25. Theorems concerning Poisson Integrals.

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1. A Lemma.

1. We will prove a lemma which is used frequently in this paper.

Lemma. *Let C be a curve in $|z| < 1$, which passes through $z=1$ with a tangent which makes an angle ω ($0 \leq \omega \leq \pi$) with the upper semi-circle of $|z|=1$ and ζ be a point on C .*

Let by a transformation;

$$(1) \quad x = \frac{z - \zeta}{1 - \bar{\zeta}z}$$

$z_1 = e^{i\theta_1}$, $z_0 = 1$, $z_2 = e^{-i\theta_1}$, ($0 < \theta_1 < \pi$) be transformed into $x_1 = e^{i\phi_1}$, $x_0 = e^{i\phi_0}$, $x_2 = e^{i\phi_2}$. Then

$$x_1 \rightarrow -1, \quad x_0 \rightarrow e^{-i(\pi-2\omega)}, \quad x_2 \rightarrow -1,$$

as $\zeta \rightarrow 1$ along C .

Proof.

Put $x = e^{i\phi}$, $z = e^{i\theta}$, $\zeta = ae^{i\psi}$ in (1), then

$$e^{i\phi} = \frac{e^{i\theta} - ae^{i\psi}}{1 - ae^{i(\theta-\psi)}} = \frac{[e^{i\frac{\theta}{2}} - ae^{i(\psi-\frac{\theta}{2})}]^2}{|1 - ae^{i(\phi-\psi)}|^2}.$$

Hence if we put

$$e^{i\frac{\theta}{2}} - ae^{i(\psi-\frac{\theta}{2})} = Ae^{i\chi} \quad (A > 0),$$

we get $\phi = 2\chi$, where

$$(2) \quad \tan \chi = \frac{\tan \frac{\theta}{2} \cdot \frac{1 + a \cos \psi}{1 - a \cos \psi} - \frac{a \sin \psi}{1 - a \cos \psi}}{1 - \frac{a \sin \psi}{1 - a \cos \psi} \cdot \tan \frac{\theta}{2}}.$$

In (2) we make $\zeta \rightarrow 1$ along C , then

(i) if $0 < |\theta| < \pi$, $\lim \tan \chi = \pm \infty$, $\chi \rightarrow \frac{\pm \pi}{2}$, $\phi \rightarrow \pm \pi$, so that

$$x \rightarrow -1.$$

$$(ii) \quad \text{if } \theta=0, \quad \lim \tan \chi = -\lim \frac{a \sin \psi}{1-a \cos \psi} = -\tan \left(\frac{\pi}{2} - \omega \right),$$

$$\chi \rightarrow -\left(\frac{\pi}{2} - \omega \right), \quad \phi \rightarrow -(\pi - 2\omega),$$

so that

$$x \rightarrow e^{-i(\pi-2\omega)}, \quad \text{q. e. d.}$$

2. Poisson Integral.

2. Theorem 1. Let $u(z)$ ($z=re^{i\theta}$) be a harmonic function which is regular in $|z| < 1$ and is given by a Poisson Integral;

$$(3) \quad u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1-r^2}{1-2r \cos(\theta-\phi)+r^2} d\phi$$

where $u(e^{i\theta})$ is an integrable function in Lebesgue's sense.

By a transformation;

$$(4) \quad x = \frac{z-a}{1-\bar{a}z} \quad (|a| < 1)$$

$u(z)$ becomes a harmonic function $v(x)$ in $|x| < 1$.

Then $v(x)$ can be expressed by a Poisson integral of the form (3).

Proof.

We use Fichtenholz's theorem⁽¹⁾: let $u(z)$ be a harmonic function regular in $|z| < 1$, then a necessary and sufficient condition that $u(z)$ can be expressed by a Poisson integral is that

⁽¹⁾ Fichtenholz: Sur l'intégrale de Poisson et quelques questions qui s'y rattachent. Fundamenta Math. **13** (1929) 1-33.

$$F(r, \theta) = \int_0^\theta u(re^{i\lambda}) d\lambda \quad (0 \leq r < 1)$$

is uniformly absolutely continuous, *i. e.* for a given ε , we can find δ , independently of r , such that

$$\sum_{i=1}^n |F(r, \theta_i) - F(r, \theta'_i)| < \varepsilon$$

if
$$\sum_{i=1}^n |\theta_i - \theta'_i| < \delta.$$

Suppose that $v(x)$ can not be expressed by a Poisson integral, then by this Fichtenholz's theorem there exist

$$(5) \quad x_n^{(k)} = \rho_n e^{i\theta_n^{(k)}}, \quad x_n'^{(k)} = \rho_n e^{i\theta_n'^{(k)}} \quad (k=1, 2, \dots, k_n)$$

$$(0 < \rho_1 < \rho_2 < \dots < \rho_n < \dots \rightarrow 1)$$

such that

$$(6) \quad \sum_{k=1}^{k_n} |\theta_n^{(k)} - \theta_n'^{(k)}| \rightarrow 0 \quad \text{or} \quad \sum_{k=1}^{k_n} \int_{x_n^{(k)}}^{x_n'^{(k)}} |dx| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \text{ and}$$

$$(7) \quad \left| \sum_{k=1}^{k_n} \int_{\theta_n^{(k)}}^{\theta_n'^{(k)}} v(\rho_n e^{i\theta}) d\theta \right| \geq \delta \quad (> 0).$$

Since $d\theta = \frac{dx}{ix} = \frac{1}{i} \frac{1-|a|^2}{(z-a)(1-\bar{z}\bar{a})} dz$, (7) becomes

$$(8) \quad \left| \sum_{k=1}^{k_n} \int_{z_n^{(k)}}^{z_n'^{(k)}} u(z) \frac{1-|a|^2}{(z-a)(1-\bar{z}\bar{a})} dz \right| \geq \delta$$

where $z_n^{(k)} = r_n^{(k)} e^{i\phi_n^{(k)}}$, $z_n'^{(k)} = r_n'^{(k)} e^{i\phi_n'^{(k)}}$ correspond to $x_n^{(k)}$, $x_n'^{(k)}$ respectively and the path of integration is the circle K_n : $\left| \frac{z-a}{1-\bar{z}\bar{a}} \right| = \rho_n$.

Since on K_n , $|z-a| \geq \eta (> 0)$, $|1-\bar{z}\bar{a}| \geq \eta$, when $n \geq n_0$, (8) becomes

$$(9) \quad \sum_{k=1}^{k_n} \int_{z_n^{(k)}}^{z_n'^{(k)}} |u(z)| |dz| \geq \delta_1 \quad (\delta_1 = \eta^2 \delta)$$

and (6) becomes

$$(10) \quad \sum_{k=1}^{k_n} \int_{z_n^{(k)}}^{z_n'^{(k)}} |dz| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

the path of integration being K_n .

We will show that (9) and (10) lead to a contradiction.

For simplicity, we assume $0 < a < 1$.

Let K_n meet the real axis at q_n and q'_n , then

$$q_n = \frac{a + \rho_n}{1 + \rho_n a}, \quad q'_n = \frac{a - \rho_n}{1 - \rho_n a}.$$

We draw a circle $\bar{K}_n: |z| = q_n$, then $\bar{K}_n$ touches K_n at q_n externally and contains K_n in it.

Let a radius of the unit circle meet K_n and $\bar{K}_n$ at z and ξ , then

$$q_n - r = |\xi - z| \leq q_n + q'_n = \frac{2a(1 - \rho_n)(1 + \rho_n)}{(1 - \rho_n a)(1 + \rho_n a)}, \quad \text{where } r = |z|,$$

$$1 - q_n = \frac{(1 - a)(1 - \rho_n)}{1 + \rho_n a}.$$

Hence

$$(11) \quad q_n - r \leq \frac{2a(1 - q_n)(1 + \rho_n)}{(1 - a)^2(1 + \rho_n a)} \leq \frac{4}{(1 - a)^2}(1 - q_n) \quad \text{or}$$

$$\frac{q_n - r}{1 - q_n} \leq \frac{4}{(1 - a)^2}.$$

Further plainly there exists a constant K such that on K_n ($n \geq n_0$)

$$(12) \quad |dz| \leq K d\theta \quad \text{where } z = re^{i\theta}.$$

Hence (9) and (10) become:

$$(13) \quad \sum_{k=1}^{l_n} \int_{\phi_n^{(k)}}^{\phi_n^{(k)}} |u(z)| d\theta \geq \frac{\delta_1}{K} \quad (z \text{ being on } K_n)$$

$$(14) \quad \sum_{k=1}^{l_n} \int_{\phi_n^{(k)}}^{\phi_n^{(k)}} d\theta \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since, by the hypothesis,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\phi \quad (z = re^{i\theta})$$

(13) becomes

$$(15) \quad \frac{\delta_1}{K} \leq \int_0^{2\pi} |u(e^{i\phi})| d\phi \sum_{k=1}^{k_n} \int_{\phi_n^{(k)}}^{\phi_n'^{(k)}} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\theta.$$

Next consider

$$\begin{aligned} I &= \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \Big/ \frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2} \\ &= \frac{1-r^2}{1-q_n^2} \cdot \frac{1-2q_n\cos(\theta-\phi)+q_n^2}{1-2r\cos(\theta-\phi)+r^2} = \frac{1-r^2}{1-q_n^2} \left[\frac{q_n}{r} - \frac{1}{2r} \frac{(q_n-r)(1-q_n r)}{\frac{1+r^2}{2r} - \cos(\theta-\phi)} \right]. \end{aligned}$$

Since $q_n \geq r$, $\frac{1+r^2}{2r} > 1$, the term in the bracket attains its maximum value when $\cos(\theta-\phi) = -1$.

