12. *On the Fundamental Domains and the Groups for Meromorphic Functions. II.*

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VIII

In the previous paper⁽²⁶⁾ I have defined the group for a meromorphic function by

$$f(z') = f(z) \tag{1}$$

and discussed some properties of it. I will here consider some more remarkable properties of it, considered as a function of z .

I have there denoted by $z' = \varphi(z)$ the functions defined by $g(z', z) = 0$, where $f(z') - f(z) = (z' - z)g(z', z)$.⁽²⁷⁾

In general the function $z' = \varphi(z)$ is reducible into a finite or an infinite number of different functions as I have shown before.

Let $z' = \Phi(z)$ be an irreducible function among the functions $z' = \varphi(z)$, that is, only one function which is a factor of the functions, that is the same thing, the Riemann surface of the function $\Phi(z)$ is connected.

Now we shall give a theorem.

Theorem 10: *Let $f(z)$ be a meromorphic (transcendental and not rational) function and if a function $z' = \Phi(z)$ which is uniform in some domain satisfies (1), then $\Phi(z)$ must be a liner function of the form $e^{i\theta\pi}z + b$, where θ is a rational number and b is a constant. If a function $\Phi(z)$ which is finite valued in some domain satisfies (1), then $\Phi(z)$ must be a finite⁽²⁸⁾ algebraic function of z .*

In order to prove this theorem we shall give the following series of theorems: theorems 11, 12, 13, 14, 15, which have some interest by themselves.

It must be remarked that we can not at once deduce that the inverse

⁽²⁶⁾ T. Shimizu: On the fundamental domains and the groups for meromorphic functions, I. This Journal **8** (1931) 175-236. Especially 228-236.

⁽²⁷⁾ T. Shimizu: *ibid* 229-230.

⁽²⁸⁾ We mean by a finite algebraic function an algebraic function without poles and algebraic poles in the finite part of the z -plane.

function $z = \Phi^{-1}(z')$ is uniform (or finite valued) from $f(z') = f(z)$, if the function $z' = \Phi(z)$ is uniform (or finite valued), but we can only conclude that there must exist a function $z = \Phi(z')$ of z' which is uniform (or finite valued).

Theorem 11: *Let $z' = \Phi(z)$ be a function defined by $g_n(z', z) = 0$, where $g_n(z', z)$ is an integral function with respect to z' and z .*

Further if $z' = \Phi(z)$ satisfies $f(z') = f(z)$, where $f(z)$ is a meromorphic (not rational) function.

Then $\Phi(z)$ must be an integral (transcendental or not) function if $\Phi(z)$ is uniform in some domain of the z -plane. And $\Phi(z)$ must be a finite⁽²⁹⁾ algebroidal function (algebraic function included) if $\Phi(z)$ is finite valued in some domain of the z -plane.

By the theorem⁽³⁰⁾ given in the section VII deduced from Gronwall-Hahn's theorem it is evident that if a function $z' = \Phi(z)$ satisfies (1), then $\Phi(z)$ can be defined by $g_n(z', z) = 0$, where $g_n(z', z)$ is an integral function with respect to z' and z , and it must be a factor of the decomposition of $f(z') = f(z)$ into factor-manifolds (varieties)

$$\prod_{i=1}^{\infty} (g_i(z', z))^{\alpha_i} = 0.$$

From the theorem 11 it follows that if a function $\Phi(z)$ satisfies (1), $f(z)$ being a transcendental meromorphic function, then $\Phi(z)$ and its inverse function both must be infinitely many valued functions and they have an infinite number of branches for any point set lying everywhere dense in the planes respectively.

Proof: Since $\Phi(z)$ satisfies (1), $\Phi(z)$ must have some regular element.

Now if the domain of existence does not cover a continuum we can not prolong any element through the boundary of the continuum. By Julia's theorem⁽³¹⁾ however there exists no set of values containing continuum which lies outside the domain of existence for a function defined by $g_n(z', z) = 0$.

⁽²⁹⁾ We mean by a finite algebroidal function a function which is finite valued and has no other singularity than algebraic branch points (without any algebraic poles) in the finite part of the z -plane. See p. 244 ⁽³⁵⁾ and p. 249 ⁽⁴³⁾ of this paper.

⁽³⁰⁾ T. Shimizu: loc. cit. line 7-line 15, p. 230.

⁽³¹⁾ Julia: Sur le domaine d'existence d'une fonction implicite définie par une relation entière $G(x, y) = 0$. Bull. Soc. Math. France, **54** (1926) 29-37.

Hence the domain of existence covers almost all the z -plane except a point set not containing continuum.

Now assume that there exist two domains D_1 and D_2 in which $\Phi(z)$ is finite valued and let n_1 and n_2 be the numbers of their branches.

Since n_1 and n_2 are finite we can choose n_1 elements $\mathfrak{E}_1, \mathfrak{E}_2, \dots, \mathfrak{E}_{n_1}$ and n_2 elements $\mathfrak{E}'_1, \mathfrak{E}'_2, \dots, \mathfrak{E}'_{n_2}$ such that the projections of $\mathfrak{E}_1, \mathfrak{E}_2, \dots, \mathfrak{E}_{n_1}$ coincide with each other and lie within D_1 and those of $\mathfrak{E}'_1, \mathfrak{E}'_2, \dots, \mathfrak{E}'_{n_2}$ coincide and lie within D_2 . For, first there must exist regular elements $\mathfrak{E}_1$ and $\mathfrak{E}'_1$ on the Riemann surface, whose projections lie in D_1 and D_2 respectively, hence we choose next other elements $\mathfrak{E}_2$ and $\mathfrak{E}'_2$ such that they lie within the projections of $\mathfrak{E}_1$ and $\mathfrak{E}'_1$ respectively in D_1 and D_2 . In these ways obtaining $\mathfrak{E}_1, \mathfrak{E}_2, \dots, \mathfrak{E}_{n_1}$, $\mathfrak{E}'_1, \mathfrak{E}'_2, \dots, \mathfrak{E}'_{n_2}$ and considering the common parts of the two sequences of elements respectively we can obtain the required elements $\mathfrak{E}_1, \mathfrak{E}_2, \dots, \mathfrak{E}_{n_1}$ and $\mathfrak{E}'_1, \mathfrak{E}'_2, \dots, \mathfrak{E}'_{n_2}$ considered only in the common parts respectively, since the processes end in a finite number of steps.

