24. On the Theory of Surfaces in the Affine Space :
X. Studies on the Surface Σ and some allied Problems.

By Buchin SU.

(Received May 9, 1929.)

XLI.

Some Characteristic Properties of the Surface Σ .

74. In Part XXV I have defined a surface of the class Σ by the fact that all the axes of Čech are affine normals and deduced a characteristic property for these surfaces: *A surface is of the class Σ when and only when the curves of Segre are affine geodesics.* Here I will give some other properties which characterize the surface Σ .

First we prove

Theorem. *A surface is of the class Σ when and only when the curves of Darboux are affine geodesics.*

In fact, taking the asymptotic lines for parametric curves, we get the equation of the affine geodesic with the direction $u' = \frac{du}{dv}$:

$$(1) \quad \frac{u''}{u'} = -\frac{F_u}{F}u' + \frac{F_v}{F}.$$

If the curves of Darboux be affine geodesics, then we have

$$u' = -\varepsilon^w \sqrt[3]{\frac{D}{A}}, \quad (r=0, 1, 2),$$

whence

$$(2) \quad \frac{u''}{u'} = \frac{1}{3} \left\{ \left(\frac{D_u}{D} - \frac{A_u}{A} \right) u' + \left(\frac{D_v}{D} - \frac{A_v}{A} \right) \right\}.$$

From (1) and (2) it follows that

$$(3) \quad \left(\frac{D_u}{D} - \frac{A_u}{A} + 3 \frac{F_u}{F} \right) u' + \left(\frac{D_v}{D} - \frac{A_v}{A} - 3 \frac{F_v}{F} \right) = 0.$$

Therefore the necessary and sufficient condition is of the form :

$$(4) \quad \begin{aligned} \frac{D_u}{D} - \frac{A_u}{A} + 3 \frac{F_u}{F} &= 0, \\ \frac{D_v}{D} - \frac{A_v}{A} - 3 \frac{F_v}{F} &= 0, \end{aligned}$$

or

$$(5) \quad \begin{aligned} \frac{DF^3}{A} &= f(v), \\ \frac{D}{AF^3} &= \varphi(u), \end{aligned}$$

where $f(v)$ and $\varphi(u)$ are functions of their arguments respectively.

By a convenient transformation of variables u, v we may render that

$$f(v) = \varphi(u) = 1,$$

the ruled surface being excluded *a priori*. Thus the condition in question becomes

$$(6) \quad A = D, \quad F = 1,$$

which is nothing but the one for a surface Σ ⁽¹⁾.

Thus the theorem is completely demonstrated.

From (1) and (6) we immediately conclude that *the surface Σ is the only isothermal-asymptotic surface whose affine geodesics may be represented by straight lines of a plane.*

75. In virtue of the correspondence of Segre the three points corresponding to the three osculating planes of curves of Segre at a point of a surface lie in the tangent plane of the point and they are colinear, since the osculating planes pass through the axis of Čech, we call this line the conjugate axis of Čech. Suggested by Prof. T. Takasu, I will prove the following

Theorem. *If the conjugate axis of Čech is always in the plane at infinity, then the surface must necessarily be of the class Σ and conversely.*

Proof. In projective differential geometry we have a theorem of Fubini ⁽²⁾:

To a plane π corresponds the same point by the correspondence of Segre and by the polarity with respect to the quadric of Lie when and only when the plane π passes through one direction of Segre.

Therefore, if the conjugate axis of Čech always lies in the plane at infinity, then the axis of Čech must pass through the centre of the quadric of Lie at the same point. After a geometrical interpretation of the affine normal due to Prof. L. Berwald ⁽³⁾ it follows that the axis of Čech of the surface in question coincides with the affine normal. Q. E. D.

⁽¹⁾ Cf. Part XXV.

⁽²⁾ Cf. G. Fubini-E. Čech, *Geometria proiettiva differenziale*, I, p. 133

⁽³⁾ *Math. Zeits.*, **10** (1921).

76. The theorem above quoted may be stated in the following

Theorem. *The surface Σ is characterized by the property that all the curves of Segre are lines of shadow.*

For, if all the curves of Segre be lines of shadow, then the conjugate axis of Čech lies in the plane at infinity and accordingly our surface must necessarily be of the class Σ .