Hence by (11),

$$\begin{aligned} I &\leq \frac{1-r^2}{1-q_n^2} \left(\frac{1+q_n}{1+r} \right)^2 = \frac{1-r}{1-q_n} \cdot \frac{1+q_n}{1+r} \leq 2 \frac{1-r}{1-q_n} = 2 \left(1 + \frac{q_n-r}{1-q_n} \right) \\ &\leq 2 \left(1 + \frac{4}{(1-a)^2} \right) = M \quad \text{say.} \end{aligned}$$

Hence

$$\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \leq M \frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2}$$

so that (15) becomes

$$\frac{\delta_1}{MK} \leq \frac{1}{2\pi} \int_0^{2\pi} |u(e^{i\phi})| d\phi \sum_{k=1}^{k_n} \int_{\phi_n^{(k)}}^{\phi_n'^{(k)}} \frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2} d\theta$$

or

$$(16) \quad \frac{\delta_1}{MK} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1-q_n^2}{1-2q_n\cos\lambda+q_n^2} d\lambda \sum_{k=1}^{k_n} \int_{\phi_n^{(k)}}^{\phi_n'^{(k)}} |u(e^{i(\phi+\lambda)})| d\phi.$$

But from (14) for $n \geq n_0$, $\sum_{k=1}^{k_n} |\phi_n'^{(k)} - \phi_n^{(k)}| < \varepsilon'$, consequently by the absolute

continuity of an indefinite integral,

$$\sum_{k=1}^{k_n} \int_{\phi_n^{(k)}}^{\phi_n'^{(k)}} |u(e^{i(\phi+\lambda)})| d\phi < \varepsilon$$

so that (16) becomes

$$(17) \quad \frac{\delta_1}{MK} \leq \varepsilon.$$

For a sufficiently small ε , (17) is evidently impossible. Hence our hypothesis (9) and (10) leads to a contradiction. Thus the theorem is proved.

3. We will prove a theorem concerning the order of the derivatives of the harmonic function expressed by a Poisson integral.

Let

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\phi \quad (z=re^{i\theta})$$

and $0 < a < 1$.

Put

$$\rho = \sqrt{1-a^2}, \quad \theta = \tan^{-1} \frac{\rho}{a}$$

then

$$(18) \quad u(a) \leq \frac{1-a^2}{2\pi} \int_{-\theta}^{\theta} |u(e^{i\phi})| \frac{d\phi}{(1-a)^2} + \frac{1-a^2}{2\pi} \int_{\theta}^{2\pi-\theta} |u(e^{i\phi})| \frac{d\phi}{\rho^2} \leq \\ \frac{1+a}{2\pi(1-a)} \int_{-\theta}^{\theta} |u(e^{i\phi})| d\phi + \frac{1+a}{2\pi(1-a)} \left(\frac{1-a}{\rho} \right)^2 \int_0^{2\pi} |u(e^{i\phi})| d\phi.$$

Since $\int_{-\theta}^{\theta} |u(e^{i\phi})| d\phi \rightarrow 0, \quad \frac{1-a}{\rho} \rightarrow 0 \quad \text{as } a \rightarrow 1,$ we have

$$u(a) = o\left(\frac{1}{1-a}\right)$$

or

$$u(z) = o\left(\frac{1}{1-r}\right) \quad (r = |z|).$$

This was proved by Ostrowski⁽²⁾.

Next we will evaluate $\left| \frac{\partial u}{\partial x} \right|, \left| \frac{\partial u}{\partial y} \right| \quad (z = x + iy).$

⁽²⁾ A. Ostrowski, Ueber quasianalytische Funktionen und Bestimmtheit asymptotischer Entwicklungen. Acta Math. **53**. (1929), 181–266.

By $w = \frac{z-a}{1-\bar{a}z}$, ($0 < a < 1$), $u(z)$ becomes $v(w)$, which, by Theorem 1, can be expressed by a Poisson integral:

$$v(w) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\phi}) \frac{1-\rho^2}{1-2\rho\cos(\psi-\phi)+\rho^2} d\phi \quad (w = \rho e^{i\psi}).$$

Hence

$$\begin{aligned} (19) \quad \left| \left(\frac{\partial v}{\partial \rho} \right)_{\rho=0} \right| &\leq \frac{1}{\pi} \int_0^{2\pi} |v(e^{i\phi}) \cos(\psi-\phi)| d\phi \leq \frac{1}{\pi} \int_0^{2\pi} |v(e^{i\phi})| d\phi \\ &= \frac{1}{\pi} \int_0^{2\pi} |v(w)| \left| \frac{dw}{w} \right| = \frac{1}{\pi} \int_0^{2\pi} |u(z)| \frac{1-a^2}{|z-a|^2} |dz| \\ &= \frac{1}{\pi} \int_0^{2\pi} |u(e^{i\phi})| \frac{1-a^2}{1-2a\cos\phi+a^2} d\phi = o\left(\frac{1}{1-a}\right). \\ \left| \left(\frac{\partial v}{\partial \rho} \right)_0 \right| &= \left(\left| \frac{dv}{dw} \right| \right)_0 = \left(\left| \frac{du}{dz} \right| \right)_a \left| \frac{dz}{dw} \right|_0 = \left(\left| \frac{du}{dz} \right| \right)_a (1-a^2). \end{aligned}$$

Hence

$$\left(\left| \frac{du}{dz} \right| \right)_a = \frac{1}{1-a^2} o\left(\frac{1}{1-a}\right) = o\left(\frac{1}{(1-a)^2}\right)$$

or

$$\left| \frac{\partial u}{\partial x} \right| = o\left(\frac{1}{(1-|z|^2)^2}\right), \quad \frac{\partial u}{\partial y} = o\left(\frac{1}{(1-|z|^2)^2}\right).$$

If $|u(z)| \leq M$ for $|z| < 1$, then from (19),

$$\begin{aligned} (20) \quad \left| \left(\frac{\partial v}{\partial \rho} \right)_0 \right| &= \frac{1}{\pi} \left| \int_0^{2\pi} v(e^{i\phi}) \cos(\psi-\phi) d\phi \right| \leq \frac{M}{\pi} \int_0^{2\pi} |\cos(\psi-\phi)| d\phi \\ &= \frac{4M}{\pi}. \end{aligned}$$

We can easily find a function $v(x)$ such that $|v(x)| \leq M$ in $|x| < 1$ and $\left| \left(\frac{\partial v}{\partial \rho} \right)_0 \right| = \frac{4M}{\pi}$, so that the above $\frac{4M}{\pi}$ can not be replaced by any smaller one.

From (20) we get,

$$\left(\left| \frac{du}{dz} \right| \right)_a \leq \frac{4M}{\pi(1-|z|^2)} \quad \text{or} \quad \left| \frac{\partial u}{\partial x} \right| \leq \frac{4M}{\pi(1-|z|^2)}, \quad \left| \frac{\partial u}{\partial y} \right| \leq \frac{4M}{\pi(1-|z|^2)}.$$

Hence we have the theorem:

Theorem 2. *Let*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1-r^2}{1-2r \cos(\theta-\phi)+r^2} d\phi \quad (z=re^{i\theta}=x+iy)$$

where $u(e^{i\phi})$ is an integrable function in Lebesgue's sense. Then

$$(21) \quad u(z) = o\left(\frac{1}{1-|z|}\right)$$

$$(22) \quad \left| \frac{\partial u}{\partial x} \right| = o\left(\frac{1}{(1-|z|)^2}\right), \quad \left| \frac{\partial u}{\partial y} \right| = o\left(\frac{1}{(1-|z|)^2}\right).$$

If further $|u(z)| \leq M$ in $|z| < 1$, then

$$(23) \quad \left| \frac{\partial u}{\partial x} \right| \leq \frac{4M}{\pi(1-|z|^2)}, \quad \left| \frac{\partial u}{\partial y} \right| \leq \frac{4M}{\pi(1-|z|^2)}.$$

The right hand side of (23) can not be replaced by any smaller one.