Now consider on the z -plane a band B connecting the projections of $\mathfrak{E}_1$ and $\mathfrak{E}'_1$, then we can prolong the element $\mathfrak{E}_1$ up to some element, say $\mathfrak{E}'_1$, among $\mathfrak{E}'_1, \mathfrak{E}'_2, \dots, \mathfrak{E}'_{n_2}$ within the band B .

For, otherwise, let S be the set of points which corresponds to the limiting points of all possible prolongation of $\mathfrak{E}_1$ within B .

Since S must be a continuum connecting two boundaries of the band B , if we prolong each element up to S , the function must tend to ∞ by Julia's theorem.⁽³²⁾ Since the zeros of $\Phi(z)$ are the roots of $g_n(0, z) = 0$ and they have only one limiting point at ∞ , we can consider a small circle C such that $\frac{1}{\Phi(z)}$ is regular on the Riemann surface whose projection lies in a domain bounded by some arcs of C and S , and further it tends to 0 when z tends to S .

Hence the function must be a constant by Gross' theorem⁽³³⁾ which

⁽³²⁾ Julia: loc. cit.

⁽³³⁾ W. Gross: Zur Theorie der Differentialgleichungen mit festen kritischen Punkten. Math. Annalen **78** (1918) 332-342.

We can prove it in another way, perfectly similar to Iversen's method. Iversen: Recherches des fonctions inverse des fonctions meromorphes, Thèse, Helsingfors (1914).

By Julia's theorem $\Phi(z)$ tends to ∞ when z tends to any accessible boundary points of the Riemann surface, and the Häufungsbereich of a point on the Riemann surface whose projection is a finite point on the z -plane consists of a discrete point set having only one limiting point at ∞ as shown in the proof of theorem 19 in the section XII.

Julia has applied for his theorem. Thus such a set S can not exist.

Now let $\mathfrak{B}_1$ be such a prolongation of the element $\mathfrak{C}_1$ to the element $\mathfrak{C}_1'$.

Here $\mathfrak{B}_1$ may overlap itself, that is, may have its projection intersecting itself in a finite number of times at a finite number of parts.

Next considering the projection B_1 of $\mathfrak{B}_1$ on the z -plane which is some sort of band, we prolong the element $\mathfrak{C}_2$ up to $\mathfrak{C}_2'$, one of $\mathfrak{C}_i'$ ($i = 2, 3, \dots, n_2$), within B_1 along the exactly same path with the prolongation $\mathfrak{B}_1$, that is, we prolong after the exactly same manner with $\mathfrak{B}_1$ and hence go round along circuits, if $\mathfrak{B}_1$ has any, in the same sense and in the same number of times with $\mathfrak{B}_1$ (See the figure below).

The new prolongation between $\mathfrak{C}_2$ and $\mathfrak{C}_2'$ is always possible as before, and in this time there may appear new loops of paths of prolongation.

Let this new prolongation be denoted by $\mathfrak{B}_2$ and its projection by B_2 .

Now $\mathfrak{C}_2'$ can not be equal to $\mathfrak{C}_1'$. For otherwise we would arrive at $\mathfrak{C}_1$, if we prolong inversely from $\mathfrak{C}_1'$ along $\mathfrak{B}_2$ since $\mathfrak{B}_2$ lies wholly within $\mathfrak{B}_1$.

Since $\mathfrak{C}_2$ differs from $\mathfrak{C}_1$, this is a contradiction.

Now we proceed to prolong $\mathfrak{C}_3$ in the same way as before within $\mathfrak{B}_2$, and we arrive at an element $\mathfrak{C}_3'$ different from $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$.

In these ways considering the prolongation lying within the band $\mathfrak{B}_i$ before obtained we can make the correspondence between the elements $\mathfrak{C}_1$ and $\mathfrak{C}_2'$, $\mathfrak{C}_2$ and $\mathfrak{C}_2'$, $\dots$, and $\mathfrak{C}_i$ and $\mathfrak{C}_i'$.

Thus if $n_1 \neq n_2$, n_1 and n_2 being finite, there would exist some prolongation which does not correspond to any element in D_1 or D_2 .

Thus there would exist a boundary for some prolongation $\mathfrak{B}_j$ through which we can not prolong some element $\mathfrak{C}_j$.

Hence we can take a point outside the Wertbereich of the point on the surface and apply the same proof as Iversen's for the theorem that we can prolong an arbitrary element of the inverse function of a meromorphic function within a circle K , containing the element, up to the origin of K . (his Thèse. p. 18).

From this we can at once prove that there exists no boundary set S in the above argument for our prolongation.

Since as we have shown before such a natural boundary can not exist, we must have $n_1 = n_2$.

The relation $n_1 = n_2$ must hold good for every domain of z in which $\Phi(z)$ has some element if n_1 be finite. If there exists some domain in which $\Phi(z)$ has an infinite number of branches, by considering some elements sufficiently large but finite in number among them and applying the same argument as before, we can at once show that such a domain can not exist, if there exists some domain in which $\Phi(z)$ is n_1 valued.

Thus if $\Phi(z)$ be finite valued, n being the number, in some domain of the z -plane, then $\Phi(z)$ must be n valued in the whole plane except the set of points not containing continuum which corresponds to the accessible or non accessible singular points of $\Phi(z)$.