Conversely, a surface Σ may be represented by the solutions of the differential equations

$$(7) \quad \begin{aligned} \mathfrak{x}_{uu} &= A\mathfrak{x}_v, \\ \mathfrak{x}_{uv} &= \mathfrak{y}, \\ \mathfrak{x}_{vv} &= A\mathfrak{x}_u, \end{aligned}$$

where u, v are asymptotic parameters.

Transform the variables u, v to

$$\begin{aligned} \alpha &= \varepsilon^{2r}u + \varepsilon^r v, \\ \beta &= \varepsilon^{2r}u - \varepsilon^r v, \end{aligned} \quad (r=0, 1, 2),$$

then we get

$$\mathfrak{x}_{\alpha\beta} = -\frac{1}{2}A\mathfrak{x}_\beta$$

or

$$(8) \quad \mathfrak{x}_\beta = \mathfrak{B}(\beta) \exp\left(-\frac{1}{2} \int A d\alpha\right)$$

which shows that the curves of Segre are lines of shadow.

Thus the theorem is completely demonstrated.

Combining the above results with the theorem of Čech, we get:

Theorem. *If three systems of curves of Segre of a surface Σ' be curves of contact of enveloping cones, then all the vertices of the enveloping cones lie in a plane π . Such a surface Σ' may be transformed to a surface Σ when the former undertakes a projective transformation which brings the plane π to the plane at infinity. The surface Σ' is characterized by the fact that the conjugate axis of Čech always lies in a fixed plane. The vertices of the enveloping cones describe an algebraic curve of the third order in the plane π .*

The last part of this theorem will be demonstrated in the following paragraphs.

XLII.

**The Curve described by the Vertices
of the Enveloping Cones along Curves of Segre of a Surface Σ' .**

77. We will now demonstrate the remaining part of the theorem in the last paragraph, viz.

The curve in question is an algebraic curve of the third order.

In order to simplify the calculation we transform the surface Σ' to a surface Σ by a certain projective transformation which brings the fixed plane π to the plane at infinity and thence show that the vertices of the enveloping cylinders along curves of Segre of Σ lie on an algebraic curve of the third order in the plane at infinity.

Putting

$$\mathfrak{X} = -\frac{1}{F}(\mathfrak{x}_u \times \mathfrak{x}_v)$$

or
$$X = \frac{1}{F} \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \quad Y = \dots, \quad Z = \dots,$$

and using the conditions

$$A = D, \quad F = 1,$$

we get

$$\begin{aligned} \mathfrak{X}_{uu} &= -A\mathfrak{X}_v + A_v\mathfrak{X}, \\ \mathfrak{X}_{uv} &= A^2\mathfrak{X}, \\ \mathfrak{X}_{vv} &= -A\mathfrak{X}_u + A_u\mathfrak{X}, \end{aligned} \tag{1}$$

where u, v are asymptotic parameters and the function A satisfies the equations⁽⁴⁾

$$\begin{aligned} A_{uu} &= 3AA_v, \\ A_{vv} &= 3AA_u. \end{aligned} \tag{2}$$

Now consider the surface

$$\begin{aligned} 0 = W &= (A_{uv} - 2A^3)(\xi X + \eta Y + \zeta Z)^3 \\ &\quad + 2[(\xi X_u + \eta Y_u + \zeta Z_u)^3 + (\xi X_v + \eta Y_v + \zeta Z_v)^3] \\ &\quad + 6A(\xi X + \eta Y + \zeta Z)(\xi X_u + \eta Y_u + \zeta Z_u)(\xi X_v + \eta Y_v + \zeta Z_v) \\ &\quad - 3A_v(\xi X + \eta Y + \zeta Z)^2(\xi X_u + \eta Y_u + \zeta Z_u) \\ &\quad - 3A_u(\xi X + \eta Y + \zeta Z)^2(\xi X_v + \eta Y_v + \zeta Z_v) - \tau^3, \end{aligned} \tag{3}$$

⁽⁴⁾ Cf. Part XXVI.

where ξ, η, ζ denote the current coordinates of a point and $\tau=1$ or 0 according as the point is a proper or an improper point.