4. We will prove the following fundamental

Theorem 3. *Let $u(z)$ be a harmonic function which is regular in $|z| < 1$ and is expressed by a Poisson integral:*

$$(24) \quad u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1-r^2}{1-2r \cos(\theta-\phi)+r^2} d\phi \quad (z=re^{i\theta})$$

where $u(e^{i\phi})$ is an integrable function in Lebesgue's sense, and G be a simply connected domain in $|z| < 1$ whose boundary Γ consists of a finite number of analytic curves $C_i (i=1, \dots, k)$ in $|z| < 1$ and a certain number of circular arcs of the unit circle $|z|=1$. The angles at which any two C 's meet in $|z| < 1$ are different from zero and the angle at which any C meets the unit circle is different from zero and π , so that any C does not touch the unit circle.

We map G conformally in the unit circle $|x| < 1$. Then $u(z)$ becomes a harmonic function $v(x)$ regular in $|x| < 1$.

Then $v(x)$ can be expressed by a Poisson integral of the form (24) in $|x| < 1$.

Proof.

Suppose that $v(x)$ can not be expressed by a Poisson integral, then by Fichtenholz's theorem, we can find.

$$x_n^{(k)} = \rho_n e^{i\theta_n^{(k)}}, \quad x_n'^{(k)} = \rho_n e^{i\theta_n'^{(k)}} \quad (k=1, \dots, k_n)$$

$$(\rho_1 < \rho_2 < \dots < \rho_n < \dots \rightarrow 1)$$

such that

$$(25) \quad \sum_{k=1}^{l_n} \int_{x_n^{(k)}}^{x_n'^{(k)}} |dx| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$(26) \quad \sum_{k=1}^{l_n} \int_{x_n^{(k)}}^{x_n'^{(k)}} |v(x)| |dx| \geq \delta (> 0),$$

the path of integration being the circle $C_n: |x| = \rho_n$.

Let $z_n^{(k)}$, $z_n'^{(k)}$ correspond to $x_n^{(k)}$, $x_n'^{(k)}$, then from (25) (26) we have

$$(27) \quad \sum_{k=1}^{l_n} \int_{z_n^{(k)}}^{z_n'^{(k)}} \left| \frac{dx}{dz} \right| |dz| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$(28) \quad \sum_{k=1}^{l_n} \int_{z_n^{(k)}}^{z_n'^{(k)}} |u(z)| \left| \frac{dx}{dz} \right| |dz| \geq \delta,$$

the path of integration being the curve which corresponds to C_n .

Since the mapping function $x = \psi(z)$ and $u(z)$ are regular at every point of Γ in $|z| < 1$, which is not an angular point of Γ , any limit point of $z_k^{(n)}$, $z_n'^{(k)}$ ($k = 1, \dots, l_n$, $n = 1, 2, \dots$) will be an angular point of Γ or a point of Γ on $|z| = 1$.

Hence we have to investigate the behaviour of the integral on the left hand side of (28) in the vicinity of the following points:

- (i) Point A , which is an angular point of Γ in $|z| < 1$.
- (ii) Point B , which is an angular point of Γ on $|z| = 1$, where two C 's meet at an angle different from zero and π .
- (iii) Point C , which is a point of Γ , where one C_i and a circular arc of $|z| = 1$ which belongs to Γ , meet at an angle different from zero and π .
- (iv) Point D , which is a point of a circular arc of $|z| = 1$, which belongs to Γ and is not an angular point of Γ .

Let the affix of any one of the points A, B, C, D be denoted by the same letter z_0 and we assume that z_0 corresponds to $x = 1$ on the x -plane.

We will prove that the integral on the left hand side of (28)

tends to zero with the length of the path of integration in the vicinity of the points A, B, C, D .

(i) Point A ($z=z_0$).

Since $u(z)$ is regular at z_0 , $|v(x)| < M$ in the vicinity of $x=1$. Hence

$$\sum \int |v(x)| |dx| \leq M \sum \int |dx|$$

so that

$$(29) \quad \sum \int |v(x)| |dx| \rightarrow 0 \quad \text{as} \quad \sum \int |dx| \rightarrow 0$$

in the vicinity of $x=1$.

(ii) Point B ($z=z_0$).

We use Osgood and Taylor's theorem⁽³⁾: let G be a simply connected domain in the z -plane bounded by a finite number of analytic curves, any two of which meet at an angle different from zero. We map G conformally in the unit circle $|x| < 1$.

Let z_0 be an angular point of the boundary of G which corresponds to $x=1$. Then in the vicinity of z_0 and $x=1$,

$$(30) \quad z - z_0 = A(x-1)^k(1+o(1)), \quad x-1 = \frac{1}{A}(z-z_0)^{\frac{1}{k}}(1+o(1))$$

$$(31) \quad \frac{dz}{dx} = Ak(x-1)^{k-1}(1+o(1)), \quad \frac{dx}{dz} = \frac{1}{Ak}(z-z_0)^{\frac{1}{k}-1}(1+o(1))$$

where $k = \frac{\alpha}{\pi}$, α being the angle at z_0 , and $o(1)$ is the quantity which tends uniformly to zero as $x \rightarrow 1$ (or $z \rightarrow z_0$).

When x lies on the circle $C_n: |x| = \rho_n$, then, by a simple calculation, we get

$$(32) \quad |dx| = \frac{2\rho_n \rho d\rho}{\sqrt{[\rho^2 - (1-\rho_n)^2][(1+\rho_n)^2 - \rho^2]}} \leq \frac{K\rho d\rho}{\sqrt{\rho^2 - (1-\rho_n)^2}}$$

where K is a constant and $\rho = |x-1|$.

Since by (21), $u(z) = o\left(\frac{1}{|z-z_0|}\right)$, we have from (30)

$$(33) \quad v(x) = o\left(\frac{1}{\rho^k}\right) \text{ in the vicinity of } x=1, \text{ where } 0 < k < 1.$$

⁽³⁾ Osgood and Taylor, Conformal transformations on the boundaries of the regions of definition. Trans. Amer. Math. Soc. **14** (1913), 277—298.

Let C_n be an arc of $|x|=\rho_n$ which lies in the domain; $|x-1|\leq\epsilon$, then by (32) and (33),

$$\int_{C_n} |v(x)| |dx| = o\left(\int_{1-\rho_n}^{\epsilon} \frac{\rho d\rho}{\rho^k \sqrt{\rho^2 - (1-\rho_n)^2}}\right).$$

By putting $\rho=(1-\rho_n)t$,

$$\begin{aligned} \int_{1-\rho_n}^{\epsilon} \frac{\rho^{1-k} d\rho}{\sqrt{\rho^2 - (1-\rho_n)^2}} &= (1-\rho_n)^{1-k} \int_1^{\frac{\epsilon}{1-\rho_n}} \frac{t^{1-k} dt}{\sqrt{t^2 - 1}} \sim (1-\rho_n)^{1-k} \int_0^{\frac{\epsilon}{1-\rho_n}} t^{-k} dt \\ &= \epsilon^{1-k} \quad \text{as } \rho_n \rightarrow 1. \end{aligned}$$

Hence $\int_{C_n} |v(x)| |dx| = o(\epsilon^{1-k}) = o(1)$.

(iii) Point C ($z=z_0$).

At z_0 , an analytic curve C_1 and an arc of $|z|=1$ meets in an angle α ($0 < \alpha < \pi$). We draw a straight line L through z_0 , making an angle $\frac{\alpha}{4}$ with $|z|=1$. Then the vicinity U of z_0 is divided by L into two parts U_1 , U_2 , where U_1 is the part of U which is bounded by L and $|z|=1$ and U_2 is the part of U which is bounded by C_1 and L .

Let U , U_1 , U_2 correspond to U' , U_1' , U_2' on the x -plane. Then the integral $\int |v(x)| |dx|$ along the arc of $|x|=\rho_n$, which lies in U_2' tends to zero with $\int |dx|$ as in the case (ii), so that we have only to consider $\int |v(x)| |dx|$ along the arc of $|x|=\rho_n$ in U_1' .

We denote the arc of $|x|=\rho_n$ which lies in U_1' by C_n' and let C_n correspond to C_n' on the z -plane.