By applying the theorem that a function $\Phi(z)$, defined by $g_n(z', z) = 0$, tends to ∞ ⁽³⁴⁾ if z tends to any accessible boundary of $\Phi(z)$, we shall show that the function $\Phi(z)$ defined by $g_n(z', z) = 0$ must be a finite algebroidal function defined by the equation

$$z'^\nu + A_1(z)z'^{\nu-1} + \dots + A_\nu(z) = 0,$$

where $A_1(z), A_2(z), \dots, A_\nu(z)$ are all integral functions (transcendental or not) if $\Phi(z)$ satisfies (1) and is finite valued, $f(z)$ being a transcendental meromorphic function.

Remark: If $f(z)$ is a rational function, we can not deduce that $\Phi(z)$ must be a finite algebroidal function in the same way as we shall show in the following, but we can directly from $f(z') = f(z)$ deduce that $\Phi(z)$ must be an algebraic function (not necessarily a finite algebraic function) since $f^{-1}f(z)$ is an algebraic function of z .

Now the singular points of $\Phi(z)$ may be isolated or not.

1) Let c be an isolated singular point of $\Phi(z)$, then $\Phi(z)$ being finite valued, c must be a pole or an algebraic pole.

⁽³⁴⁾ We can not deduce at once from this and the finite valuedness of $\Phi(z)$ that the singular points of $\Phi(z)$ must be poles or algebraic poles. For, there may appear a perfect nowhere dense set of singular points and it is not yet known whether there exists or not a function which is continuous in a domain and regular except at a perfect nowhere dense set within it and has a value α as the limit of values on these singular points.

By Pompeiu and Denjoy a function is known which is regular in a domain except at a perfect nowhere dense set in it and is continuous at these points

Pompeiu: Sur la continuité des fonctions de variables complexes, Ann. d. Toulouse, Ser. 2. 7 (1903) 264-315.

Denjoy: Sur les fonctions analytiques uniformes à singularités discontinues non isolées. C. R. Paris 150 (1910) 32-34.

Let c be a pole, then for $|z - c| < \varepsilon_1$ the image of it covers a domain $|\Phi(z)| > \frac{1}{\varepsilon_2}$ on the z' -plane, ε_1 and ε_2 being positive small numbers.

Since $f(z)$ takes every value except two outside any large circle,

$$f(\Phi(z)) = f(z)$$

can not hold good, the left hand side being indeterminate, while the right hand side being determinate in the limit.

Let c be an algebraic pole then the image of a n -ply piled up circle about c covers the m -ply piled up part outside a large circle.

Hence as before $f(\Phi(z)) = f(z)$ can not hold good.

2) Let c be a limiting point of singular points. It c does not belong to a perfect set, $\Phi(z)$ being finite valued, c can not be a singular point of determinate character, that is, $\Phi(z)$ can not tend to ∞ when z tends to c along all accessible paths.

Hence c must belong to a perfect set, if it exists.

Let C be a partial set of a perfect nowhere dense set of singular points of $\Phi(z)$.

Consider a closed curve V containing C within it and having no singular point on it, which can be always chosen since the set of singular points is nowhere dense and $\Phi(z)$ is finite valued.

Then we shall show that the mapped image of the inner part of V contains at least almost the whole vicinity of ∞ without leaving any continuum.

Consider all closed curves W on the Riemann surface whose projections all coincide with V on the z -plane.

W may consist of a finite number of closed curves on the surface lying over V , each going round c a finite number of times.

Since W define domains $\mathfrak{D}$ bounded by them we consider the mapped images $\mathfrak{E}$ of $\mathfrak{D}$.

Since W pass through only regular points and are closed curves or a curve by the finite valuedness of $\Phi(z)$, the images X of W can not run to ∞ , and there may appear only a finite number of separated images of W by the finite valuedness of $\Phi(z)$.

Let these curves be denoted by $X_1, X_2, \dots, X_p$.

Since they are all closed, we can define the part lying outside all of $X_1, X_2, \dots, X_p$.

Now $\Phi(z)$ takes a value larger than any large number, and hence the images $\mathfrak{C}$ contain ∞ within or on the boundary of them.

Now assume that there would exist an infinite number of different continua $R_1, R_2, \dots, R_n, \dots$ lying outside all $X_1, X_2, \dots, X_p$ and clustering to ∞ or there would exist some continuum R_ω extending itself to ∞ such that $R_1, R_2, \dots, R_n, \dots$ and R_ω are the exceptional values of $\Phi(z)$ in $\mathfrak{D}$.

Let E_1 and E_2 be two values lying outside all X_i on the z' -plane, which $\Phi(z)$ takes in $\mathfrak{D}$. Since X_i lie in some finite part and $\Phi(z)$ takes any large values in $\mathfrak{D}$, the existence of such E_1 and E_2 is evident.

Now let e_1 be a point in $\mathfrak{D}$ such that $\Phi(e_1) = E_1$.

Now consider a prolongation along a curve L from E_1 to E_2 on the z' -plane with the initial element $e_1 = \Phi^{-1}(E_1)$ in such a manner that L passes through some of R_i , ($i = 1, 2, \dots, n, \dots$), or R_ω and lies outside all X_i .

Since there exists no continuum outside the existence domain of the function $z = \Phi^{-1}(z')$ defined by $g_n(z', z) = 0$ by Julia's theorem and R_i and R_ω are some continua (not a discontinuous set), we can always choose such a L .

Now there corresponds at least a curve l running from e_1 to some point e_2 on the z -plane such as $\Phi(e_2) = E_2$.

Since L does not cross any of X_i , l can not cross W .

Hence l must lie within $\mathfrak{D}$. Now the prolongation along L consisting of a sequence of regular elements, all the values of L must be taken at the points on l by $\Phi(z)$, which is contradictory, for, we have assumed that R_i and R_ω should be the exceptional values of $\Phi(z)$ in $\mathfrak{D}$.