By means of (1) and (2) we can easily show that

$$(4) \quad \frac{\partial W}{\partial u} = \frac{\partial W}{\partial v} = 0$$

i. e. *the surface is fixed in space.*

On the other hand, the vertex of the enveloping cylinder may be represented by the equations

$$(5) \quad x'\bar{X} + y'\bar{Y} + z'\bar{Z} = \begin{vmatrix} X \varepsilon X_u + X_v \bar{X} \\ Y \varepsilon Y_u + Y_v \bar{Y} \\ Z \varepsilon Z_u + Z_v \bar{Z} \end{vmatrix} = 0, \quad t' = 0.$$

From (5) we have

$$(6) \quad \begin{aligned} x'X_u + y'Y_u + z'Z_u &= -(X_uX_vX), \\ x'Y_v + y'Y_v + z'Z_v &= \varepsilon(X_uX_vX), \end{aligned}$$

whence we conclude that $W=0$ for $\xi=x', \eta=y', \zeta=z', \tau=t'=0$.

Therefore, considering ξ, η, ζ as the homogeneous coordinates of a point in the plane at infinity, we obtain the required algebraic curve of the third order:

$$(7) \quad (A_{uv} - 2A^3)y^3 + 2y_1^3 + 2y_2^3 + 6Ayy_1y_2 - 3A_\varepsilon y^2y_1 - 3A_\varepsilon y^2y_2 = 0,$$

where

$$(8) \quad \begin{aligned} y &= \xi X + \eta Y + \zeta Z, \\ y_1 &= \xi X_u + \eta Y_u + \zeta Z_u, \\ y_2 &= \xi X_v + \eta Y_v + \zeta Z_v. \end{aligned}$$

Thus our theorem is completely established.

78. The above method of deducing the curve W , being quite a similar one due to Prof. Čech in his paper on surfaces with plane curves of Segre⁽⁵⁾, suggests us to discuss the curve W for the cases of surface $\Sigma_1, \Sigma_2, \dots, \Sigma_6$.

For the surface Σ_6 we have that $A=1$ and accordingly (7) becomes

$$(9) \quad \begin{aligned} y - y_1 - y_2 &= 0, \\ y - \varepsilon y_1 - \varepsilon^2 y_2 &= 0, \\ y - \varepsilon^2 y_1 - \varepsilon y_2 &= 0. \end{aligned}$$

(⁵) E. Čech, Sur les surfaces dont toutes les courbes de Segre sont planes, Publications de la Faculté des Sciences de l'Université Masaryk, (1922), 1-35.

That is, the curve W breaks up into three straight lines.

In the case of surface Σ_5 we get that

$$A = -\frac{2}{3} \frac{1}{u+v}$$

and (7) becomes

$$(10) \quad \frac{2}{3} \frac{1}{u+v} y + y_1 + y_2 = 0, \\ \frac{5}{9} \frac{1}{(u+v)^2} y^2 - y_1^2 - y_2^2 + \frac{2}{3} \frac{1}{u+v} y y_1 + \frac{2}{3} \frac{1}{u+v} y y_2 + y_1 y_2 = 0.$$

That is, the curve W breaks up into a conic and a straight line touching the conic.

Similarly, we can, for surface Σ_4 , deduce that the curve W breaks up into a conic

$$\left[1 + \frac{5}{9} \cot^2(u+v) \right] y^2 - y_1^2 - y_2^2 + y_1 y_2 \\ + \frac{2}{3} \cot(u+v) y y_1 + \frac{2}{3} \cot(u+v) y y_2 = 0$$

and a straight line

$$\frac{2}{3} \cot(u+v) y + y_1 + y_2 = 0.$$

In the case of surface Σ_3 we have

$$A = \frac{2uv}{u^3 + v^3}$$

and

$$(11) \quad W = -\frac{2(u^6 + v^6 - 3u^3v^3)}{(u^3 + v^3)^3} y + y_1^3 + y_2^3 + \frac{6uv}{u^3 + v^3} y y_1 y_2 \\ - \frac{3u(u^3 - 2v^3)}{(u^3 + v^3)^2} y^2 y_1 + \frac{3v(2u^3 - v^3)}{(u^3 + v^3)^2} y^2 y_2.$$