If $z=re^{i\theta}$ be a point on C_n , then it is easily seen that there is a constant K such that

$$(34) \quad |dz| \leq K d\theta.$$

Let z be a point on C_n and the radius of $|z|=1$ through z meet $|z|=1$ at z' and x, x' correspond to z, z' respectively. Then by (30) and (31),

$$\begin{aligned} 1-|z| &= |z-z'| = \left| \int_{x'}^x \frac{dz}{dx} dx \right| = |Ak| \left| \int_{x'}^x (x-1)^{k-1} (1+o(1)) dx \right| \\ &\leq A_1 |x-x'| |x-1|^{k-1} \leq A_2 (1-\rho_n) |z-z_0|^{\frac{1}{k}-1}. \\ |x-x'| &= \left| \int_{z'}^z \frac{dx}{dz} dz \right| = \left| \frac{1}{Ak} \int_{z'}^z (z-z_0)^{\frac{1}{k}-1} (1+o(1)) dz \right| \\ &\leq B_1 |z-z'| |z-z_0|^{\frac{1}{k}-1} \end{aligned}$$

or

$$B_2(1-\rho_n) \leq (1-|z|) |z-z_0|^{\frac{1}{k}-1}, \text{ where } A_1, A_2, B_1, B_2, \text{ are constants.}$$

Hence

$$(35) \quad B_2(1-\rho_n) \leq (1-|z|) |z-z_0|^{\frac{1}{k}-1} \leq A_2(1-\rho_n).$$

Let z_1, z_2 be any two points on C_n , then by (35), we have

$$(36) \quad \frac{(1-|z_1|) |z_1-z_0|^{\frac{1}{k}-1}}{(1-|z_2|) |z_2-z_0|^{\frac{1}{k}-1}} = O(1).$$

Now by (31) and (34),

$$\begin{aligned} \int_{C_n'} |v(x)| |dx| &= \int_{C_n} |u(z)| \left| \frac{dx}{dz} \right| |dz| \leq K \int_{C_n} |u(z)| |z-z_0|^{\frac{1}{k}-1} d\theta \\ &= K \int_0^{2\pi} |u(e^{i\phi})| d\phi \int_{C_n} \frac{(1-|z|^2) |z-z_0|^{\frac{1}{k}-1}}{1-2|z|\cos(\theta-\phi)+|z|^2} d\theta \\ &\leq 2K \int_0^{2\pi} |u(e^{i\phi})| d\phi \int_{C_n} \frac{(1-|z|) |z-z_0|^{\frac{1}{k}-1}}{1-2|z|\cos(\theta-\phi)+|z|^2} d\theta \end{aligned}$$

Now

$$\frac{(1-|z|)|z-z_0|^{\frac{1}{k}-1}}{1-2|z|\cos(\theta-\phi)+|z|^2} \bigg/ \frac{(1-|z'|)|z'-z_0|^{\frac{1}{k}-1}}{1-2|z'|\cos(\theta-\phi)+|z'|^2}$$

$$= \frac{(1-|z|)|z-z_0|^{\frac{1}{k}-1}}{(1-|z'|)|z'-z_0|^{\frac{1}{k}-1}} \cdot \frac{1-2q_n\cos(\theta-\phi)+q_n^2}{1-2|z|\cos(\theta-\phi)+|z|^2}.$$

Since

$$\frac{1-2q_n\cos(\theta-\phi)+q_n^2}{1-2|z|\cos(\theta-\phi)+|z|^2} \leq \frac{1+2q_n+q_n^2}{1+2|z|+|z|^2} = \left(\frac{1+q_n}{1+|z|} \right)^2 \leq 4$$

we have by (36)

$$\frac{(1-|z|)|z-z_0|^{\frac{1}{k}-1}}{1-2|z|\cos(\theta-\phi)+|z|^2} = O\left(\frac{(1-|z'|)|z'-z_0|^{\frac{1}{k}-1}}{1-2|z'|\cos(\theta-\phi)+|z'|^2} \right)$$

$$= o\left(\frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2} \right)$$

so that

$$\int_{C_n'} |v(x)| |dx| = o(1) \int_0^{2\pi} |u(e^{i\phi})| d\phi \int_0^{2\pi} \frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2} d\theta$$

$$= o(1) \int_0^{2\pi} |u(e^{i\phi})| d\phi = o(1).$$

(iv) Point D ($z=z_0$).

Let the vicinity of z_0 be denoted by U which corresponds to U' on the x -plane, and the arc of $|x|=\rho_n$ which lies in U' be denoted by C_n' corresponding to C_n on the z -plane. Let $1-|z|$ attain its minimum value at z' on C_n and put $|z'|=q_n$. We draw a circle K_n ; $|z|=q_n$, then K_n contains C_n inside of it. Let z be any point on C_n and the radius through z meet K_n at ζ , then $|\zeta|=q_n$.

where K is constant and $z=re^{i\theta}$. Let $1-|z|$ be minimum at z' ($|z'|=q_n$) on C_n . We draw a circle K_n ; $|z|=q$, then K_n touches C_n at z' externally and contains C_n in it.

Let z be any point on C_n and the radius of $|z|=1$ through z meet K_n at ζ , then $|\zeta|=q_n$.

Since the mapping functions $z=\phi(x)$, $x=\psi(z)$ are regular at $x=1$ and $z=z_0$ respectively and $\phi'(1)\neq 0$, $\psi'(z_0)\neq 0$, we have

$$1-|z|=(1-|z'|)(1+o(1))$$

or

$$1-|z|=(1-q_n)(1+o(1)).$$

Hence as in the proof of Theorem 1,

$$\frac{1-|z|^2}{1-2|z|\cos(\theta-\phi)+|z|^2}=O(1)\cdot\frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2}\quad (z=re^{i\theta}).$$

Evidently there exists a constant K such that for $z=re^{i\theta}$ on C_n ,

$$|dz|\leq K d\theta.$$

Hence we have

$$\begin{aligned}\int_{C'_n}|v(x)||dx| &= \int_{C_n}|u(z)|\left|\frac{dx}{dz}\right||dz|=O(1)\int_{C_n}|u(z)|d\theta \\ &= O(1)\int_0^{2\pi}|u(e^{i\phi})|d\phi\int_{C_n}\frac{1-|z|^2}{1-2|z|\cos(\theta-\phi)+|z|^2}d\theta \\ &= O(1)\int_0^{2\pi}|u(e^{i\phi})|d\phi\int_{C_n}\frac{1-q_n^2}{1-2q_n\cos(\theta-\phi)+q_n^2}d\theta=o(1).\end{aligned}$$

From the above investigation we see

$$\int|v(x)||dx|\rightarrow 0\quad\text{as}\quad\int|dx|\rightarrow 0$$

the path of the integration being the arc of $|x|=\rho_n$ which lies in the vicinity of $x=1$.

Hence our hypothesis (25) (26) is contradictory, and the theorem is completely proved.

5. We will apply the above theorem to prove the following

Theorem 4. Let $u(z)$ be a harmonic function which is regular in $|z| < 1$ and is expressed by a Poisson integral,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1-r^2}{1-2r \cos(\theta-\phi)+r^2} d\phi \quad (z=re^{i\theta})$$

where $u(e^{i\phi})$ is an integrable function in Lebesgue's sense.

(A) Let $C(\theta)$ be a circular arc through $z=-1$, $z=1$, making an angle θ ($0 \leq \theta \leq \pi$) with the upper semi-circle of $|z|=1$. We put

$$\overline{\lim} u(z) = \overline{L}(\theta), \quad \underline{\lim} u(z) = \underline{L}(\theta), \quad \text{when } z \rightarrow 1 \text{ along } C(\theta).$$

Then $\overline{L}(\theta)$ is a convex and $\underline{L}(\theta)$ is a concave function of θ ($0 \leq \theta \leq \pi$), i. e.

$$(37) \quad \overline{L}(\theta_2) \leq \frac{(\theta_3 - \theta_2)\overline{L}(\theta_1) + (\theta_2 - \theta_1)\overline{L}(\theta_3)}{\theta_3 - \theta_1} \left\{ \begin{array}{l} \\ \end{array} \right. \quad (0 \leq \theta_1 < \theta_2 < \theta_3 \leq \pi).$$

$$(38) \quad \underline{L}(\theta_2) \geq \frac{(\theta_3 - \theta_2)\underline{L}(\theta_1) + (\theta_2 - \theta_1)\underline{L}(\theta_3)}{\theta_3 - \theta_1} \left\{ \begin{array}{l} \\ \end{array} \right.$$

(B) Let $K(\theta)$ be any curve through $z=1$ and lies between two circular arcs which touch $C(\theta)$ at $z=1$, then

$$\overline{\lim} u(z) = \overline{L}(\theta), \quad \underline{\lim} u(z) = \underline{L}(\theta), \quad (0 < \theta < \pi)$$

when $z \rightarrow 1$ along $K(\theta)$.