Thus the images of $\mathfrak{D}$ cover almost all some vicinity of ∞ except a point set not containing continuum.

Now such a function $\Phi(z)$ can not satisfy

$$f(\Phi(z)) = f(z).$$

For, $f(\Phi(z))$ takes a value arbitrarily near to every value in some

vicinity of $z = c$ since the image of an arbitrarily small circle about $z = c$ contains almost every point in some vicinity of ∞ except a set of points not containing any continuum, and $f(Z)$ takes every value in any vicinity of ∞ if Z takes all the points of the vicinity of ∞ .

Thus the left hand side of the above relation being indeterminate about $z = c$, the right hand side tends to $f(c)$ in an arbitrarily small vicinity of c when z tends to c .

Hence we have shown that $\Phi(z)$ can not have any transcendental singular point in any finite part of the z -plane.

Thus if $\Phi(z)$ is uniform in a domain, then $\Phi(z)$ must be uniform in the whole plane and can not have any singular point in the finite part, and hence $\Phi(z)$ must be an integral function.

If $\Phi(z)$ is finite valued in a domain, then $\Phi(z)$ must be finite valued in the whole plane and can not have no other singular point than algebraic branch points (excluding algebraic poles) in the finite part.

Such a function $\Phi(z)$, as well known, can be defined by the equation

$$A_0(z)z^\nu + A_1(z)z^{\nu-1} + \dots + A_\nu(z) = 0,$$

where $A_0(z), A_1(z), \dots, A_\nu(z)$ as symmetric functions of ν branches of $\Phi(z)$ are regular functions in the whole plane, that is, integral functions.

$\Phi(z)$ being finite except at ∞ in our case, $A_0(z)$ must be a constant, and such a function is nothing but a finite algebroidal function.⁽³⁵⁾

Further we can assume that the equation which defines $\Phi(z)$ is irreducible, that is, the left hand side of it can not be decomposed into factors of the same form so that the coefficients, which appear in the factors, may be uniform and regular in the whole plane.

Theorem 12: *Any transcendental integral function $\Phi(z)$ can not satisfy (1).*

Proof. We will give here two different proofs for it.

(I) By Fatou's theorem⁽³⁶⁾ we have that, for some values of r not belonging to certain intervals of values, the circle $|z| \leq r$ has as the mapped image, putting

$$z' = \Phi(z),$$

a domain on the z' -plane containing the circumference $|z'| = r^q$ within

⁽³⁵⁾ For the term an algebroidal function and some properties of such a function see Remoundos: Extensions aux fonctions algebroides multiformes des théorème de M. Picard et de ses généralisations.

⁽³⁶⁾ P. Fatou: Sur l'itération des fonctions transcendentes entières. Acta Mathematica. 47 (1926) 337-370.

it for any large but fixed positive number q .

Or by Valiron's theorem⁽³⁷⁾ the equation

$$z' = \Phi(z)$$

has at least a root in the ring-region

$$M_{\Phi}^{-1}(|z'|) \leq |z| \leq (1 + \varepsilon) M_{\Phi}^{-1}(|z'|),$$

$r = M_{\Phi}^{-1}(R)$ denoting the inverse function of $R = M_{\Phi}(r)$, and $M_{\Phi}(r)$ the maximum modulus of $\Phi(z)$ for $|z| \leq r$, however small ε may be, except possibly for the values of r which can be contained in a sequence of circles C_n , $n = 1, 2, \dots$ of the z' -plane so that, c_n being the affix of the center and ρ_n the radius of C_n , we may have

$$\lim c_n/c_{n+1} = 0; \quad \lim \rho_n/|c_n| = 0.$$

From this the image of the circle $|z| \leq r$ must contain the circumference $|z| = M_{\Phi}((1 - \varepsilon)r)$ except for certain values of $M_{\Phi}(r)$.⁽³⁸⁾

Now suppose that $f(\Phi(z)) = f(z)$ for an integral function $f(z)$, then $M_f(r)$ being truly increasing, when r increases, we would have

$$M_{f\Phi}(r) \geq M_f(r^q) > M_f(r),$$

$M_{f\Phi}(r)$ being the maximum modulus of $f(\Phi(z))$ for $|z| \leq r$, for sufficiently large values of r which is evidently a contradiction.

Now let $f(z)$ be a meromorphic but not integral function, and be written in the quotient of two integral functions

$$\frac{f_1(z)}{f_2(z)},$$

where $f_2(z)$ is a canonical product of primary factors with respect to the poles of $f(z)$, of possibly least genus, defined by Borel, or by Denjoy for a meromorphic function of infinite order.

Then by a theorem which I have given in a paper⁽³⁹⁾ the function

$$\mathfrak{M}_f(r) = \max_{|z| \leq r} \log \sqrt{|f_1(z)|^2 + |f_2(z)|^2}$$

is a steadily increasing function of r , since it is an increasing convex function of $\log r$.

⁽³⁷⁾ G. Valiron: Sur un théorème de M. Fatou. Bull. d. sci. math. France. **46** (1922) 200.

⁽³⁸⁾ I have given a more precise theorem. T. Shimizu: On the function-group for a meromorphic function II. Proc. Phys.-Math. Soc. Japan. Ser. 3. **14** (1932). 38-40.

⁽³⁹⁾ T. Shimizu: Some theorems on meromorphic functions cited at ⁽⁴⁶⁾, or On the theory of meromorphic functions. This Journal. **6** (1929) 162.

Thus if we had $f(\Phi(z)) = f(z)$, for a meromorphic function $f(z)$, then we would have

$$\frac{f_1(\Phi(z))}{f_2(\Phi(z))} = \frac{f_1(z)}{f_2(z)}.$$

Now by the theorem above cited we would have

$$\begin{aligned} \max_{|z| \leq r} \log \sqrt{|f_1(\Phi(z))|^2 + |f_2(\Phi(z))|^2} &\geq \max_{0 \leq \theta < 2\pi} \log \sqrt{|f_1(M_\Phi((1-\varepsilon)r)e^{i\theta})|^2} \\ &\quad + |f_2(M_\Phi((1-\varepsilon)r)e^{i\theta})|^2} > \max_{|z| \leq r} \log \sqrt{|f_1(z)|^2 + |f_2(z)|^2}, \end{aligned}$$

for sufficiently large values of r which is evidently a contradiction. Thus the theorem is proved.

