29. *On the Theory of Surfaces in the Affine Space:*

V. Studies on Affine Moulding Surfaces.

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XVI.

A Theorem on Affine Surfaces of Revolution.

37. Theorem 12 shows that the affine surface-normals along the parallel curves of an affine surface of revolution all intersect at a point. This suggests us to inquire: Is there any surface, other than the affine surfaces of revolution in the class of affine moulding surfaces, which possesses the same property? To this question I will answer by the following

Theorem 33. *If affine surface-normals along the parallel curve of an affine moulding surface all intersect at a point, then the surface must necessarily be an affine surface of revolution.*

Proof. Take the equations II (10) for our surface in consideration. The parallel curves $v=\text{const.}$ and accordingly the meridian curves are necessarily affine lines of curvature.

By the hypothesis we may put

$$(1) \quad \mathfrak{x} + \alpha \tilde{\mathfrak{x}} = \mathfrak{y}(v),$$

where $\mathfrak{y}(v) \equiv y_1(v), y_2(v), y_3(v)$ denotes the point where all affine surface-normals $\tilde{\mathfrak{x}}$ along $v=\text{const.}$ intersect.

Differentiating (1) with respect to u and v and using the formulae

$$(2) \quad \tilde{\mathfrak{x}}_u = -\frac{1}{R_1} \mathfrak{x}_u, \quad \tilde{\mathfrak{x}}_v = -\frac{1}{R_2} \mathfrak{x}_v,$$

we are immediately led to

$$(3) \quad \begin{aligned} \mathfrak{x}_u \left(1 - \frac{\alpha}{R_1} \right) + \alpha_u \tilde{\mathfrak{x}} &= 0, \\ \mathfrak{x}_v \left(1 - \frac{\alpha}{R_2} \right) + \alpha_v \tilde{\mathfrak{x}} &= \mathfrak{y}'(v), \end{aligned}$$

whence, after the independence of the vectors $\mathfrak{x}_u$, $\mathfrak{x}_v$ and $\tilde{\mathfrak{x}}$, follows

$$(4) \quad \alpha_u = 0, \quad R_1 = \alpha,$$

that is, R_1 is a function of v alone.

But from the equations to an affine moulding surface:

$$\begin{aligned} x &= \exp(-\int \phi dv) [\int \xi \phi \exp(\int \phi dv) dv + U_1], \text{ etc.} \\ (5) \quad \phi &= (a\xi + b\eta + c\zeta - v)^{-1}, \\ (aU) &\equiv aU_1 + bU_2 + cU_3 = 0, \end{aligned}$$

we obtain

$$\begin{aligned} (6) \quad \xi - x &= \exp(-\int \phi dv) [\int \xi' \exp(\int \phi dv) dv - U_1], \text{ etc.} \\ x_v &= \phi(\xi - x), \text{ etc.} \end{aligned}$$

Substituting (1), (6) into the second equation of (3) and rearranging the terms, we get

$$\begin{aligned} (7) \quad & \phi \exp(-\int \phi dv) [\int \xi' \exp(\int \phi dv) dv] \left(1 - \frac{\alpha}{R_2}\right) \\ & + \frac{\alpha'}{\alpha} [y_1(v) - \exp(-\int \phi dv) (\int \xi \phi \exp(\int \phi dv) dv) - y_1'(v)] \\ & = \exp(-\int \phi dv) \left[\phi \left(1 - \frac{\alpha}{R_2}\right) + \frac{\alpha'}{\alpha} \right] U_1, \end{aligned}$$

and two similar expressions.

Hence it follows:

$$\begin{aligned} (8) \quad & \phi \exp(-\int \phi dv) [\int (a\xi') \exp(\int \phi dv) dv] \left(1 - \frac{\alpha}{R_2}\right) \\ & + \frac{\alpha'}{\alpha} [(ay) - \exp(-\int \phi dv) \{ \int (a\xi) \phi \exp(\int \phi dv) dv \}] = (ay'). \end{aligned}$$

In the last relation all the quantities except R_2 being known as functions of v alone, we conclude that R_2 is also a function of v alone.