(C) Let $\{z_n\}$ be a sequence of points such that $z_n \rightarrow 1$ and z_n ($n=1, 2, \dots$) lie between two circular arcs which touch $C(\theta)$ at $z=1$. Then

$$\overline{\lim}_{n \rightarrow \infty} u(z_n) \leq \overline{L}(\theta), \quad \underline{\lim}_{n \rightarrow \infty} u(z_n) \geq \underline{L}(\theta) \quad (0 \leq \theta \leq \pi).$$

(D) Let $k(\theta)$ be a simple Jordan curve which touches $C(\theta)$ at $z=1$, and put

$$\overline{\lim} u(z) = \overline{l}(\theta), \quad \underline{\lim} u(z) = \underline{l}(\theta), \quad (0 < \theta < \pi)$$

when $z \rightarrow 1$ along $k(\theta)$.

Then

$$(39) \quad \overline{l}(\theta_2) \leq \frac{(\theta_3 - \theta_2)\overline{l}(\theta_1) + (\theta_2 - \theta_1)\overline{l}(\theta_3)}{\theta_3 - \theta_1} \left\{ \begin{array}{l} \\ \end{array} \right. \quad (0 < \theta_1 < \theta_2 < \theta_3 < \pi).$$

$$(40) \quad \underline{l}(\theta_2) \geq \frac{(\theta_3 - \theta_2)\underline{l}(\theta_1) + (\theta_2 - \theta_1)\underline{l}(\theta_3)}{\theta_3 - \theta_1} \left\{ \begin{array}{l} \\ \end{array} \right.$$

Further if $u(z)$ is bounded in $|z| < 1$, then (B), (C), (D) can be replaced by a more general (B'), (C'), (D'):

(B') Let $K'(\theta)$ be a simple Jordan curve which touches $C(\theta)$ at $z=1$.

We put

$\overline{\lim} u(z) = \overline{L}'(\theta)$, $\underline{\lim} u(z) = \underline{L}'(\theta)$, when $z \rightarrow 1$ along $K'(\theta)$. Then

$$\overline{L}'(\theta) = \overline{L}(\theta), \quad \underline{L}'(\theta) = \underline{L}(\theta) \quad (0 < \theta < \pi).$$

$$(D') \quad \left. \begin{aligned} \overline{L}'(\theta_1) &\leq \frac{(\theta_3 - \theta_2)\overline{L}'(\theta_1) + (\theta_2 - \theta_1)\overline{L}'(\theta_3)}{\theta_3 - \theta_1} \\ \underline{L}'(\theta_2) &\geq \frac{(\theta_3 - \theta_2)\underline{L}'(\theta_1) + (\theta_2 - \theta_1)\underline{L}'(\theta_3)}{\theta_3 - \theta_1} \end{aligned} \right\} \quad (0 \leq \theta_1 < \theta_2 < \theta_3 \leq \pi).$$

(C') Let $\{z_n\}$ be a sequence of points such that $z_n \rightarrow 1$ and the line which connects z_n and $z=1$ converges to the tangent of $C(\theta)$ at $z=1$, then

$$\overline{\lim}_{n \rightarrow \infty} u(z_n) \leq \overline{L}(\theta), \quad \underline{\lim}_{n \rightarrow \infty} u(z_n) \geq \underline{L}(\theta) \quad (0 \leq \theta \leq \pi).$$

Proof.

We will prove the part of the theorem which concerns $\overline{\lim} u(z)$. Another part can be treated similarly.

(A) First we will prove (37) for $\theta_1=0$, $\theta_3=\pi$.

Since if one of $\overline{L}(0)$, $\overline{L}(\pi)$ is $+\infty$, (37) is evident, we will assume that $\overline{L}(0) < +\infty$, $\overline{L}(\pi) < +\infty$. We take $z_1 = e^{i\theta_1}$, $z_2 = e^{-i\theta_1}$ ($0 < \theta_1 < \pi$) so near to $z=1$ that for $0 < \theta \leq \theta_1$,

$$u(e^{i\theta}) \leq \overline{L}(0) + \varepsilon \quad \text{and} \quad u(e^{-i\theta}) \leq \overline{L}(\pi) + \varepsilon.$$

Let a be a point on $C(\theta)$ ($0 < \theta < \pi$), then we transform $u(z)$ into $v(x)$ by the transformation,

$$(41) \quad x = \frac{z-a}{1-\bar{a}z}$$

then by Theorem 1, $v(x)$ can be expressed by a Poisson integral;

$$v(x) = \int_0^{2\pi} v(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\phi \quad (x=re^{i\theta}).$$

Let by (41), $z_1 = e^{i\theta_1}$, $z_0=1$, $z_2 = e^{-i\theta_1}$ be transformed into $x_1 = e^{i\phi_1}$, $x_0 = e^{i\phi_0}$, $x_2 = e^{i\phi_2}$, then by the lemma,

$$(42) \quad \phi_1 \rightarrow \pi, \quad \phi_0 \rightarrow -(\pi-2\theta), \quad \phi_2 \rightarrow -\pi.$$

Now

$$\begin{aligned}
 (43) \quad u(a) &= v(0) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\phi}) d\phi \\
 &= \frac{1}{2\pi} \left[\int_{\phi_0}^{\phi_1} v(e^{i\phi}) d\phi + \int_{\phi_1}^{\phi_2} v(e^{i\phi}) d\phi + \int_{\phi_2}^{\phi_0} v(e^{i\phi}) d\phi \right] \\
 &\leq \frac{1}{2\pi} \left[(\phi_1 - \phi_0)(\bar{L}(0) + \varepsilon) + \int_{\phi_1}^{\phi_2} v(e^{i\phi}) d\phi + (\phi_0 - \phi_2)(\bar{L}(\pi) + \varepsilon) \right].
 \end{aligned}$$

Since on the arc: $z = e^{i\theta}$ ($\theta_1 \leq \theta \leq 2\pi - \theta_1$), $|z - a| \geq \delta (> 0)$, when a is very near to $z=1$, we have

$$\begin{aligned}
 (44) \quad \left| \int_{\phi_1}^{\phi_2} v(e^{i\phi}) d\phi \right| &= \left| \int_{x_1}^{x_2} v(x) \frac{dx}{x} \right| \leq \int_{x_1}^{x_2} |u(z)| \left| \frac{1 - a^2}{z - a} \right|^2 |dz| \\
 &\leq \frac{1 - |a|^2}{\delta^2} \int_{x_1}^{x_2} |u(z)| |dz| \leq \frac{1 - |a|^2}{\delta^2} \int_0^{2\pi} |u(e^{i\phi})| d\phi = o(1)
 \end{aligned}$$

as $a \rightarrow 1$.

Hence

$$(45) \quad \bar{L}(\theta) = \overline{lim} \ u(a) \leq \frac{(\pi - \theta)\bar{L}(0) + \theta\bar{L}(\pi)}{\pi}.$$

Hence (37) is proved for $\theta_1=0$, $\theta_3=\pi$.

Next $0 \leq \theta_1 < \theta_3 \leq \pi$. We map the domain bounded by $C(\theta_1)$ and $C(\theta_3)$ conformally on the unit circle $|x| < 1$, so that $C(\theta_1)$ and $C(\theta_3)$ go over the upper and lower halves of the unit circle $|x|=1$ and $z=1$ becomes $x=1$.

Then $C(\theta_2)$ becomes a circular arc through $x=-1$, $x=1$, making an angle $\pi \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1}$ with the upper semi-circle of $|x|=1$, and $u(z)$ becomes $v(x)$ which can be expressed by a Poisson integral, so that we can apply the above result to $v(x)$ and get,

$$\begin{aligned}
 \bar{L}(\theta_2) &\leq \frac{\left(\pi - \pi \frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \right) \bar{L}(\theta_1) + \pi \left(\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1} \right) \bar{L}(\theta_3)}{\pi} \\
 &\leq \frac{(\theta_3 - \theta_2) \bar{L}(\theta_1) + (\theta_2 - \theta_1) \bar{L}(\theta_3)}{\theta_3 - \theta_1}, \text{ q. e. d.}
 \end{aligned}$$

(B) Let $K(\theta)$ be a curve through $z=1$ which lies between two circular arcs which touch $C(\theta)$ at $z=1$. From a point t on $K(\theta)$ we erect a perpendicular on $C(\theta)$, its foot being z . Then $|z-t|=O(|t-1|^2)$, so that by (22)

$$(46) \quad |u(z)-u(t)| \leq \int_z^t \frac{|du|}{|d\zeta|} |d\zeta| = o(1) \int_z^t \frac{1}{|\zeta-1|^2} |d\zeta| \\ = o(1) \frac{|z-t|}{|t-1|^2} = o(1).$$

(B) is an immediate consequence of (46).