We say that the curve W has a cusp. For, from the equations

$$\frac{\partial W}{\partial y_1} = \frac{\partial W}{\partial y_2} = \frac{\partial^2 W}{\partial y_1^2} = \frac{\partial^2 W}{\partial y_2^2} - \left(\frac{\partial^2 W}{\partial y_1 \partial y_2} \right)^2 = 0$$

we can deduce that the coordinates of the cusp are given by

$$(12) \quad y : y_1 : y_2 = -(u^3 + v^3) : u^2 : v^2.$$

In the case of surface Σ_2 we have

$$A = -\frac{2}{3}(\xi_0 + \xi_1 + \xi_2),$$

where

$$\xi_0 = \cot(u + v), \quad \xi_1 = \cot(\varepsilon^2 u + \varepsilon v), \quad \xi_2 = \cot(\varepsilon u + \varepsilon^2 v),$$

whence the cubic form W takes the form:

$$(13) \quad \begin{aligned} W = & -\frac{1}{27}[10(\xi_0^3 + \xi_1^3 + \xi_2^3) - 6(\xi_0 + \xi_1 + \xi_2) + 24\xi_0\xi_1\xi_2]y^3 \\ & + y_1^3 + y_2^3 - 2(\xi_0 + \xi_1 + \xi_2)yy_1y_2 \\ & - (\xi_0^2 + \varepsilon\xi_1^2 + \varepsilon^2\xi_2^2)y^2y_1 - (\xi_0^2 + \varepsilon^2\xi_1^2 + \varepsilon\xi_2^2)y^2y_2. \end{aligned}$$

The curve W has a double point whose coordinates may be found:

$$(14) \quad y : y_1 : y_2 = -3 : (\xi_0 + \varepsilon^2\xi_1 + \varepsilon\xi_2) : (\xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2).$$

Finally, for the surface Σ_1 it can be shown that the curve W is of genus one.

In conclusion:

For the different kinds $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6$ of a surface Σ the curve W

(Σ_1) is of genus one,

(Σ_2) has a double point,

(Σ_3) has a cusp,

(Σ_4 and Σ_5) breaks up into a conic and a straight line, where the line, in the case Σ_5 , touches the conic,

(Σ_6) breaks up into three straight lines.

XLIII.

Deduction of Equations to a Surface Σ by Means of Čech's Method.

79. In order to obtain the equations to a surface Σ we have to integrate the following system of partial differential equations for $\mathfrak{X}$:

$$(1) \quad \begin{aligned} \mathfrak{X}_{uu} &= -A\mathfrak{X}_v + A_v\mathfrak{X}, \\ \mathfrak{X}_{uv} &= A^2\mathfrak{X}, \\ \mathfrak{X}_{vv} &= -A\mathfrak{X}_u + A_v\mathfrak{X}. \end{aligned}$$

The equations to the surface are given by formulas of Lelievre:

$$(2) \quad \mathfrak{r} = \int (\mathfrak{X} \times \mathfrak{X}_u \, du + \mathfrak{X}_v \times \mathfrak{X} \, dv).$$

Putting $A = 2\varphi$ and denoting the general solution of (1) by $\mathfrak{X} = y$, we get

$$(3) \quad \begin{aligned} \frac{\partial y}{\partial u} &= y_1, & \frac{\partial y}{\partial v} &= y_2, \\ \frac{\partial y_1}{\partial u} &= 2 \frac{\partial \varphi}{\partial v} y - 2\varphi y_2, & \frac{\partial y_1}{\partial v} &= 4\varphi^2 y, \\ \frac{\partial y_2}{\partial u} &= 4\varphi^2 y, & \frac{\partial y_2}{\partial v} &= 2 \frac{\partial \varphi}{\partial u} y - 2\varphi y_1. \end{aligned}$$

where the function φ satisfies the conditions

$$(4) \quad \frac{\partial^2 \varphi}{\partial u^2} = 6\varphi \frac{\partial \varphi}{\partial v}, \quad \frac{\partial^2 \varphi}{\partial v^2} = 6\varphi \frac{\partial \varphi}{\partial u}.$$

The solutions of system (3) have been obtained by Prof. Čech⁽⁶⁾. Using his beautiful results⁽⁷⁾, we can also deduce the equations to a surface Σ , which I have obtained by the method stated in Part XXVII.

In the following lines I put

$$x_0 = u + v, \quad x_1 = \varepsilon^2 u + \varepsilon v, \quad x_2 = \varepsilon u + \varepsilon^2 v$$

so that $x_0 + x_1 + x_2 = 0$ and consider the equations to the different kinds $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6$.

