4. *Line Congruences and L-minimal Surfaces, I.*

By Buchin Su.

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As an extension of ordinary minimal surfaces, de Montcheuil⁽¹⁾ firstly studied the surfaces defined by the equation

$$(1) \quad R_1 + R_2 = F(u) + \Phi(v),$$

where R_1, R_2 denote the principal radii of the surface and u, v stand for the quantities for which

$$(2) \quad X = \frac{u+v}{1+uv}, \quad Y = \frac{i(v-u)}{1+uv}, \quad Z = \frac{uv-1}{1+uv}$$

represent the direction-cosines of the surface-normal. If Δ be Beltrami's second differential parameter with respect to the spherical representation of the surface, then from (1) and (2), we get:

$$(3) \quad \Delta(R_1 + R_2) = 0,$$

which was found by J. Weingarten⁽²⁾ early in 1888. That the middle envelopes of such surface is a minimal surface was shown by K. Kommerell⁽³⁾, who, making use of line congruence, has proved the following interesting theorems:

I. One and the same minimal surface can be generated as middle envelope of a great many number of line congruences.

II. Every minimal surface can be generated as middle envelope of a great many number of normal system, while the orthogonal surfaces of such system are nothing but those studied by de Montcheuil.

⁽¹⁾ de Montcheuil, Étude des surfaces définies par l'équation $R + R' = F(u) + \Phi(u_1)$, Bull. de la Société Math. de France, **26** (1898), 103-114; Étude sur les surfaces réelles définies par l'équation $\xi_{u^2u_1}^{IV} = 0$, The same Journ. **27** (1899), 114-129.

⁽²⁾ Cf. W. Blaschke, Ueber die Geometrie von Laguerre, III. Beiträge zur Flächentheorie, Hamburger Abhandlungen, **4** (1925), 1-12.

⁽³⁾ K. Kommerell, Strahlensysteme und Minimalflächen, Math. Annalen, **70** (1911), 143-169.

The name " L -minimal surface" was given by W. Blaschke⁽⁴⁾ to whom we owe much to know that these surfaces are very important in Laguerre-geometry and they include the ordinary minimal surfaces.

In the following lines we will establish some results, quite similar to Kommerell's, for L -minimal surfaces and line congruences and introduce the surfaces defined by the equation

$$(4) \quad \Delta\Delta(R_1 + R_2) = 0,$$

which obviously include L -minimal surfaces. Accordingly we will give the name "generalized L -minimal surfaces" or simply " GL -surface", though these surfaces are not invariant by Laguerre's transformations as will be seen in the later paragraph. The method of treatment is that due to Kommerell with some modifications.

1. The surface enveloped by the symmetrical plane with regard to the centres of principal curvature of a surface is called the middle envelope of the latter. We begin with the

Theorem: If the original surface is a GL -surface, then its middle envelope is L -minimal surface and vice versa.

Proof: Let W be the distance of the tangent plane of the original surface from the origin and let P be that of its middle envelope. By the definition we have

$$(5) \quad 2W - (R_1 + R_2) = 2P,$$

where R_1, R_2 denote the principal radii of the original surface.

But by a well-known formula of Weingarten⁽⁵⁾, the relation

$$(6) \quad R_1 + R_2 = (2 + \Delta)W$$

holds good. Hence it follows:

$$(7) \quad (2 + \Delta)P = -\frac{1}{2}\Delta(R_1 + R_2).$$

Denoting the principal radii of the middle envelope by r_1, r_2 , then from (6) and (4), we arrive at

$$(8) \quad (r_1 + r_2) = -\frac{1}{2}\Delta(R_1 + R_2),$$

whence

⁽⁴⁾ W. Blaschke, Ueber die Geometrie von Laguerre, II. Flächentheorie in Ebenenkoordinaten, Hamburger Abhandlungen, **3** (1924), 195-212.

⁽⁵⁾ Cf. e.g. L. Bianchi, Lezioni di geometria differenziale I. p. 259.

$$(9) \quad \Delta(r_1 + r_2) = -\frac{1}{2} \Delta \Delta(R_1 + R_2).$$

The last formula shows us the truth of our theorem.

