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I.
Under the title "On a Proof of Picard's General Theorem" in this journal 4 (1927), 311-317 I have given a proof of Picard's theorem by the use of a hyper-elliptic integral which is not perhaps without interest. In the process of that proof we must replace "uniform and regular" in the line 18 on the page 312 by "finite valued with no branch point except at \( z=a \)," for the domain \( 0<|z-a|<\rho \) is doubly connected. (1) Consequently we have to replace in several lines properties of a uniform function by those of a finite valued function (2). The proof is not at all affected by these corrections, and on the other hand these suggest us some easy extensions of Picard's theorem.

In the above proof of Picard's theorem we have seen that the relation \( u^2 = v^5 + 1 \) can not be satisfied by two finite valued functions

\[(1)\) For the proof of Picard's theorem on a meromorphic or an integral function we have no need of these corrections.

\[(2)\) In the lines 23 (page 311), 3, 18, 27, (page 312), and 23, 30 (page 313), "uniform and regular" must be replaced by "finite valued with no branch point"; in the line 24 (page 311) "then" by "then at least a branch of"; in the line 4 (page 312) "Weierstrass' theorem" by "extended Weierstrass' theorem"; in the line 22 (p. 312) "is regular" by "has no pole"; in the line 23 (p. 312) "Laurent's expansion" by "extended Laurent's expansion";

in the lines 24 (p. 312), and 19 (p. 313) \( \sum_{k=-\infty}^{\infty} A_k(z-a)^k \) by \( \sum_{k=-\infty}^{\infty} A_k(z-a)^{\lambda k} \), where \( \lambda \) denotes the number of valuedness of \( \frac{du}{dz}/v \); in the line 7 (p. 313) "a value" by "a suitably chosen value"; in the line 13 (p. 313) "it does not cover" by "it covers at most a finite number of times"; in the line 15 (p. 313) "many" by "many and the finite-valuedness of \( G(y) \)"; in the line 24 (p. 313) "\( z_{\alpha}, z_{\alpha + 1}, \ldots \)" by "an infinite partial sequence of \( z_{\alpha}, z_{\alpha + 1}, \ldots \)"; in the line 28 (p. 313) "these \( z_{\alpha}, z_{\alpha + 1}, \ldots \)" by "the infinite partial sequence of these \( z_{\alpha}, z_{\alpha + 1}, \ldots \)".

That "we can take all the values of some circuits..." (line 4, page 313) is very evident by a theorem of Painlevé. Cf. Zoretti: Sur les fonctions analytiques uniformes...... Journal de math. 6e série 1 (1905), 9.
Antoine; Sur l'homéomorphie de deux figures et de leurs voisinages. ibid 8e série 4 (1921), 296.
u(z) and \( v(z) \), each of which has no branch point in \( 0 < |z-a| < \rho \) and has \( z=a \) as an isolated essential singularity.

Thus we have obtained that a function which is finite valued with no branch point in \( 0 < |z-a| < \rho \) can not have more than two exceptional values. This of course can be reduced to Picard's theorem by a simple transformation \( t-a = (z-a)^{1/\alpha} \).

If \( \int \frac{du}{dv} \) has more than three periodicity-modulii, where \( P(u, v) = 0 \), \( P \) being a polynomial, then we can use such a pair of \( u \) and \( v \) for the proof of Picard's theorem.

We can similarly prove that a meromorphic function about an isolated essential singularity can not have two exceptional values (in Picard's sense) and a value for which the function has only multiple roots of the same multiplicity larger than one.

For Picard's theorem on an integral function the above proof becomes very simple (3).

II.

Introducing the functions \( T(r, f) \) and \( N(r, \alpha) \) for a meromorphic function of \( z \) Prof. R. Nevanlinna (4) has defined \( \alpha \) as an exceptional value, when \( \alpha \) satisfies

\[
\lim_{r \to \infty} \frac{N(r, \alpha)}{T(r, f)} < 1,
\]

and obtained an important relation

\[
2 \geq \sum_{\nu=1}^{q} \left( 1 - \lim_{r \to \infty} \frac{N(r, \alpha_{\nu})}{T(r, f)} \right).
\]

From this he has deduced many remarkable theorems, for instance:
A meromorphic function has at most an enumerable number of exceptional values.


I have defined (5) the functions \( A(r, f) \) and \( n(r, \alpha) \) for a meromorphic function \( f(z) \) of \( z \) in the following way:

\[
A(r, f) = \frac{1}{\pi} \int_0^{2\pi} \int_0^r \frac{|f'(pe^{i\theta})|^2}{(1 + |f(pe^{i\theta})|^2)^2} \rho d\rho d\theta
\]

which expresses a mean number of sheets of the Riemann surface of the inverse function of \( f(z) \) in \( |z| < r \), and \( n(r, \alpha) \) denotes the number of \( \alpha \)-points of \( f(z) \) in \( |z| < r \).

I have obtained some theorems, one of which is the following: (6)

For a sequence of infinite number of intervals tending to \( \infty \) we have for \( q \) values \( \alpha_v(v=1,2,\ldots,q) \), different from each other,

\[
(1-\varepsilon)(q-2)A(r,f) < \sum_{v=1}^{q} n(r, \alpha_v)
\]

where \( \varepsilon \) denotes an arbitrarily given positive number.

From this we have

\[
(1-\varepsilon)(q-2) \leq \sum_{v=1}^{q} \frac{n(r, \alpha_v)}{A(r,f)}
\]

for a sequence of \( r \), hence

\[
q-2 \leq \sum_{v=1}^{q} \lim_{r \rightarrow \infty} \frac{n(r, \alpha_v)}{A(r,f)},
\]

thus we have a theorem which is parallel to that of Nevanlinna:

For \( q \) values \( \alpha_v(v=1,2,\ldots,q) \), different from each other,

\[
2 \geq \sum_{v=1}^{q} \left(1 - \lim_{r \rightarrow \infty} \frac{n(r, \alpha_v)}{A(r,f)}\right).
\]

When we define \( \alpha \) as an exceptional value for which

\[
\lim_{r \rightarrow \infty} \frac{n(r, \alpha)}{A(r,f)} < 1,
\]

then we can obtain: A meromorphic function has at most an enumerable number of exceptional values (in our sense).

This definition of an exceptional value is of some interest, for \( n(r, \alpha) \) denotes just the number of \( \alpha \)-points and \( A(r, f) \) a mean
number of sheets of the Riemann surface, and it has, I think, a more
direct meaning than that by $\lim_{r \to \infty} \frac{N(r, \alpha)}{T(r, f)} < 1$.

By a theorem obtained by me (7) we have at once

$$\lim_{r \to \infty} \frac{n(r, \alpha)}{A(r, f)} < 1.$$  

If $\lim_{r \to \infty} \frac{n(r, \alpha)}{A(r, f)} = 1$ we shall call $\alpha$ a regular value, and we can show

that $sn z$ (Jacobi's $sn$-function) has only regular values. In fact, for

sufficiently large $r$

$$\pi(r/c)^2 - 2\pi(r/c) < A(r, sn) < \pi(r/c)^2 + 2\pi(r/c)$$

$$\pi(r/c)^2 - 2\pi(r/c) < n(r, \alpha) < \pi(r/c)^2 + 2\pi(r/c)$$

hence

$$\lim_{r \to \infty} \frac{n(r, \alpha)}{A(r, sn)} = 1 \text{ for any } \alpha.$$  

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(7) T. Shimizu: loc. cit. 146.