Case Σ_6 . The general solution of (3) takes the form

$$(5) \quad \mathfrak{X} = c_0 e^{-2v_0} + c_1 e^{-2v_1} + c_2 e^{-2v_2}$$

and accordingly the equations to the surface Σ_6 become

$$\mathfrak{r} = a_0 e^{2v_0} + a_1 e^{2v_1} + a_2 e^{2v_2},$$

or

$$(6) \quad xyz = \text{const.}$$

Case Σ_5 . In this case we have that

$$(7) \quad \mathfrak{X} = a \frac{u^2 + uv + v^2}{\sqrt[3]{u+v}} + b \frac{u-v}{\sqrt[3]{u+v}} + c \frac{1}{\sqrt[3]{u+v}},$$

a, b, c being arbitrary constants. From (2) and (7) it follows immediately that

$$(8) \quad \begin{aligned} x &= (u+v)^{\frac{1}{3}}, \\ y &= (u+v)^{\frac{1}{3}}(u-v), \\ z &= (u+v)^{\frac{1}{3}}\{7(u-v)^2 - 3(u+v)^2\}. \end{aligned}$$

Case Σ_4 . Put

(⁶) Čech, loc. cit.

(⁷) We remark that these results may also be deduced by our method stated in Part XXVII.

$$\alpha = u + v, \quad \beta = u - v,$$

then we get

$$(9) \quad \mathfrak{X} = \sin^{-\frac{1}{3}} \alpha (a \cos \alpha + b e^{\frac{\beta}{\sqrt[3]{3}}} + c e^{-\frac{\beta}{\sqrt[3]{3}}}),$$

and (2) becomes

$$\mathfrak{z} = \frac{1}{2} \int (\mathfrak{X} \times \mathfrak{X}_\beta d\alpha + \mathfrak{X} \times \mathfrak{X}_\alpha d\beta).$$

After some easy calculations we arrive at the following equations to the surface Σ_4 :

$$(10) \quad \begin{aligned} x &= \sin^{\frac{1}{3}} \alpha e^{\frac{\beta}{\sqrt[3]{3}}}, \\ y &= \sin^{\frac{1}{3}} \alpha e^{-\frac{\beta}{\sqrt[3]{3}}}, \\ z &= \int_{\infty}^{\alpha} \frac{d\alpha}{\sin^{\frac{2}{3}} \alpha}. \end{aligned}$$

Case Σ_3 . We have in this case

$$(11) \quad \mathfrak{X} = a \frac{u^3 - v^3}{\sqrt[3]{u^3 + v^3}} + b \frac{uv}{\sqrt[3]{u^3 + v^3}} + c \frac{1}{\sqrt[3]{u^3 + v^3}}.$$

Similarly, the equations to the surface Σ_3 have the forms:

$$(12) \quad \begin{aligned} x &= (u^3 + v^3)^{\frac{1}{3}}, \\ y &= (u^3 + v^3)^{\frac{1}{3}} uv, \\ z &= \int_{\infty}^{\frac{u}{v}} \frac{dt}{(1+t^3)^{\frac{2}{3}}}, \end{aligned}$$

or

$$\begin{aligned} x &= \alpha (1 + \beta^3)^{\frac{1}{3}}, \\ y &= \alpha^3 \beta (1 + \beta^3)^{\frac{1}{3}}, \\ z &= \int_{\infty}^{\beta} \frac{dt}{(1+t^3)^{\frac{2}{3}}}. \end{aligned}$$

Case Σ_2 . In this case we put

$$(13) \quad \xi_0 = \cot(u + v), \quad \xi_1 = \cot(\varepsilon^2 u + \varepsilon v), \quad \xi_2 = \cot(\varepsilon u + \varepsilon^2 v)$$

and introduce two new variables α and β , where

$$(14) \quad \alpha = \frac{\xi_0 + \xi_1 + \xi_2 + \sqrt{3}}{\xi_0 + \varepsilon^2 \xi_1 + \varepsilon \xi_2} = \frac{\xi_0 + \varepsilon \xi_1 + \varepsilon^2 \xi_2}{\xi_0 + \xi_1 + \xi_2 - \sqrt{3}},$$

$$\beta = \frac{\xi_0 + \xi_1 + \xi_2 + \sqrt{3}}{\xi_0 + \varepsilon \xi_1 + \varepsilon^2 \xi_2} = \frac{\xi_0 + \varepsilon^2 \xi_1 + \varepsilon \xi_2}{\xi_0 + \xi_1 + \xi_2 - \sqrt{3}},$$