From (9) we further get

$$(10) \quad \Delta^{n-1}(r_1 + r_2) = -\frac{1}{2} \Delta^n(R_1 + R_2)$$

or in other words, our theorem may be extended for the general case. But here we will not enter further.

In order to rewrite the equation (4), we make use of the equation to the tangent plane in the form

$$(11) \quad (u+v)x + i(v-u)y + (uv-1)z + w = 0.$$

From the well-known formula

$$R_1 + R_2 = w - uw_u - vw_v + (1+uv)w_{uv},$$

where $w_u = \frac{\partial w}{\partial u}$, $w_{uv} = \frac{\partial^2 w}{\partial u \partial v}$, etc., we obtain

$$\Delta(R_1 + R_2) = (1+uv)^3 w_{uuvv},$$

$$\Delta \Delta(R_1 + R_2) = (1+uv)^2 \{(1+uv)^3 w_{uuvv}\}_{uv}.$$

Thus the equation (4) is equivalent to

$$(12) \quad \{(1+uv)^3 w_{uuvv}\}_{uv} = 0,$$

whence

$$(13) \quad w_{uuvv} = (1+uv)^{-3} (F(u) + \Phi(v)),$$

$F(u)$, $\Phi(v)$ being arbitrary functions.

Specially, we get the equation to L -minimal surface⁽⁶⁾ when $F(u)$, $\Phi(v)$ are put to zero, i.e.

$$(15) \quad w = A(u)v + A_1(v)u + B(u) + B_1(v).$$

2. From (15) we get

$$w_u = A'(u)v + A_1(v) + B'(u),$$

$$w_v = A_1'(v)u + A(u) + B_1'(v),$$

whence follows the expression of the point coordinates P of L -minimal surface :

(⁶) Montcheuil, loc. cit. p. 193.

$$\begin{aligned}
(16) \quad \xi &= -\frac{1}{2(1+uv)} \left[(u+v)(Av + A_1u + B + B_1) + (1-u^2)(A'v + A_1 + B') \right. \\
&\quad \left. + (1-v^2)(A_1'u + A + B_1') \right], \\
\eta &= +\frac{i}{2(1+uv)} \left[(u-v)(Av + A_1u + B + B_1) - (1+u^2)(A'v + A_1 + B) \right. \\
&\quad \left. + (1+v^2)(A_1'u + A + B_1') \right], \\
\zeta &= +\frac{1}{(1+uv)} \left[Av + A_1u + B + B_1 - uv(A' + A_1') - u(A_1 + B) \right. \\
&\quad \left. - v(A + B_1') \right],
\end{aligned}$$

whence

$$\begin{aligned}
(17) \quad \xi_u &= -\frac{1-v^2}{2}P - \frac{1-u^2}{2}Q, & \xi_v &= -\frac{1-u^2}{2}P - \frac{1-v^2}{2}\bar{Q}, \\
\eta_u &= +\frac{i(1+v^2)}{2}P - \frac{i(1+u^2)}{2}Q, & \eta_v &= -\frac{i(1+u^2)}{2}P + \frac{i(1+v^2)}{2}\bar{Q}, \\
\zeta_u &= -vP - uQ, & \zeta_v &= -uP - v\bar{Q},
\end{aligned}$$

where

$$\begin{aligned}
(18) \quad P &= (A' + B - uB' + A_1' + B_1 - vB_1')(1+uv)^{-2}, \\
Q &= (A''v + B'')(1+uv)^{-1}, \quad \bar{Q} = (A_1''u + B_1'')(1+uv)^{-1}.
\end{aligned}$$

Thus we obtain the fundamental quantities of the first form E, F, G :

$$\begin{aligned}
(19) \quad E &= PQ(1+uv)^2, \\
F &= (P^2 + Q\bar{Q})(1+uv)^2/2, \\
G &= P\bar{Q}(1+uv)^2, \\
W &= \sqrt{EG - F^2} = i(1+uv)^2(Q\bar{Q} - P^2)/2;
\end{aligned}$$

and the surface-normal

$$(20) \quad X = \frac{u+v}{1+uv}, \quad Y = i\frac{v-u}{1+uv}, \quad Z = \frac{uv-1}{1+uv}.$$