(II) Fatou has shown that⁽⁴⁰⁾ the set of points in the vicinity of which the iterated functions

$\Phi(z), \Phi(\Phi(z)) \equiv \Phi_2(z), \Phi(\Phi_2(z)) \equiv \Phi_3(z), \dots, \Phi(\Phi_{n-1}(z)) \equiv \Phi_n(z), \dots$ of an integral function $\Phi(z)$ do not form a normal family has at least a point in the finite part of the z -plane, and all points z_0 of it are the limiting points of the antecedents of an arbitrary point of the z -plane except possibly a remarkable point $\bar{z}$ which is independent of z_0 .

On the other hand if we have

$$f(\Phi(z)) = f(z),$$

then we would have

$$\begin{aligned} f(z) &= f(\Phi(z)) = f(\Phi_2(z)) = \dots = f(\Phi_n(z)) = \dots \\ &= f(\Phi_{-1}(z)) = f(\Phi_{-2}(z)) = \dots = f(\Phi_{-n}(z)) = \dots \end{aligned}$$

where $\Phi_{-1}(z)$ denotes the inverse function of $\Phi(z)$ and $\Phi_{-n}(z)$ the $n-1$ -th iterated function of $\Phi_{-1}(z)$.

Thus any point of the above set is a limiting point of equivalent points of $f(z)$. Since there exists at least such a point in the finite part of the z -plane $f(z)$ can not be a meromorphic function.

Another proof also using the iteration will be given as follows.

As Fatou⁽⁴¹⁾ has shown in the same paper, in a simple way, that the iteration of a transcendental integral function must have an infinite number of fixed points, it is sufficient to consider the behavior of the iterated functions in the vicinities of them.

⁽⁴⁰⁾ Fatou: loc. cit. Acta math. **47**, 351 §11 Théorème.

⁽⁴¹⁾ Fatou: loc. cit. §9, 347; §10, 348.

Let z_0 be a fixed point of the iteration, then it has been shown by the researches of König, Leau, Fatou and Julia:

1) In the vicinity of z_0 such that $|\Phi'(z_0)| \geq 1$ the antecedents or consequents of a point converge to z_0 .

2) In the vicinity of z_0 such that $\Phi'(z_0) = e^{i\theta\pi}$, θ being a rational number there are at least two parts in one of which the consequents converge to z_0 and in the other part they diverge.

3) In the vicinity of z_0 such that $\Phi'(z_0) = e^{i\theta\pi}$, θ being an irrational number, either the antecedents or consequents of a point converge to z_0 , or there exists at least a closed curve round z_0 in the finite part which is a set of limiting points of them.

In our case there exists a limiting point of the equivalent points of $f(z)$ in the finite part of the z -plane and hence $f(z)$ can not be a meromorphic function.

Theorem 13: *A rational integral function $\Phi(z)$ can not satisfy (1) for a meromorphic (transcendental) function $f(z)$, unless $\Phi(z)$ is a linear function of the form $e^{i\theta\pi}z + b$, θ being a rational number.*

Proof: Let $n \geq 2$ be the degree of $\Phi(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$, then for sufficiently large values of $|z|$ we have

$$|a_n z^n| > |a_{n-1} z^{n-1} + \dots + a_0|$$

$$|\Phi(z)| = |a_n z^n| (1 + o(1)).$$

Hence the mapped image of the circle $|z| \leq r$ for such large r contains the ring-region $(1 - \varepsilon) |a_n| r^n < |\Phi(z)| < (1 + \varepsilon) |a_n| r^n$.

Thus similarly as before by comparing the maximum moduli of both sides we can show that the identity (1) can not hold good.

In stead of the above proof we can give another proof owing to the theory of the iteration of a rational function of degree not less than 2.

By Fatou-Julia's theorem⁽⁴²⁾ on the iteration there exists always a set of points to each of which the consequents or antecedents of a point converge in the finite part of the z -plane.

In the vicinity of such a point the equivalent points of $f(z)$ converge to it which is evidently impossible.

Now let $\Phi(z)$ be of degree 1 and $\Phi(z) = az + b$, $|a| \geq 1$, then by the relations

⁽⁴²⁾ G. Julia: Sur l'itération des fonctions rationnelles. Jour. de. math. pur. et appl. Ser. 8. I (1918) 47-245.

P. Fatou: Bull. Soc. Math. France 47-48 (1919-1920).

$$\begin{aligned} f(z) &= f(\Phi(z)) = f(\Phi_2(z)) = \dots \\ &= f(\Phi_{-1}(z)) = f(\Phi_{-2}(z)) = \dots \end{aligned}$$

we have
$$\begin{aligned} f(z) &= f(a^n z + b(1 + a + a^2 + \dots + a^{n-1})) \\ &= f\left(a^{-n} z - \frac{b}{a}\left(1 + \frac{1}{a} + \dots + \frac{1}{a^{n-1}}\right)\right) \end{aligned}$$

according as $|a| < 1$ or $|a| > 1$.

If $|a| < 1$, then the equivalent points of $f(z)$ converge to

$$b(1 + a + \dots + a^n + \dots).$$

If $|a| > 1$, then those of $f(z)$ converge to

$$-\frac{b}{a}\left(1 + \frac{1}{a} + \dots + \frac{1}{a^n} + \dots\right).$$

If $a = e^{i\theta\pi}$, θ being an irrational number, then, putting $z' = \Phi(z)$ in the form $\left(z' - \frac{b}{1-a}\right) = a\left(z - \frac{b}{1-a}\right)$, we can see the limiting points of the equivalent points of $f(z)$ lie on every circle with the center at $\frac{b}{1-a}$.