(C) We prove the case $\theta=0$.

When $\bar{L}(0)=+\infty$, (C) is evident, so we assume $\bar{L}(0) < +\infty$.

Let C be a circular arc which touches $C(0)$ at $z=1$ and z_n ($n=1, 2, \dots$) lie between C and $C(0)$. By the transformation,

$$(47) \quad x = \frac{z-z_n}{1-\bar{z}_n z}$$

$u(z)$ becomes $v(x)$ which can be expressed by a Poisson integral.

We take $\zeta_1=e^{i\theta_1}$ ($0 < \theta_1 < \pi$) so near to $z=1$ that for $z=e^{i\theta}$ ($0 < \theta \leq \theta_1$),

$$u(e^{i\theta}) \leq \bar{L}(0) + \varepsilon.$$

By (47), $\zeta_1, \zeta_0=1$ correspond to $x_1=e^{i\phi_1}, x_0=e^{i\phi_0}$ so that by the lemma

$$\phi_1 \rightarrow \pi, \quad \phi_0 \rightarrow -\pi.$$

$$(48) \quad u(z_n)=v(0)=\frac{1}{2\pi} \int_0^{2\pi} v(e^{i\phi}) d\phi \\ = \frac{1}{2\pi} \int_{\phi_0}^{\phi_1} v(e^{i\phi}) d\phi + \frac{1}{2\pi} \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi \\ \leq \frac{1}{2\pi} (\bar{L}(0) + \varepsilon) (\phi_1 - \phi_0) + \frac{1}{2\pi} \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi.$$

Now

$$(49) \quad \left| \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi \right| = \left| \int_{x_1}^{x_0} v(x) \frac{dx}{x} \right| \leq \int_{\zeta_1}^1 |u(z)| \frac{1-|z_n|^2}{|z-z_n|^2} |dz|.$$

We take $\zeta_2=e^{-i\theta_2}$ ($0 < \theta_2 < \pi$) so near to $z=1$ that for

$$z=e^{-i\theta} \quad (0 < \theta \leq \theta_2),$$

$$\frac{1-|z_n|^2}{|z-z_n|^2} \leq K \quad (n=1, 2, \dots)$$

this being possible by the hypothesis that z_n lie between C and $C(0)$.

$$\text{Since for } z=e^{i\theta} \quad (\theta_1 \leq \theta \leq 2\pi-\theta_2),$$

$$|z-z_n| \geq k > 0 \quad (n=1, 2, \dots)$$

we have from (49),

$$\left| \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi \right| \leq \frac{1-|z_n|^2}{k^2} \int_{\xi_1}^{\xi_2} |u(z)| |dz| + K \int_{\xi_2}^1 |u(z)| |dz|.$$

We take ξ_2 so near to $z=1$, that $K \int_{\xi_2}^1 |u(z)| |dz| < \varepsilon$, and then n_0

so large that for $n \geq n_0$,

$$\frac{1-|z_n|^2}{k^2} \int_{\xi_1}^{\xi_2} |u(z)| |dz| < \varepsilon$$

so that

$$\left| \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi \right| < 2\varepsilon.$$

Hence from (48), we get

$$\overline{\lim}_{n \rightarrow \infty} u(z_n) \leq \bar{L}(0), \quad \text{q. e. d.}$$

The general case can be reduced to this case $\theta=0$ by a conformal transformation.

We will prove (D) after (B'), (C'), (D').

(B') The proof is similar with (B). We use the relation:

$$\left| \frac{du}{dz} \right| = O\left(\frac{1}{1-|z|} \right) \text{ instead of } \left| \frac{du}{dz} \right| = o\left(\frac{1}{(1-|z|)^2} \right).$$

(D') can be proved easily, so we do not enter into the detail of the proof.

(C') We prove the case $\theta=0$, when $\bar{L}(0) < \infty$.

As in (C), we reach to the relation;

$$u(z_n) \leq \frac{1}{2\pi} (\bar{L}(0) + \varepsilon) (\phi_1 - \phi_0) + \frac{1}{2\pi} \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi.$$

$$\text{Since } |v(e^{i\phi})| \leq M, \quad \frac{1}{2\pi} \int_{\phi_1}^{\phi_0} v(e^{i\phi}) d\phi \rightarrow 0.$$

$$\lim_{n \rightarrow \infty} \overline{u}(z_n) \leq \overline{L}(0).$$

The general case can be reduced to the case $\theta=0$ by a conformal transformation.

(D) An arc of the circle $|z-1| = \varepsilon$ and $k(\theta_1)$, $k(\theta_3)$ form a boundary of a simply connected domain G . If one of $\overline{l}(\theta_1)$, $\overline{l}(\theta_3)$ is $+\infty$, then (D) is evident, so we assume that $\overline{l}(\theta_1) < +\infty$, $\overline{l}(\theta_3) < +\infty$.

Let $v(z)$ be the conjugate harmonic function of $u(z)$, then $f(z) = e^{u+iv}$ is an analytic function of z in $|z| < 1$. Since $|f(z)| = e^u < e^{\frac{K}{1-|z|}}$ by (21) and $f(z)$ is bounded on the boundary of G , $f(z)$ is bounded in G by Lindelöf-Phragmén's theorem. Hence $u(z) < K$ in G . If $\overline{l}(\theta_1) = -\infty$, then $f(z)$ tends to zero along $k(\theta_1)$, so that by Montel's theorem, $f(z)$ tends to zero along $k(\theta_2)$, or $\overline{l}(\theta_2) = -\infty$. Hence in this case (D) is proved. If $-\infty < \overline{l}(\theta_1) < +\infty$, $-\infty < \overline{l}(\theta_3) < +\infty$, we consider $g(z) = e^{-u-iv}$ and as $f(z)$ we see that $g(z)$ is bounded in G , so that $u(z) > -K$ in G . Hence $|u(z)| < K$ in G .

We map G conformally in the unit circle $|x| < 1$ and apply (D'), then we see easily that (D) is true.

Hence our theorem is completely proved.

6. By a conformal transformation we get from Theorem 4 the following

Theorem 5. Let $u(z)$ be a harmonic function which is regular and bounded in the domain G :

$$G: \alpha \leq \arg z \leq \beta, \quad 0 < |z| \leq r.$$

Put

$$\overline{L}(\theta) = \overline{\lim}_{r \rightarrow 0} u(re^{i\theta}), \quad \underline{L}(\theta) = \lim_{r \rightarrow 0} u(re^{i\theta}) \quad (\alpha \leq \theta \leq \beta).$$

Then $\overline{L}(\theta)$ is a convex and $\underline{L}(\theta)$ be a concave function of θ .

Theorem 6. Let $u(s)$ ($s = \sigma + it$) be a harmonic function which is regular and bounded in a strip S :

$$S: \alpha \leq \sigma \leq \beta, \quad 0 \leq t < \infty.$$

Put

$$\overline{L}(\sigma) = \overline{\lim}_{t \rightarrow \infty} u(\sigma + it), \quad \underline{L}(\sigma) = \lim_{t \rightarrow \infty} u(\sigma + it) \quad (\alpha \leq \sigma \leq \beta)$$

Then $\overline{L}(\sigma)$ is a convex and $\underline{L}(\sigma)$ is a concave function of σ .

Theorem 6 contains as a special case the following theorem of Hardy⁽⁴⁾:

Let $u(s)$ be a harmonic function which is regular and bounded in a strip S : $\alpha \leq \sigma \leq \beta$, $0 \leq t < \infty$, and $\lim_{t \rightarrow \infty} u(\alpha + it) = A$, $\lim_{t \rightarrow \infty} u(\beta + it) = B$ exist, then $\lim_{t \rightarrow \infty} u(\sigma + it) = C$ exists for all σ ($\alpha \leq \sigma \leq \beta$) and

$$(50) \quad C = \frac{(\beta - \sigma)A + (\sigma - \alpha)B}{\beta - \alpha}.$$

7. We deduce from (C) of Theorem 4 the following corollary.

Theorem 7. *Let*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\phi$$

and $\lim_{\theta \rightarrow +0} u(e^{i\theta}) = A$, where A may be finite or infinite, then

$$\lim u(z) = A$$

when z tends to $z=1$ lying between the upper semi-circle of $|z|=1$ and a circle C which touches the upper semi-circle.