For the coordinates x, y, z of any point Q in the tangent plane of the surface at P we can put

$$\begin{aligned}
(21) \quad x &= \xi + \lambda\xi_u + \mu\xi_v, \\
y &= \eta + \lambda\eta_u + \mu\eta_v, \\
z &= \zeta + \lambda\zeta_u + \mu\zeta_v,
\end{aligned}$$

λ, μ being arbitrary functions of u, v .

Passing through $Q(x, y, z)$ draw a straight line (u, v) parallel to the surface-normal at P so that its equations are

$$(22) \quad x_1 = x + tX, \quad y_1 = y + tY, \quad z_1 = z + tZ,$$

where t is a parameter.

If we determine λ, μ so that the point Q is the middle point of the line (22), then the middle envelope of the line congruence (22) is the given L -minimal surface; and then the equations (21) become those to the middle surface of the line congruence.

3. Let Kummer's fundamental quantities of the line congruence be

$$(23) \quad \begin{aligned} \sum X_u^2 &= E, & \sum X_u X_v &= F, & \sum X_v^2 &= G, \\ -\sum X_u x_u &= \bar{D}, & -\sum X_u x_v &= \bar{D}', & -\sum X_v x_u &= \bar{D}_1', & -\sum X_v x_v &= \bar{D}''. \end{aligned}$$

From (20) we get

$$(24) \quad E_0 = 0, \quad F_0 = 2(1 + uv)^{-2}, \quad G_0 = 0.$$

If r be the value of the abscissa t for a limiting point, it is well-known that

$$(25) \quad (E_0 G_0 - F_0^2) r^2 - (\bar{D}' E_0 - (\bar{D} + \bar{D}_1') F_0 + \bar{D} G_0) r + \bar{D} \bar{D}'' - (\bar{D} + \bar{D}_1')^2 / 4 = 0.$$

Now let (x, y, z) be the middle point of the line (u, v) . We have, by (24),

$$(26) \quad \bar{D} + \bar{D}_1' = 0.$$

For our surface the fundamental quantities of the second form are

$$(27) \quad \begin{aligned} \sum \xi_u X_u &= -D = -Q, & \sum \xi_u X_v &= \sum \xi_v X_u = -D' = -P, \\ \sum \xi_v X_v &= -D'' = -\bar{Q}. \end{aligned}$$

Differentiating the equations (21) and deducing the expressions by virtue of the well-known formulae

$$(28) \quad \begin{aligned} \xi_{uu} &= \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \xi_u + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \xi_v + DX, \\ \xi_{uv} &= \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \xi_u + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \xi_v + D'X, \\ \xi_{vv} &= \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \xi_u + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \xi_v + D''X; \end{aligned}$$

and similar equations for η and ζ , we get

$$\begin{aligned}
(29) \quad x_u &= \left[1 + \lambda \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \mu \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \lambda_u \right] \xi_u + \left[\lambda \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} + \mu \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \mu_u \right] \xi_v + (\lambda D + \mu D') X, \\
x_v &= \left[\lambda \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \mu \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} + \lambda_v \right] \xi_u + \left[1 + \lambda \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \mu \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} + \mu_v \right] \xi_v + (\lambda D' + \mu D'') X;
\end{aligned}$$

and similar expressions for y and z , whence

$$\begin{aligned}
(30) \quad \bar{D}' &= \left[\lambda \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \mu \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} + \lambda_v \right] Q + \left[1 + \lambda \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \mu \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} + \mu_v \right] P, \\
\bar{D}_1' &= \left[1 + \lambda \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \mu \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \lambda_u \right] P + \left[\lambda \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} + \mu \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \mu_u \right] \bar{Q}.
\end{aligned}$$