Thus the theorem is proved.

Theorem 14: *There exists a transcendental integral function and a meromorphic function for which*

$$f(e^{i\theta\pi} z + b) = f(z)$$

where θ is an arbitrary rational number, and b is an arbitrary constant.

For, let $\theta = \frac{p}{q}$ and $\Phi(z) = e^{\frac{p}{q}\pi} z + b$, $0 \leq \frac{p}{q} < 2$,

then choosing β such that

$$b = (1 - e^{\frac{p}{q}\pi})\beta, \text{ if } \frac{p}{q} \neq 0,$$

we have an integral function or a meromorphic function

$$f(z) = f_1((z - \beta)^{2q})$$

for which

$$f(\Phi(z)) = f_1((e^{\frac{p}{q}\pi} z + b - \beta)^{2q}) = f_1((z - \beta)^{2q}) = f(z),$$

where $f_1(z)$ is an arbitrary (integral or meromorphic) function.

If $p = 0$, that is, $\Phi(z) = z + b$, then we have an integral function, choosing $\beta b = 2\pi i$,

$$f(z) = e^{\beta z},$$

and a meromorphic function (Weierstrass' $\wp$ -function), for $\beta b = \omega$,

$$f(z) = \wp(\beta z),$$

where ω denotes a period of $\wp(z)$.

These satisfy the required relation.

IX

In order to prove the following theorem we must use the lemma which will be proved in the next section, since the properties of an algebroidal function which will be used are not yet known.

Theorem 15: *Any finite algebroidal function $z' = \Phi(z)$ which is not an algebraic function⁽⁴³⁾ can not satisfy $f(z') = f(z)$ for a meromorphic function $f(z)$.*

Proof: Let $f(z)$ be an integral function, then by the theorem 17⁽⁴⁴⁾ the mapped image of a circle $|z| \leq r$ by the function $\Phi(z)$ contains at least one of the following rings $|\Phi(z)| = \phi_i(r^{1+\delta}) \equiv R_i$, ($i = 0, 1, 2, \dots, \mu$), where $R_\mu > R_{\mu-1} > \dots > R_2 > R_1 > R_0 \equiv r^{1+\delta}$ for an infinitely increasing sequence of sufficiently large values of $r > r_0$, δ being a positive number.

As shown in the theorem 12 above proved, $M_f(r)$ being a steadily increasing function of r , we can prove it similarly.

In this case $f(\Phi(z))$ may be a finite algebroidal (not integral) function, but considering the integral function $f(Z)$ of Z , we have

$$\text{Max}_{|z| \leq r} |f(\Phi(z))| \geq \text{Max}_{|Z| = R} |f(Z)|, \quad R \text{ denoting one of } R_i, \quad (i = 0, 1, 2, \dots, \mu)$$

since the image of $|z| \leq r$ contains $|Z| = R$. On the other hand

$$\text{Max}_{|Z| = R} |f(Z)| > \text{Max}_{|z| \leq r} |f(z)|$$

since $M_f(r)$ is a steadily increasing function and $M_f(r)$ is taken at the boundary of the domain.

Hence we have a contradiction.⁽⁴⁵⁾

(43) An algebroidal function which is not an algebraic function may be called a transcendental algebraic function. A transcendental algebraic function must have an essential singular point at ∞ .

(44) The theorem 17 and the proof of it will be given in the next section X.

(45) It is not so immediate to prove theorem 15 from the theory of iteration of algebroidal functions, since we have no literature on it, and it will be not so easy to show the existence of some fixed points in the finite part for the iterated functions of any algebroidal function.

Let $f(z)$ be a meromorphic function but not an integral function, then putting $f(z) = \frac{f_1(z)}{f_2(z)}$ as in the theorem 12 we have

$$\log \sqrt{|f_1(\Phi(z))|^2 + |f_2(\Phi(z))|^2}.$$

By comparing as in the theorem 12 the both sides of (1) we have the following contradiction.

In this case $f_1(\Phi(z))$ and $f_2(\Phi(z))$ may not be integral functions but considering integral functions $f_1(Z)$ and $f_2(Z)$ of Z , we have

$$\max_{|z| \leq r} \log \sqrt{|f_1(\Phi(z))|^2 + |f_2(\Phi(z))|^2} \geq \max_{|Z| = R} \log \sqrt{|f_1(Z)|^2 + |f_2(Z)|^2}$$

since the image of $|z| \leq r$ contains the circumference $|Z| = R$.

On the other hand

$$\max_{|Z| = R} \log \sqrt{|f_1(Z)|^2 + |f_2(Z)|^2} > \max_{|z| \leq r} \log \sqrt{|f_1(z)|^2 + |f_2(z)|^2}$$

since $\mathfrak{M}_r(r)$ is a steadily increasing function and $\mathfrak{M}_r(r)$ is taken at the boundary of the domain.

Thus in all cases the theorem is proved.

We can further show that

$$\mathfrak{M}^*(r) = \max_{|z| \leq r} \log \sqrt{|g_1(z)|^2 + |g_2(z)|^2}$$

for two finite algebroidal functions $g_1(z)$ and $g_2(z)$ is a steadily increasing function of r (more precisely, it is an increasing convex function of $\log r$ unless $g_1(z)$ and $g_2(z)$ are both of the form az^n).

This can be obtained similarly as we have proved for two integral functions.⁽⁴⁶⁾

For, considering a point z_0 which may be a branch point of $g_1(z)$ or of $g_2(z)$ or of both, we put

$$K(z) = g_1(z)^2 e^{i\alpha_1} + g_2(z)^2 e^{i\alpha_2},$$

where $g_1(z_0)^2 e^{i\alpha_1} = |g_1(z_0)|^2$ and $g_2(z_0)^2 e^{i\alpha_2} = |g_2(z_0)|^2$.

