous ones which have appeared in *Japanese Journal of Mathematics* IV, No. 4 give rise to an interesting question:—whether there exist or not any remarkable properties of the sections outside the circle of convergence which are characterised by the properties of the functions themselves.

It is evident that the sections of a power series are not related but a negligible quantity $O(\eta^n)$ with the conditions of the functions outside the unit circle. What parts do the conditions on the unit circle play? My other results, it seems to me, suggest a certain relation between the singularities on the unit circle and the behaviour of the sections on it. But as for the behaviour outside the unit circle I have no opinion.