4. In order to obtain the simple expressions for $\bar{D}'$ and $\bar{D}_1'$, we differentiate (18):

$$\begin{aligned}
(31) \quad P_u &= \frac{A'' - uB''}{(1+uv)^2} - \frac{2Pv}{1+uv}, & P_v &= \frac{A_1'' - vB_1''}{(1+uv)^2} - \frac{2Pu}{1+uv}; \\
Q_u &= \frac{A'''v + B''' - vQ}{1+uv}, & Q_v &= \frac{A'' - uQ}{1+uv}; \\
\bar{Q}_u &= \frac{A_1'' - v\bar{Q}}{1+uv}, & \bar{Q}_v &= \frac{A_1'''u + B_1''' - u\bar{Q}}{1+uv};
\end{aligned}$$

and hence

$$\begin{aligned}
(32) \quad E_u &= (A'' - uB'')Q + (1+uv)(A'''v + B''' - vQ)P, \\
E_v &= (A_1'' - vB_1'')Q + (1+uv)(A'' - uQ)P; \\
F_u &= (A'' - uB'')P - v(1+uv)P^2 + (1+uv)((A'''v + B''')\bar{Q} + A_1''Q)/2, \\
F_v &= (A_1'' - vB_1'')P - u(1+uv)P^2 + (1+uv)((A_1'''u + B_1''')Q + A''\bar{Q})/2; \\
G_u &= (A'' - uB'')\bar{Q} + (1+uv)(A_1'' - v\bar{Q})P, \\
G_v &= (A_1'' - vB_1'')\bar{Q} + (1+uv)(A_1'''u + B_1''' - u\bar{Q})P.
\end{aligned}$$

From (18) follow the expressions

$$(33) \quad (1+uv)(A'' - uQ) = A'' - uB'', \quad (1+uv)(A_1'' - v\bar{Q}) = A_1'' - vB_1''.$$

Further we easily get

$$\begin{aligned}
(34) \quad 2(GP - F\bar{Q}) &= (1+uv)^2(P^2 - Q\bar{Q})\bar{Q}, \\
2(FP - E\bar{Q}) &= (1+uv)^2(P^2 - Q\bar{Q})P, \\
2(GQ - FP) &= (1+uv)^2(Q\bar{Q} - P^2)P, \\
2(FQ - EP) &= (1+uv)^2(Q\bar{Q} - P^2)Q;
\end{aligned}$$

and whence

$$\begin{aligned}
(35) \quad & \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\} P + \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} \bar{Q} = + \frac{1}{(1+uv)^2(Q\bar{Q}-P^2)} (\bar{Q}E_u - 2PF_u + PE_v), \\
& \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\} P + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\} \bar{Q} = + \frac{1}{(1+uv)^2(Q\bar{Q}-P^2)} (\bar{Q}E_v - PG_u), \\
& \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\} Q + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\} P = - \frac{1}{(1+uv)^2(Q\bar{Q}-P^2)} (PE_v - QG_u), \\
& \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} Q + \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\} P = - \frac{1}{(1+uv)^2(Q\bar{Q}-P^2)} (-QE_v + 2PF_v - PG_u).
\end{aligned}$$

But the relations (32) and (33) give the expressions :

$$\begin{aligned}
(36) \quad & QG_u - PE_v = (Q\bar{Q} - P^2)(A'' - uB''), \\
& \bar{Q}E - PG_u = (Q\bar{Q} - P^2)(A_1'' - vB_1''), \\
& QG_v - 2PF_v + PG_u = (Q\bar{Q} - P^2)(A_1'' - vB_1'' - 2u(1+uv)P), \\
& \bar{Q}E_u - 2PF_u + PE_v = (Q\bar{Q} - P^2)((A'' - uB'' - 2v(1+uv)P).
\end{aligned}$$