In a sufficiently small $|z - z_0| \leq \delta$ there exists at least a point z_1 , $|z_1 - z_0| = \delta$, such that

$$|g_1(z_1)|^2 + |g_2(z_1)|^2 \geq |K(z_1)|^2 \geq |K(z_0)|^2 = |g_1(z_0)|^2 + |g_2(z_0)|^2.$$

⁽⁴⁶⁾ T. Shimizu: Some theorems on meromorphic functions. Proc. Phys.-Math. Soc. Japan Ser. 3. **11** (1929) 28-35.

This follows immediately from the following relations.

For a regular point z_0 we have⁽⁴⁷⁾

$$|K(z_0)| < \frac{1}{2\pi} \int_0^{2\pi} |K(z_0 + \delta e^{i\theta})| d\theta \quad (2)$$

and for a branch point z_0 , at which s -branches of $K(z)$ are connected,

$$|K(z_0)| < \frac{1}{2\pi} \int_0^{2\pi} |K(z_0 + \delta^s e^{is\theta})| d\theta, \quad (3)$$

$K(z_0 + Z^s)$ being regular for Z in $|Z| < \delta_1$.

$K(z)$ being finite valued the maximum value of the function $|z^\beta| (|g_1(z)|^2 + |g_2(z)|^2)$ which is uniform on the Riemann surface of $K(z)$ must be taken at least at a point on the Riemann surface for $r_1 \leq |z| \leq r_3$, which is a closed set.

That the maximum can not occur within the ring-region $r_1 < |z| < r_3$ on the Riemann surface follows immediately from (2) and (3).

Thus for the function $z^\beta g_1(z)^2 + z^\beta g_2(z)^2$ in $r_1 \leq |z| \leq r_3$, the maximum of $(|g_1(z)|^2 + |g_2(z)|^2)$ is taken at a point of $|z| = r_1$ or of $|z| = r_3$.

Hence, for example,

$$r_2^\beta (|g_1(r_2 e^{i\theta_2})|^2 + |g_2(r_2 e^{i\theta_2})|^2) \leq r_3^\beta (|g_1(r_3 e^{i\theta_3})|^2 + |g_2(r_3 e^{i\theta_3})|^2).$$

Putting

$$M(r) = \text{the maximum of } |g_1(r e^{i\theta})|^2 + |g_2(r e^{i\theta})|^2 \text{ for } |z| = r$$

and choosing β such that

$$r_3^\beta M(r_3) = r_1^\beta M(r_1)$$

we have

$$r_2^\beta M(r_2) < r_1^\beta M(r_1) = r_3^\beta M(r_3)$$

from which the convexity of $M(r)$ and hence of $\Re^*(r)$ follows immediately as usually known.

Now there exist algebraic functions which satisfy (1). In order that an algebraic function $z' = \Phi(z)$ may satisfy (1) the function must be of the form

$$z'^\nu + az^\nu + p(z', z) = 0, \quad |a| = 1,$$

(47) G. Valiron: Remarques sur certaines fonctions convexes. Proc. Phys.-Math. Soc. Japan Ser. 3. **13** (1931) 19-33.

Remoundos: loc. cit.

where $p(z', z)$ is a polynomial of z' and z with degree not larger than ν with respect to both z' and z , and less than ν with respect to each of z' and z .

There exist algebraic functions of the above form, for examples,

$$P(z') = P(z) + 2\pi i$$

which satisfy (1), that is, $e^{P(z')} = e^{P(z)}$, $P(z)$ being a polynomial of z .

We shall discuss some properties of algebraic functions which satisfy (1) in the section XII.

We have hitherto considered some necessary conditions for which a function may satisfy (1)—only one of the functions satisfying (1).

If we suppose further that all the functions $z' = \varphi(z)$ which satisfy (1) are uniform in the whole plane, then we can obtain a theorem analogous to the theorem 8.⁽⁴⁸⁾

Theorem 16: *If all the functions $z' = \varphi(z)$ which satisfy (1) for a transcendental meromorphic function $f(z)$ are uniform in a domain, then all the functions $z' = \varphi(z)$ must be linear functions of the form $e^{t\frac{p}{q}\pi}z + b$, where $q = 1, 2$ or 3 and $p = 0, \pm 1, \pm 2, \dots$ and b a constant. Further in order that all $z' = \varphi(z)$ satisfying (1) for a meromorphic function (not necessarily transcendental) $f(z)$ are uniform in some domain it is necessary and sufficient that $f(z)$ should be of the form $L_1(u(L_2(u)))$, where $u(z)$ is the function defined on the pages 254-255 and $L_1(z)$ and $L_2(z)$ are linear functions, and if $f(z)$ is transcendental, $L_2(z)$ must be of the form $az + b$. The forms of $z' = \varphi(z)$ thus are determined completely.*

Proof: Since $z' = \varphi(z)$ are all uniform in a domain, $\varphi(z)$ are uniform in the whole plane by the theorem 11.

Now by the symmetry of the relation $f(z') = f(z)$ if a function $z' = \Phi(z)$ satisfies (1) then there must appear the same function $z = \Phi(z')$ which satisfies (1).

$z' = \Phi(z)$ being uniform, $z = \Phi(z')$ is a uniform function of z' .

Now the inverse function $z' = \Phi^{-1}(z)$ of $z = \Phi(z')$ must satisfy (1).

$z' = \Phi^{-1}(z)$ belonging to one of the functions $z' = \varphi(z)$ it is also uniform.

⁽⁴⁸⁾ T. Shimizu: On the fundamental domains and the groups.....I. (This Journal, this volume) loc. cit. Sect. VI, p. 220. In it as shown in the above theorem 16 we must read $(az + b)/(cz + d)$ for $az + b$ on the page 220, the last but one line, and on the page 227, the lines 5, 7 and 27.