so that

$$(15) \quad \begin{aligned} du &= \frac{\sqrt{3}}{2} \left(-\frac{d\alpha}{\alpha^3-1} + \frac{\beta d\beta}{\beta^3-1} \right), \\ dv &= \frac{\sqrt{3}}{2} \left(\frac{\alpha d\alpha}{\alpha^3-1} - \frac{d\beta}{\beta^3-1} \right) \end{aligned}$$

and

$$(16) \quad \begin{aligned} \frac{\partial \alpha}{\partial u} &= \frac{2}{\sqrt{3}} \frac{\alpha^3-1}{\alpha\beta-1}, & \frac{\partial \beta}{\partial u} &= \frac{2}{\sqrt{3}} \frac{\alpha(\beta^3-1)}{\alpha\beta-1}, \\ \frac{\partial \alpha}{\partial v} &= \frac{2}{\sqrt{3}} \frac{\beta(\alpha^3-1)}{\alpha\beta-1}, & \frac{\partial \beta}{\partial v} &= \frac{2}{\sqrt{3}} \frac{\beta^3-1}{\alpha\beta-1}. \end{aligned}$$

Substituting these values in (2), we get

$$(17) \quad \begin{aligned} \mathfrak{r} = - \int & \left[\left(\frac{\alpha\beta+1}{\alpha\beta-1} (\mathfrak{X} \times \mathfrak{X}_\alpha) + \frac{2\alpha(\beta^3-1)}{(\alpha^3-1)(\alpha\beta-1)} (\mathfrak{X} \times \mathfrak{X}_\beta) \right) d\alpha \right. \\ & \left. - \left(\frac{2\beta(\alpha^3-1)}{(\beta^3-1)(\alpha\beta-1)} (\mathfrak{X} \times \mathfrak{X}_\alpha) + \frac{\alpha\beta+1}{\alpha\beta-1} (\mathfrak{X} \times \mathfrak{X}_\beta) \right) d\beta \right]. \end{aligned}$$

The general solution of (3) for this case is, of the form

$$(18) \quad \mathfrak{X} = a \frac{\alpha^2 - \beta}{(\alpha\beta-1)\sqrt[3]{\alpha^3-1}} + b \frac{\beta^2 - \alpha}{(\alpha\beta-1)\sqrt[3]{\beta^3-1}} + c \frac{\sqrt[3]{(\alpha^3-1)(\beta^3-1)}}{\alpha\beta-1}.$$

After some lengthy calculations we get the equations to the surface Σ_2

$$(19) \quad \begin{aligned} x &= \int_{\infty}^{\alpha} \frac{dt}{(t^3-1)^{\frac{2}{3}}}, \\ y &= \int_{\infty}^{\beta} \frac{dt}{(t^3-1)^{\frac{2}{3}}}, \\ z &= \frac{\alpha\beta-1}{\sqrt[3]{(\alpha^3-1)(\beta^3-1)}}. \end{aligned}$$

Case Σ_1 . Put

$$\begin{aligned} z_0 &= m(x_1 + a_1 - x_2 - a_2), \\ z_1 &= m(x_2 + a_2 - x_0 - a_0), & (m, a_i: \text{Constants}) \\ z_2 &= m(x_0 + a_0 - x_1 - a_1) & a_0 + a_1 + a_2 = 0, \end{aligned}$$

and

$$\begin{aligned}
 (20) \quad D &= (\wp z_1 - \wp z_2)(\wp z_2 - \wp z_0)(\wp z_0 - \wp z_1), \\
 A &= \zeta z_0 + \zeta z_1 + z_2, \\
 B &= \wp' z_0 + \wp' z_1 + \wp' z_2.
 \end{aligned}$$

The solution of (3) is given by the equation

$$(21) \quad \mathfrak{K} = D^{-\frac{1}{3}}(aA + bB + c).$$

Therefore the surface Σ_1 may be represented by the following equations:

$$\begin{aligned}
 (22) \quad x &= \int D^{-\frac{2}{3}} \left(\frac{\partial A}{\partial u} du - \frac{\partial A}{\partial v} dv \right), \\
 y &= \int D^{-\frac{2}{3}} \left(\frac{\partial B}{\partial u} du - \frac{\partial B}{\partial v} dv \right), \\
 z &= \int A^2 D^{-\frac{2}{3}} \left(\frac{\partial}{\partial u} \left(\frac{B}{A} \right) du - \frac{\partial}{\partial v} \left(\frac{B}{A} \right) dv \right).
 \end{aligned}$$

XLIV.