Substituting (35) and (36) into (30), we arrive at

$$\begin{aligned}
(37) \quad & \bar{D}' = P + \mu_v P + \lambda_v Q + \frac{1}{(1+uv)^2} ((A'' - uB'')\lambda + (A_1'' - vB_1'') \\
& \quad \quad \quad - 2u(1+uv)P)\mu, \\
& \bar{D}_1' = P + \mu_u \bar{Q} + \lambda_u P + \frac{1}{(1+uv)^2} ((A'' - uB'') - 2v(1+uv)P)\lambda \\
& \quad \quad \quad + (A_1'' - vB_1'')\mu),
\end{aligned}$$

which reduce, by virtue of (31) and (33), to the expressions :

$$\begin{aligned}
(38) \quad & \bar{D}' = P + (\mu P + \lambda Q)_v, \\
& \bar{D}_1' = P + (\mu \bar{Q} + \lambda P)_u.
\end{aligned}$$

Thus the condition (26) becomes

$$(39) \quad 2P + (\mu P + \lambda Q)_v + (\mu \bar{Q} + \lambda P)_u = 0.$$

Specially for the ordinary minimal surfaces,

$$A' + B = uB', \quad A_1' + B_1 = vB_1',$$

so that $P=0$, $Q=B''=-F(u)$, $\bar{Q}=B_1''=-\Phi(v)$, and (39) becomes

$$(40) \quad F(u)\lambda_v + \Phi(v)\mu_u = 0,$$

which is nothing but the result due to Kommerell.

Since there exist infinitely many solutions λ, μ of the equation (39), we conclude

Theorem : *One and the same L -minimal surface can be generated as the middle envelope of an infinitely many system of line congruence.*

5. If the line congruence thus characterized be normal, then

$$\bar{D}' = \bar{D}_1'$$

and hence

$$\bar{D}' = \bar{D}_1' = 0,$$

i.e.

$$(41) \quad (\mu P + \lambda Q)_v = (\mu \bar{Q} + \lambda P)_u = -P.$$

Let t be the abscissa of the point where the line of the congruence meets one of its normal surface. Then

$$\sum X dx_1 = 0,$$

whence

$$t_u = -\sum X x_u, \quad t_v = -\sum X x_v,$$

which, by (29), reduce to

$$(42) \quad t_u = -(\lambda Q + \mu P), \quad t_v = -(\lambda P + \mu \bar{Q}),$$

the expressions being co-existent by virtue of (41).

But

$$(43) \quad |t| = |(R_1 + R_2)/2|,$$

where R_1, R_2 denote the principal radii of the orthogonal surface. Thus putting

$$(44) \quad \phi = \pm (R_1 + R_2)/2,$$

we get

$$(45) \quad \phi_u = \lambda Q + \mu P, \quad \phi_v = \lambda P + \mu \bar{Q},$$

whence

$$\Delta \phi = \frac{2}{F_0} \phi_{uv} = \frac{2}{F_0} \phi_{vu},$$

or

$$(46) \quad \Delta \phi = -(1 + uv)^2 P = (uB' - A' - B) + (vB_1' - A_1' - B_1)$$

by means of (24), (41) and (18)₁

From (46), it follows that

$$\Delta(R_1 + R_2) = F(u) + \Phi(v)$$

i.e.

$$\Delta\Delta(R_1 + R_2) = 0,$$

or in other words,

Theorem: The orthogonal surfaces of any normal line congruence, whose middle envelope is L -minimal surface, are identical with the generalized L -minimal surfaces.

From the above discussion, we get

Theorem: Every L -minimal surface can be generated as the middle envelope of infinitely many normal line congruences, while the orthogonal surfaces of such system are nothing but the generalized L -minimal surfaces.