This generalizes a theorem of Fatou⁽⁵⁾ who proved this theorem when $A = +\infty$ and C is a circle whose diameter is the segment through $z=0$ and $z=1$.

3. Limit Ovals of an Analytic Function.

8. Let $w=f(z)$ be an analytic function regular on the curve C , $z=z(t)$ ($0 < t \leq 1$) except at one end $z=z(0)$ and consider the smallest convex set $\mathfrak{R}_\delta$, which contains the set $w=f(z)$ ($0 < z \leq \delta$) and its limit points on the w -plane.

We will call $\lim_{\delta \rightarrow 0} \mathfrak{R}_\delta = \mathfrak{R}$ the limit oval of $f(z)$ at $z=z(0)$ along C .

Evidently the diameter of $\mathfrak{R}_\delta$ is the oscillation of $f(z)$ along the curve $z=z(t)$ ($0 < t \leq \delta$) and hence the diameter of $\mathfrak{R}$ is the oscillation of $f(z)$ at $z=z(0)$ along C .

Let $\mathfrak{R}$, $\mathfrak{R}_1$, $\mathfrak{R}_2$ be the point sets on the z -plane. For brevity we use the following notations:

(⁴) C. Hardy. A theorem concerning harmonic functions. The Journ. London Math. Soc. **1**. (1926) 130—131.

(⁵) P. Fatou. Séries trigonométriques et série de Taylor. Acta Math **30** (1906) 335—400.

(i) $\Re_1 \leq \Re_2$ means that $\Re_1$ is contained in $\Re_2$.

(ii) $m\Re$ (m being a complex number) means the point set which is generated by mz when z takes all possible positions in $\Re$.

(iii) $\Re_1 + \Re_2$ means the point set which is generated by $z_1 + z_2$ when $z_1 + z_2$ take all possible positions in $\Re_1$ and $\Re_2$ respectively.

Hence from $\Re_1 + \Re_2 \leq \Re_3$, it follows $\Re_1 \leq \Re_3 - \Re_2$.

Next let $\Re(\theta)$ ($\alpha \leq \theta \leq \beta$) be a one-parameter family of ovals such that

$$(51) \quad \Re(\theta_2) \leq \frac{(\theta_3 - \theta_2)\Re(\theta_1) + (\theta_2 - \theta_1)\Re(\theta_3)}{\theta_3 - \theta_1}.$$

Then we will call $\Re(\theta)$ a *convex family of ovals*.

If $\Re(\theta)$ is bounded, i. e. all $\Re(\theta)$ are contained in a certain circle, then as in the case of a convex function we deduce (51) from the relation :

$$(52) \quad \Re\left(\frac{\theta_1 + \theta_2}{2}\right) \leq \frac{\Re(\theta_1) + \Re(\theta_2)}{2}.$$

The proof is entirely similar with that of a convex function, only the symbol $\leq$ is to be understood here to mean "is contained in".

Theorem 8. If $\Re(\theta)$ ($\alpha \leq \theta \leq \beta$) is a convex family of ovals, then the diameter $d(\theta)$ of $\Re(\theta)$ is a convex function of θ .

Proof.

Let $\Re(\theta_1)$, $\Re(\theta_2)$, $\Re(\theta_3)$ ($\alpha \leq \theta_1 < \theta_2 < \theta_3 \leq \beta$) be three ovals of the family. If one of $d(\theta_1)$, $d(\theta_3)$ is infinite, then the theorem is evident, so we assume that $d(\theta_1) < +\infty$, $d(\theta_3) < +\infty$.

Let z_2, z_2' be any two points in $\Re(\theta_2)$, then by (51) we can find $z_1, z_1'; z_3, z_3'$ in $\Re(\theta_1)$ and $\Re(\theta_3)$ respectively, such that

$$z_2 = \frac{(\theta_3 - \theta_2)z_1 + (\theta_2 - \theta_1)z_3}{\theta_3 - \theta_1} \quad z_2' = \frac{(\theta_3 - \theta_2)z_1' + (\theta_2 - \theta_1)z_3'}{\theta_3 - \theta_1}.$$

$$|z_2 - z_2'| \leq \frac{(\theta_3 - \theta_2)|z_1 - z_1'| + (\theta_2 - \theta_1)|z_3 - z_3'|}{\theta_3 - \theta_1} \leq \frac{(\theta_3 - \theta_2)d(\theta_1) + (\theta_2 - \theta_1)d(\theta_3)}{\theta_3 - \theta_1}.$$

Hence

$$(53) \quad d(\theta_2) \leq \frac{(\theta_3 - \theta_2)d(\theta_1) + (\theta_2 - \theta_1)d(\theta_3)}{\theta_3 - \theta_1}, \quad \text{q. e. d.}$$

9. Let $f(z)$ be regular in $|z| < 1$ and for $0 \leq r < 1$

$$(54) \quad \int_0^{2\pi} |f(re^{i\theta})| d\theta < M$$

where M is a constant.

Then Gr. Fichtenholz⁽⁶⁾ proved that $f(z)$ can be expressed by a Poisson integral,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\phi \quad (z=re^{i\theta})$$

where $f(e^{i\phi}) = \lim_{r \rightarrow 1} f(re^{i\phi})$, which exist almost everywhere.

Hence we can apply the theorem which we have proved for Poisson integrals. We have the following

Theorem 9. Let $f(z)$ be regular in $|z| < 1$ and for $0 \leq r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta < M.$$

(A) Let $C(\theta)$ be a circular arc through $z=-1$, $z=1$, making an angle θ with the upper semi-circle of $|z|=1$, and $\Re(\theta)$ be the limit oval of $f(z)$ at $z=1$ along $C(\theta)$, then $\Re(\theta)$ ($0 \leq \theta \leq \pi$) is a convex family of ovals, i. e.

$$(55) \quad \Re(\theta_2) \leq \frac{(\theta_3 - \theta_2)\Re(\theta_1) + (\theta_2 - \theta_1)\Re(\theta_3)}{\theta_3 - \theta_1} \quad (0 \leq \theta_1 < \theta_2 < \theta_3 \leq \pi).$$

(B) Let $K(\theta)$ ($0 < \theta < \pi$) be any curve through $z=1$ and lies between two circular arcs which touch $C(\theta)$ at $z=1$, then the limit oval of $f(z)$ at $z=1$ along $K(\theta)$ is $\Re(\theta)$.

(C) Let $\{z_n\}$ be a sequence of points such that $z_n \rightarrow 1$ and z_n lie between two circular arcs which touch $C(\theta)$ ($0 \leq \theta \leq \pi$) at $z=1$, then any limit point of $w_n = f(z_n)$ ($n=1, 2, \dots$) lies in $\Re(\theta)$.

(D) Let $k(\theta)$ ($0 < \theta < \pi$) be a simple Jordan curve which touches $C(\theta)$ at $z=1$ and $\mathfrak{F}(\theta)$ be the limit oval at $z=1$ along $k(\theta)$, then

$$\mathfrak{F}(\theta_2) \leq \frac{(\theta_3 - \theta_2)\mathfrak{F}(\theta_1) + (\theta_2 - \theta_1)\mathfrak{F}(\theta_3)}{\theta_3 - \theta_1} \quad (0 < \theta_1 < \theta_2 < \theta_3 < \pi).$$

Further if $f(z)$ is bounded in $|z| < 1$, then (B), (C), (D) can be replaced by a more general (B'), (C'), (D'):

(B') Let $K'(\theta)$ be a simple Jordan curve which touches $C(\theta)$ at

⁽⁶⁾ Gr. Fichtenholz, l. c. (1)

$z=1$ and $\Re'(\theta)$ be the limit oval at $z=1$ along $K'(\theta)$, then

$$\Re'(\theta) = \Re(\theta) \quad (0 < \theta < \pi).$$

$$(D') \quad \Re'(\theta_2) \leq \frac{(\theta_3 - \theta_2)\Re'(\theta_1) + (\theta_2 - \theta_1)\Re'(\theta_3)}{\theta_3 - \theta_1} \quad (0 \leq \theta_1 < \theta_2 < \theta_3 \leq \pi).$$

(C') Let $\{z_n\}$ be a sequence of points such that $z_n \rightarrow 1$ and the line which connects z_n and $z=1$ converges to the tangent of $C(\theta)$ at $z=1$, then any limit point of $w_n = f(z_n)$ ($n=1, 2, \dots$) lies in $\Re(\theta)$ ($0 \leq \theta \leq \pi$).

Proof.