Some Allied Problems.

80. In the first place we will give an alternative proof to the first theorem stated in § 76.

Consider the line of intersection $\mathfrak{z}$ of two consecutive tangent planes along a curve of Segre. It may be easily deduced that

$$(1) \quad \mathfrak{z} = \varepsilon P \mathfrak{x}_u - \mathfrak{x}_v,$$

where $P = \sqrt[3]{D/A}$ and $\varepsilon^3 = 1$.

In order that the curves of Segre be lines of shadow it is necessary and sufficient that $\mathfrak{z}$ along the curves of Segre be parallel to each other or $\varepsilon P_{\mathfrak{z}u} + \mathfrak{z}_v$ be proportional to $\mathfrak{z}$.

Using the fundamental equations in affine differential geometry, we get that

$$(2) \quad \varepsilon P_{\mathfrak{z}u} + \mathfrak{z}_v = \left[\varepsilon^2 P^2 \left(\frac{F_u}{F} + \frac{P_u}{P} \right) + \varepsilon P_v - \frac{D}{F} \right] \mathfrak{x}_u + \left(\varepsilon^2 P \frac{A}{F} - \frac{F_v}{F} \right) \mathfrak{x}_v.$$

Since (2) is proportional to (1), it follows:

$$(3) \quad \varepsilon P \left(\frac{F_u}{F} + \frac{P_u}{P} \right) + \frac{P_v}{P} - \frac{F_v}{F} = 0.$$

Therefore, if two systems of curves of Segre be lines of shadow, then

$$\frac{F_u}{F} + \frac{P_u}{P} = 0, \quad \frac{P_v}{P} - \frac{F_v}{F} = 0$$

which become $A=D$, $F=1$, provided that the parameters u , v are suitably transformed. Thus we arrive at the required result.

By quite a similar method we can prove that *if three systems of curves of Darboux be lines of shadow, then the surface must necessarily be a ruled surface.*

81. The second allied problem is to *find all the surfaces where all the first directrices of Wilczynski are affine normals.*

This problem may be solved by a theorem of Demoulin's⁽⁸⁾:

The quadrics of Lie have an envelope which touches the quadric of Lie relative to a point P of the surface, besides at P , at four points forming the vertices of a gauche quadrilateral, of which the four sides belong to the quadric of Lie and the remaining two l , l' are reciprocal to it.

After Prof. E. Bompiani⁽⁹⁾ we further have that the first directrix of Wilczynski is the line passing through P and intersecting the lines l , l' .

Using these results and the formulas in § 82 of Blaschke's *Vorlesungen über Differentialgeometrie II*, we can prove the following

Theorem. *Only the affine spheres possess the property that the first directrices of Wilczynski are at the same time affine normals.*

82. The third problem arises from the extremals of variation-problem

$$\delta \int \varphi = 0, \quad \delta \int \psi = 0,$$

where φ , ψ denote the quadratic and cubic forms in affine differential geometry. We call the corresponding extremals the first and the second affine geodesics respectively. The Euler's equations to them are respectively

$$\begin{aligned} u'' &= -\frac{F_u}{F}u'^2 + \frac{F_v}{F}u', \\ 2\left(A + \frac{D}{u'^3}\right)u'' &= -\left(A_u - A\frac{F_u}{F}\right)u'^3 - 2\left(A_v - A\frac{F_v}{F}\right)u' \\ &\quad + 2\left(D_u - D\frac{F_u}{F}\right)\frac{1}{u'} + \left(D_v - D\frac{F_v}{F}\right)\frac{1}{u'^3}, \end{aligned}$$

⁽⁸⁾ Comptes Rendus, **147** (1908), 493--496.

⁽⁹⁾ Cf. Fubini-Čech, loc. cit., p. 147.

where u, v are asymptotic parameters.

From these we can show that *if the first and the second affine geodesics of a surface always coincide, then the surface must necessarily be the surface $xyz = \text{const.}$*

The 8th, April, 1929.