6. We will here establish the results discussed in the paragraphs 2-5 by a somewhat simpler method.

For the equations to the middle surface of the line congruence (22) we can put

$$(47) \quad \begin{aligned} x &= \xi + Yc - Zb, \\ y &= \eta + Za - Xc, \\ z &= \zeta + Xb - Ya, \end{aligned}$$

where a, b, c are the functions of u, v . This is always possible because

$$\sum X(Yc - Zb) = 0,$$

and hence (x, y, z) lies on the tangent plane of the L -minimal surface.

$$(48) \quad \begin{aligned} x_u &= \xi_u + Yc_u - Zb_u + cY_u - bZ_u, \\ x_v &= \xi_v + Yc_v - Zb_v + cY_v - bZ_v, \end{aligned}$$

whence, by (27),

$$\begin{aligned} -\bar{D}' &= -P + \sum X_u(Yc_v - Zb_v) + \sum X_u(cY_v - bZ_v), \\ -\bar{D}_1' &= -P + \sum X_v(Yc_u - Zb_u) + \sum X_v(cY_u - bZ_u), \end{aligned}$$

i.e.

$$(49) \quad \begin{aligned} -\bar{D}' &= -P + \sum a_v(ZY_u - YZ_u) + \sum a(Y_uZ_v - Y_vZ_u), \\ -\bar{D}_1' &= -P + \sum a_u(ZY_v - YZ_v) + \sum a(Y_vZ_u - Y_uZ_v). \end{aligned}$$

But in our case,

$$(50) \quad ZY_u - YZ_u = iX_v, \quad ZY_v - YZ_v = -iX_u, \quad Y_uZ_v - Y_vZ_u = -iF_0X.$$

Therefore (49) becomes

$$(51) \quad \begin{aligned} \bar{D}' &= P - i \sum a_v X_u + i F_0 \sum a X, \\ \bar{D}'_1 &= P + i \sum a_u X_v - i F_0 \sum a X, \end{aligned}$$

and (26) becomes

$$(52) \quad \sum a_v X_u - \sum a_u X_v = -2iP.$$

That is, if the functions a , b , c be so determined that (52) holds good, then the middle envelope of the line congruence (22) is a L -minimal surface.

Now consider the case of normal congruence. We have

$$(53) \quad \sum a_v X_u - F_0 \sum a X = -iP, \quad \sum a_u X_v - F_0 \sum a X = +iP,$$

or we have, by virtue of the relation

$$(54) \quad \begin{aligned} X_{uv} &= -F_0 X, \\ (\sum a X_u)_v &= -iP, \quad (\sum a X_v)_u = +iP. \end{aligned}$$

As in § 5, we put

$$\phi = \pm (R_1 + R_2)/2$$

so that

$$(55) \quad \phi_u = \sum X x_u = -i \sum a X_u, \quad \phi_v = \sum X x_v = +i \sum a X_v,$$

whence

$$\Delta \phi = -(1 + uv)^2 P = F(u) + \Phi(v),$$

i.e. the relation (46).

7. The generalized L -minimal surface is by no means invariant by the Laguerre's transformation, for from the general Laguerre's group

$$\begin{aligned} u^* &= \frac{\alpha u + \beta}{\gamma u + d}, \\ L_{10}, L'_{10} : \quad v^* &= \frac{\bar{\alpha} v + \bar{\beta}}{\gamma v + \bar{\delta}}, \quad (\alpha \delta - \beta \gamma)(\bar{\alpha} \bar{\delta} - \bar{\beta} \bar{\gamma}) = \theta^2 \neq 0 \\ w^* &= \frac{\theta w + \lambda uv + \mu u + \bar{\mu} v + \nu}{(\gamma u + \delta)(\bar{\gamma} v + \bar{\delta})}; \end{aligned}$$

or

$$\begin{aligned}
 u^* &= \frac{\alpha v + \beta}{\gamma v + \delta} \\
 L_{10}'', L_{10}''' : \quad v^* &= \frac{\bar{\alpha} u + \bar{\beta}}{\gamma u + \bar{\delta}} \quad (\alpha\delta - \beta\gamma)(\bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\gamma}) = \theta^2 \neq 0 \\
 w^* &= \frac{\theta w + \lambda uv + \mu u + \bar{\mu} v + \nu}{(\gamma u + \bar{\delta})(\gamma v + \delta)},
 \end{aligned}$$

we get respectively

$$\begin{aligned}
 (56) \quad u^*_{u^*v^*v^*} &= \theta^{-3}(\gamma u + \delta)^3(\bar{\gamma} v + \bar{\delta})^3 u_{uvv}, \\
 v^*_{u^*v^*v^*} &= \theta^{-3}(\bar{\gamma} u + \bar{\delta})^3(\gamma v + \delta)^3 v_{uvv},
 \end{aligned}$$

whence we may easily verify that in general $((1+uv)^3 w_{uvv})_{uv}$ is not invariant except when and only when the surface is L -minimal surface.