We will prove only (A), the other parts of the theorem can be proved similarly as the corresponding parts of Theorem 4.

(A) If $\int_0^{2\pi} |f(re^{i\theta})| d\theta < M$ ($0 \leq r < 1$), then $f(z)$ can be expressed

by a Poisson integral,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} d\phi \quad (z=re^{i\theta}).$$

First we will prove (55) for $\theta_1=0, \theta_3=\pi$. We take θ_1 ($0 < \theta_1 < \pi$) so small that for $0 < \theta \leq \theta_1$, the points $w=f(e^{i\theta})$ and $w=f(e^{-i\theta})$ are contained in $\Re(0)+\mathfrak{f}$ and $\Re(\pi)+\mathfrak{f}'$ respectively, where $\mathfrak{f}$ and $\mathfrak{f}'$ are point sets which vanish in the limit as $\theta_1 \rightarrow 0$.

Let a be a point on $C(\theta)$. Then by the transformation,

$$x = \frac{z-a}{1-\bar{a}z}$$

$z_1=e^{i\theta_1}, z_0=1, z_2=e^{-i\theta_1}$ are transformed into $x_1=e^{i\phi_1}, x_0=e^{i\phi_0}, x_2=e^{i\phi_2}$, so that

$$(56) \quad \phi_1 \rightarrow \pi, \quad \phi_0 \rightarrow -(\pi-2\theta), \quad \phi_2 \rightarrow -\pi$$

and $f(z)$ becomes $F(x)$ which can be expressed by a Poisson integral.

Hence

$$\begin{aligned} (57) \quad f(a) &= F(0) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\phi}) d\phi \\ &= \frac{1}{2\pi} \int_{\phi_0}^{\phi_1} F(e^{i\phi}) d\phi + \frac{1}{2\pi} \int_{\phi_1}^{\phi_2} F(e^{i\phi}) d\phi + \frac{1}{2\pi} \int_{\phi_2}^{\phi_0} F(e^{i\phi}) d\phi \end{aligned}$$

Since

$$(58) \quad \frac{\int_{\phi_0}^{\phi_1} F(e^{i\phi}) d\phi}{\phi_1 - \phi_0}$$

is a point of the smallest convex set which contains the points $w = F(e^{i\phi})$, $(\phi_0 \leq \phi \leq \phi_1)$ and the limit points, (58) is a point of $\Re(0) + \mathfrak{f}$.

Similarly

$$(59) \quad \frac{\int_{\phi_2}^{\phi_0} F(e^{i\phi}) d\phi}{\phi_0 - \phi_2}$$

is a point of $\Re(\pi) + \mathfrak{f}'$.

And

$$\int_{\phi_1}^{\phi_2} F(e^i) d\phi = \int_{z_1}^{z_2} f(z) \frac{1 - |a|^2}{|z - a|^2} |dz| \rightarrow 0 \quad \text{as } a \rightarrow 1 \text{ along } C(\theta).$$

From (57) we get

$$(60) \quad \Re(\theta) \leq \frac{(\pi - \theta)\Re(0) + \theta\Re(\pi)}{\pi}.$$

Hence (A) is proved for $\theta_1 = 0$, $\theta_3 = \pi$. The general case can be reduced to this special case by a conformal transformation.

10. By a conformal transformation we get from Theorem 9 the following theorems.

Theorem 10. Let $f(z)$ be regular and bounded in a domain G :

$$G: \alpha \leq \arg z \leq \beta, \quad 0 < |z| \leq r_0$$

and $\Re(\theta)$ be the limit oval at $z=0$ along the line $\arg z = \theta$ ($\alpha \leq \theta \leq \beta$), then $\Re(\theta)$ is a convex family of ovals.

Theorem 11. Let $f(s)$ ($s = \sigma + it$) be regular and bounded in a strip S :

$$S: \alpha \leq \sigma \leq \beta, \quad 0 \leq t < \infty$$

and $\Re(\sigma)$ be the limit oval of $f(s)$ along the line $\Re(s) = \sigma$, then $\Re(\sigma)$ is a convex family of ovals.

H. Bohr⁽⁷⁾ proved that the points $w = \log \zeta(\sigma + it)$ ($\sigma > 1$) along

(7) H. Bohr und B. Jessen, Ueber die Werteverteilung der Riemannschen Zetafunktion. Acta Math. 54 (1930), 1—35.

the line $\Re(s)=\sigma$ fill up a convex domain $\mathfrak{R}(\sigma)$ or a domain bounded by two convex curves.

In the latter case we denote $\mathfrak{R}(\sigma)$ the domain bounded by the outer convex curve, then in both cases $\mathfrak{R}(\sigma)$ is the limit oval of $\log \zeta(\sigma+it)$ along the line $\Re(s)=\sigma$. Since $\log \zeta(s)$ is bounded for $\Re(s) \geq \delta_0 (>1)$, Theorem 11 is applicable and we see that $\mathfrak{R}(\sigma)$ is a convex family of ovals.

11. We will prove the following theorem⁽⁸⁾.

Theorem 12. Let $f(z)$ be regular and bounded in $|z| < 1$. We put

$$M(\theta) = \overline{\lim}_{z \rightarrow 1} |f(z)| \quad \text{along } C(\theta)^{(9)}.$$

Then $\log M(\theta)$ is a convex function of θ ;

$$(61) \quad M(\theta_2) \leq M(\theta_1)^{\frac{\theta_3 - \theta_2}{\theta_3 - \theta_1}} M(\theta_3)^{\frac{\theta_2 - \theta_1}{\theta_3 - \theta_1}} \quad (0 \leq \theta_1 < \theta_2 < \theta_3 \leq \pi).$$

Proof.

We take θ_1 so small that for $0 < \theta < \theta_1$, $|f(e^{i\theta})| \leq M(0) + \varepsilon$, $|f(e^{-i\theta})| \leq M(\pi) + \varepsilon$. Let $0 < a < 1$. By

$$x = \frac{z-1}{1-az}$$

$z_1 = e^{i\theta_1}$, $z_0 = 1$, $z_2 = e^{-i\theta_1}$ are transformed into $x_1 = e^{i\phi_1}$, $x_0 = 1$, $x_2 = e^{-i\phi_1}$ so that $\phi_1 \rightarrow \pi$ and $f(z)$ becomes $F(x)$. Let $|f(z)| < M$ in $|z| < 1$. Then

$$\begin{aligned} |f(a)|^2 &= |F(0)|^2 = \frac{1}{2\pi} \int_0^{2\pi} F(e^{-i\phi}) \overline{F(e^{-i\phi})} d\phi \leq \frac{1}{2\pi} \int_{-\phi_1}^{\phi_1} |F(e^{i\phi}) F(e^{-i\phi})| d\phi \\ &\quad + \frac{1}{2\pi} \int_{\phi_1}^{2\pi-\phi_1} |F(e^{i\phi}) F(e^{-i\phi})| d\phi \\ &\leq \frac{\phi_1}{\pi} (M(0) + \varepsilon)(M(\pi) + \varepsilon) + \frac{\pi - \phi_1}{\pi} M^2. \end{aligned}$$

By making $a \rightarrow 1$, we get

$$M\left(\frac{\pi}{2}\right) = \overline{\lim}_{a \rightarrow 1} |f(a)| \leq \sqrt{M(0)M(\pi)}.$$

From this, (61) follows easily by a conformal transformation.

(⁸) This is also remarked by Mr. T. Shimizu independently. This proof is due to him, my original proof is a little longer.

(⁹) The meaning of $C(\theta)$ is the same as in Theorem 9.

But Theorem 12 is not new, since the following equivalent theorem was proved by Hardy and Littlewood⁽¹⁰⁾: let $f(s)$ ($s=\sigma+it$) be regular and bounded in a strip S : $\alpha \leq \sigma \leq \beta$, $0 \leq t < \infty$. We put

$$M(\sigma) = \overline{\lim}_{t \rightarrow \infty} |f(\sigma + it)| \text{ along the line } \Re(s) = \sigma,$$

then $\log M(\sigma)$ is a convex function of σ .

12. From Theorem 2, we get immediately the following theorem.

Theorem 13. *Let $f(z)$ be regular in $|z| < 1$ and*

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta < M \quad (0 \leq r < 1).$$

Then

$$f(z) = o\left(\frac{1}{1-|z|}\right), \quad f'(z) = o\left(\frac{1}{(1-|z|^2)^2}\right)$$

as $|z| \rightarrow 1$ along a radius.

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⁽¹⁰⁾ G. H. Hardy and J. E. Littlewood, Abel's theorem and its converse. *Proc. London Math. Soc. Ser. 2* **18** (1920), 235—235.