Now use the definition-equation of the affine surface-normal⁽¹⁾:

$$\tilde{x} = \frac{1}{2\sqrt{eg}} \left\{ \frac{\partial}{\partial u} \left(\sqrt{\frac{g}{e}} \frac{\partial \tilde{x}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\sqrt{\frac{e}{g}} \frac{\partial \tilde{x}}{\partial v} \right) \right\}$$

and remember in mind the relations

$$\begin{aligned} (ax) &= v, \quad (ax_u) = 0, \quad (ax_v) = 1, \\ (a\tilde{x}) &= \frac{1}{\alpha} [(ay) - v] \equiv \varphi(v) \quad \text{say.} \end{aligned}$$

Then we arrive at

$$(9) \quad \frac{1}{2\sqrt{eg}} \frac{\partial}{\partial v} \sqrt{\frac{e}{g}} = \varphi(v).$$

The quantities e and g are given by II (22), i.e.

(1) Cf. Blaschke, Vorlesungen II, p. 103.

$$e = -c^{-\frac{3}{4}} \phi^{-\frac{1}{2}} V^{\frac{5}{4}} \Delta^{-\frac{1}{4}},$$

$$g = -c^{\frac{1}{4}} \phi^{\frac{3}{2}} V^{\frac{1}{4}} \Delta^{\frac{3}{4}},$$

where

$$\Delta = \begin{vmatrix} U_1' & \xi' & \xi - x \\ U_2' & \eta' & \mu - y \\ U_3' & \zeta' & \xi - z \end{vmatrix}.$$

After substitution of these values and some calculations (9) becomes

$$(10) \quad \frac{\partial}{\partial v} (\phi^{-\frac{3}{2}} V^{\frac{3}{4}} \Delta^{-\frac{3}{4}}) = 3c^{\frac{1}{4}} V \varphi(v) \equiv \phi'(v) \quad \text{say,}$$

which by integration reduces to

$$(11) \quad \Delta = \frac{V}{\phi^2} [\phi(v) + \chi(u)]^{-\frac{4}{3}},$$

$\chi(u)$ being an arbitrary function of its argument.

In order to demonstrate our theorem we have to show that $\chi(u)$ is a constant.

For this purpose equation III (7) is utilized.

$$(12) \quad c\Delta \equiv U_1' f_1 + U_2' f_2 = \frac{cV}{\phi^2} [\phi(v) + \chi(u)]^{-\frac{4}{3}},$$

$$c\Delta_u \equiv U_1'' f_1 + U_2'' f_2 = -\frac{4}{3} \frac{cV}{\phi^2} [\phi(v) + \chi(u)]^{-\frac{7}{3}} \chi'(u),$$

where

$$(13) \quad f_1 = \begin{vmatrix} \eta' & 1 - \frac{\phi'}{\phi^2} \\ \eta - y & \frac{1}{\phi} \end{vmatrix}, \quad f_2 = \begin{vmatrix} 1 - \frac{\phi'}{\phi^2} & \xi' \\ \frac{1}{\phi} & \xi - x \end{vmatrix}.$$

Solving (12) for f_1 and comparing the result with (13) we get

$$\frac{cV}{\phi^2} \left[[\phi(v) + \chi(u)]^{-\frac{4}{3}} U_2'' + \frac{4}{3} [\phi(v) + \chi(u)]^{-\frac{7}{3}} \chi'(u) U_2' \right]$$

$$= \frac{\eta'}{\phi} - \left(1 - \frac{\phi'}{\phi^2} \right) V \int \eta' \frac{dv}{V} + \left(1 - \frac{\phi'}{\phi^2} \right) V U_2,$$

which after differentiation with regard to u , becomes

$$\frac{3U_2'''}{4U_2'} = \frac{7}{3} [\phi(v) + \chi(u)]^{-2} \chi'(u)^2 - [\phi(v) + \chi(u)]^{-1} \chi''(u)$$

$$- \frac{3}{4c} (\phi^2 - \phi') [\phi(v) + \chi(u)]^{\frac{4}{3}}.$$

For sake of brevity put

$$\phi(v) + \chi(u) = p, \quad -\frac{3}{4c}(\phi^2 - \phi') = \theta(v).$$

Then we get

$$(14) \quad \frac{7}{3}p^{-2}\chi'(u)^2 - p^{-1}\chi''(u) + \theta(v)p^{\frac{4}{3}} = F(u).$$