The generalized L -minimal surface is, however, invariant under some factor group of Laguerre's. For example, it is invariant for the groups :

$$\begin{aligned}
 L_5 \quad u^* &= \alpha u, \quad v^* = \alpha^{-1} v, \quad w^* = \epsilon w + \lambda uv + \mu u + \bar{\mu} v + \nu \\
 \bar{L}_5 \quad u^* &= \alpha v, \quad v^* = \alpha^{-1} u, \quad w^* = \epsilon w + \lambda uv + \mu u + \bar{\mu} v + \nu
 \end{aligned}$$

ϵ being ± 1 .

But it is rather a matter of course. For denoting the center of the middle sphere (Mittenkugel) by x_1, x_2, x_3 and its radius by x_4 , we have, for the correspondence of L_5 , the transformations of the point (x_1, x_2, x_3, x_4) in the four dimensional space :

$$\begin{aligned}
 x_1 &= \frac{1+\alpha^2}{2\alpha} x_1^* + i \frac{1-\alpha^2}{2\alpha} x_2^* - (\mu + \bar{\mu}), \\
 x_2 &= -i \frac{1-\alpha^2}{2\alpha} x_1^* + \frac{1+\alpha^2}{2\alpha} x_2^* + \mu - \bar{\mu}, \\
 x_3 &= x_3^* + \lambda - \nu, \\
 x_4 &= x_4^* - (\lambda + \nu).
 \end{aligned}$$

The first three relations show us that the centre of the middle sphere receives a congruence transformation, while the last represents the operation of parallel surface. Therefore the transformed surface is also a generalized L -minimal surface, the relation $\Delta\Delta(R_1 + R_2) = 0$ being conserved.

8. The associate L -minimal surface of (16) is given by the

equation⁽⁷⁾

$$(57) \quad w_\phi = e^{i\phi} A(u)v + e^{-i\phi} A_1(v)u + e^{i\phi} B(u) + e^{-i\phi} B_1(v),$$

whence follow the corresponding values P_ϕ , Q_ϕ , $\bar{Q}_\phi$ to (18):

$$(58) \quad \begin{aligned} P_\phi &= \{e^{i\phi}(A' + B - uB') + e^{-i\phi}(A_1' + B_1 - vB_1')\} / (1 + uv)^2 = e^{i\phi} p + e^{-i\phi} \bar{p} \\ \text{say,} \quad Q_\phi &= e^{i\phi} Q, \quad \bar{Q}_\phi = e^{-i\phi} \bar{Q}. \end{aligned}$$

Hence, if we determine λ , μ , as it is always possible, so that they satisfy the equations

$$(59) \quad \begin{aligned} E_1 &\equiv 2p + (\lambda p)_u + (\mu p + \lambda Q)_v = 0, \\ E_2 &\equiv 2\bar{p} + (\mu \bar{p})_v + (\lambda \bar{p} + \mu \bar{Q})_u = 0, \end{aligned}$$

then we have

$$(60) \quad e^{i\phi} E_1 + e^{-i\phi} E_2 \equiv 2P_\phi + (\mu \bar{Q}_\phi + \lambda P_\phi)_u + (\mu P_\phi + \lambda Q_\phi)_v = 0,$$

for any value of ϕ .

Thus we get:

For any given associate L -minimal surface (ξ, η, ζ) , which corresponds to the equation (57), we can determine λ and μ , *independent of* ϕ , so that the middle envelope of the line congruence (22) is the given surface.

The 7th, November, 1927.

(7) K. König, *L-Minimalflächen*, *Hamburger Mitteilungen*, 4 (1926), 189-203.