Now differentiate (14) with respect to v so that the following relation holds good:

$$(15) \quad \phi'(v) \left\{ -\frac{14}{3}p^{-2}\chi'(u)^2 + p^{-1}\chi''(u) + \frac{4}{3}\theta(v)p^{\frac{4}{3}} \right\} + \theta'(v)p^{\frac{5}{3}} = 0.$$

Here we have to distinguish the two cases:

1. $\phi'(v) = 0$. In this case $\varphi(v)$ being zero,

$$(a\tilde{x}) = 0,$$

the affine surface-normals are parallel to a given plane. After Dr. W. Süss⁽²⁾ the surface in consideration is an improper affine sphere, which is trivial. Hence we have to consider the case

2. $\phi'(v) \neq 0$. Putting

$$\theta_1(v) = \frac{\theta'(v)}{\phi'(v)},$$

we get (15) in the form:

$$(15') \quad -\frac{14}{3}p^{-2}\chi'(u)^2 + p^{-1}\chi''(u) + \theta_1(v)p^{\frac{5}{3}} + \frac{4}{3}\theta(v)p^{\frac{4}{3}} = 0.$$

The elimination of $\chi'(u)^2$ from (14) and (15') gives

$$(16) \quad -p^{-1}\chi''(u) + \frac{10}{3}\theta(v)p^{\frac{4}{3}} + \theta_1(v)p^{\frac{5}{3}} = 2F(u).$$

Similarly, by differentiating (16) with respect to v and eliminating $\chi''(u)$ from (16) and the equation thus obtained, we arrive at

$$(17) \quad \theta_2(v)p^{\frac{8}{3}} + \frac{10}{3}\theta_1(v)p^{\frac{7}{3}} + \frac{8}{3}\theta_1(v)p^{\frac{5}{3}} + \frac{70}{9}\theta(v)p^{\frac{4}{3}} = 2F(u),$$

where $\theta_2(v) = \frac{\theta_1'(v)}{\phi'(v)}$.

Further differentiation of (17) with respect to v leads us to the final expression:

$$(18) \quad \theta_3(v)p^{\frac{7}{3}} + \frac{10}{3}\theta_2(v)p^{\frac{6}{3}} + \frac{16}{3}\theta_2(v)p^{\frac{4}{3}} + \frac{140}{9}\theta_1(v)p^{\frac{3}{3}} \\ + \frac{40}{9}\theta_1(v)p^{\frac{1}{3}} + \frac{280}{27}\theta(v) = 0,$$

(²) Süss, Math. Annalen, **98** (1928).

$\theta_3(v)$ standing for $\frac{\theta_2'(v)}{\phi'(v)}$.

The last equation is a polynomial of $p^{\frac{1}{3}}$ with coefficients which are functions of v alone. Therefore $p = \phi(v) + \chi(u)$ and accordingly Δ must be a function of v alone.

From the discussion in Part III we immediately conclude that our surface is an affine surface of revolution. Q.E.D.

Remark. We may easily show that if the affine surface-normals along any surface-curve on parallel planes all intersect at a point, then it must necessarily be an affine moulding surface. Therefore we get

Theorem 33'. *If the affine normals of any surface along surface-curves on parallel planes all intersect at a point, then the surface must necessarily be an affine surface of revolution.*

XVII.

Special Class of Affine Spheres.

38. In the next place we will consider the corresponding problem with respect to the meridian curves of an affine moulding surface: To find all the affine moulding surfaces for which all the affine surface-normals along the meridian curves intersect at a point. First we prove

Theorem 34. *If the affine surface-normals along the meridian curve of an affine moulding surface all intersect at a point, then the surface must necessarily be an affine sphere.*

Proof. Let $\eta(u) \equiv y_1(u), y_2(u), y_3(u)$ be the point where all the affine surface-normals along a meridian curve u intersect. As shown in the last paragraph we may, from the equation

$$(1) \quad \mathfrak{x} + \alpha \tilde{\mathfrak{r}} = \eta(u),$$

deduce the following relations:

$$(2) \quad \begin{aligned} \mathfrak{x}_v \left(1 - \frac{\alpha}{R_2} \right) + \alpha_v \tilde{\mathfrak{x}} &= 0, \\ \mathfrak{x}_u \left(1 - \frac{\alpha}{R_1} \right) + \alpha \mathfrak{x}_u' &= \eta'(u), \end{aligned}$$

whence immediately follows

$$(3) \quad \alpha_v = 0, \quad R_2 = R_2(u) = \alpha.$$

Using (1) and the second equation of (2) we have

$$(4) \quad v + \alpha(a\tilde{x}) = (ay), \quad \alpha'(a\tilde{x}) = (ay').$$

If $\alpha' \neq 0$, then (4) gives

$$v = (ay) - \frac{\alpha}{\alpha'}(ay')$$

and accordingly both must be constant, which is impossible.

Therefore $\alpha' = 0$ and accordingly $(ay') = 0$ i.e. $R_2 = \text{const.}$ and

$$ay_1 + by_2 + cy_3 = \text{const.} = c',$$

or in other words, the curve $\eta(u)$ lies on one of the parallel planes, π .

Thus the second equation of (2) becomes

$$(5) \quad \mathfrak{x}_u \left(1 - \frac{\alpha}{R_1} \right) = \eta'(u).$$

Now (4) gives us the following relation:

$$(6) \quad (a\tilde{x}) = (c' - v)/\alpha,$$

which is evidently a function of v alone.

From what was discussed in the last part it follows that Δ is a function of v alone and accordingly *our surface must necessarily be an affine surface of revolution* with the axis l . The axis l cannot lie on the plane π . For, if l lies on π , then $\tilde{\mathfrak{x}}, \mathfrak{x}_u, \mathfrak{x}_v$ are coplanar, which is contradictory to our hypothesis $(\mathfrak{x}_u, \mathfrak{x}_v, \tilde{\mathfrak{x}}) \neq 0$.

Hence $\eta(u)$ *must be a single point, the intersection of l and π* . Thus all the affine surface-normals intersect at a single point. Q.E.D.

39. The surface in consideration is, therefore, characterized by the fact that it is at the same time the affine sphere and affine surface of revolution. In order to obtain the equation to the surface we distinguish the following three classes.

1. *Parabolic case.* The parabolic affine surface of revolution is, as shown by Süss⁽³⁾, an improper affine sphere. Therefore all the surfaces belonging to this class satisfy our condition.

2. *Elliptical case.* From what was discussed in Part III, the elliptic affine surfaces of revolution are nothing but the affine transforms of the ordinary surface of revolution. Therefore we have only to determine the ordinary surfaces of revolution which are at the same time the affine sphere. This class was, however, investigated by G.

(3) Süss, Loc. cit.

Tzitzéica⁽⁴⁾ to whom we owe to know the appearance of the affine sphere. The surface is given by the equations:

$$(7) \quad \begin{aligned} x &= \cos k\beta \exp\left(\int \frac{h'-1}{2h} d\alpha\right), \\ y &= \sin k\beta \exp\left(\int \frac{h'-1}{2h} d\alpha\right), \\ z &= \exp\left(\int \frac{h^2}{h'+1} d\alpha\right), \end{aligned}$$

where $h=h(\alpha)$ is a solution of the differential equation

$$(8) \quad hh''=h'^2+h^3-1.$$

3. *Hyperbolic case.* The hyperbolic affine surfaces of revolution are the affine transforms of the surface of the type

$$x^2-y^2=2f(z).$$

By quite a similar method to Tzitzéica's we can deduce the required surface:

$$(9) \quad \begin{aligned} x &= \cosh k\beta \exp\left(\int \frac{h'-1}{2h} d\alpha\right), \\ y &= \sinh k\beta \exp\left(\int \frac{h'-1}{2h} d\alpha\right), \\ z &= \exp\left(\int \frac{h^2}{h'+1} d\alpha\right), \end{aligned}$$

where $h=h(\alpha)$ satisfies (8).

Thus we arrived at

Theorem 35. *All the affine moulding surfaces for which the affine surface-normals along the meridian curves intersect at a point are improper affine spheres, Tzitzéica's surfaces (7) and the surfaces (9).*

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(4) G. Tzitzéica, Sur une nouvelle classe de surfaces, first part: Rend. del Circ. Mat. di Palermo, **25** (1908), 180—187; second part: The same Journal, **28** (1909), 210—216.