Hence every function $z' = \Phi(z)$ which satisfies (1) is uniform and its inverse function is also uniform.

Now by theorem 11 $z' = \Phi(z)$ can not have any other singular points than algebraic branch points in the finite part of the z -plane and hence $z' = \Phi(z)$ must be a linear function and must be of the form $z' = e^{i\frac{p}{q}\pi}z + b$ by the theorem 13.

Now all the functions $\varphi(z)$ being of the form $az + b$ there must exist an infinite number of different functions among $\varphi(z)$ if $f(z)$ is not a rational function.

Hence there exists at least a function $z + c$ or two functions $e^{i\frac{p_i}{q_i}\pi}z + b_i$, ($i = 1, 2$) where $e^{i\frac{p_i}{q_i}\pi} \neq 1$, and each of the latter two can not be expressed as the composition of the other, and hence $f(z)$ must be a periodic function.

Let one of the periods be 1 and an elliptic point of the highest order at $z = 0$, then if there exist two independent periods, then we have $z' = z + m + nb_1$, b_1 being not real, or $z' = z + m + nb_1$, $z' = -z + m + nb_1$ or $z' = e^{i\frac{p}{q}\pi}z + m + ne^{i\frac{1}{q}\pi}$, $q = 2$ or 3 , and $(p, q) = 1$, and, if only one independent period, then $z' = z + m$ or $z' = -z + m$, $z' = z + m$.

They may be decomposed into an infinite number of functions of the form $z' - \alpha_j = e^{i\frac{p}{q}\pi}(z - \alpha_j)$, $q = 1, 2$ or 3 , and $j = 1, 2, \dots$. Linear functions of other forms can not satisfy (1) for a periodic function as well known. (Cf. Ford: Automorphic functions. p. 139-p. 144).

But it is not sufficient for our case. For, a doubly periodic function $\wp^4(z)$ (Weierstrass' $\wp$ -function) can not have only linear functions $z' = e^{i\frac{p}{q}\pi}z + b$ for $\wp^4(z') = \wp^4(z)$.

In order to obtain the necessary and sufficient condition for it we can proceed similarly as in the proof of the theorem 8.

Let z_{ij} , $j = 1, 2, \dots$ be the points at which

$$f(z_{ij}) = \beta_i$$

and put

$$f(z) - \beta_i = a_{ij}(z - z_{ij})^{m_{ij}} + \dots$$

In order that all the functions $z' = \varphi(z)$ may be uniform it is necessary and sufficient that m_{ij} should be all equal to m_i .

For, $z' = \varphi(z)$ being defined by $z' = f^{-1}f(z)$ either the algebraic branch points β_i of $f^{-1}(w)$ must be the values β_i for which $f(z) - \beta_i$

must satisfy the above condition with $m_i \geq 2$ for all z_{ij} , $j = 1, 2, \dots$, or the transcendental singular points of $f^{-1}(w)$ must be the exceptional values of $f(z)$.

By Bloch's theorem⁽⁴⁹⁾ we must have

$$\frac{1}{m_1} + \frac{1}{m_2} + \dots + \frac{1}{m_q} \geq q - 2,$$

where some of m_i may be ∞ .

Hence we have at once $q \leq 4$.

Thus similarly as before we can repeat the same argument as that in the theorem 8 in the previous paper.⁽⁵⁰⁾

We have the following cases as possible.

- 1) $q = 4$; $m_1 = m_2 = m_3 = m_4 = 2$
- 2) $q = 3$; $m_1 = 3, m_2 = 3, m_3 = 3$
 $\quad \quad \quad = 2, \quad \quad = 4, \quad \quad = 4$
 $\quad \quad \quad = 2, \quad \quad = 3, \quad \quad = 6$
 $\quad \quad \quad = 2, \quad \quad = 2, \quad \quad = \infty$
- 3) $q = 3$; $\quad \quad = 2, \quad \quad = 2, \quad \quad = l, \quad l = 2, 3, 4, \dots < \infty$
 $\quad \quad \quad = 2, \quad \quad = 3, \quad \quad = 3$
 $\quad \quad \quad = 2, \quad \quad = 3, \quad \quad = 4$
 $\quad \quad \quad = 2, \quad \quad = 3, \quad \quad = 5$
- 4) $q = 2$; $m_1 = l_1, m_2 = l_2 \quad \left(\begin{matrix} l_1 = 2, 3, 4, \dots, \infty \\ l_2 = 2, 3, 4, \dots, \infty \end{matrix} \right)$

In all cases the function $f(z)$ for which all $z' = \varphi(z)$ are uniform must be of the form

$L_1(u(L_2(z)))$, and if $f(z)$ is a transcendental function, then $L_2(z)$ must be of the form $az + b$, a and b being constants, and the function $u(z)$ is defined by the following integral for each case:

- 1) $\int_{u_0}^u \frac{du}{\sqrt{(u-w_1)(u-w_2)(u-w_3)(u-w_4)}} = z.$
- 2) $\int_{u_0}^u \frac{du}{\sqrt[3]{u^2(1-u)^2}} = z, \quad \int_{u_0}^u \frac{du}{\sqrt[4]{u^2(1-u)^3}} = z.$

⁽⁴⁹⁾ T. Shimizu: I. loc. cit. p. 222. Cf. ⁽⁴⁸⁾.

⁽⁵⁰⁾ T. Shimizu: I. loc. cit. p. 222. Cf. ⁽⁴⁸⁾.

$$\int_{u_0}^u \frac{du}{\sqrt[6]{u^3(1-u)^4}} = z, \quad \int_{u_0}^u \frac{du}{\sqrt{u(1-u)}} = z.$$

$$\begin{aligned} 3) \quad u(z) = & \left. \begin{aligned} & \text{a dihedral function} \quad \frac{-4z^n}{(z^n-1)^2} \\ & \text{a tetrahedral function} \quad \frac{-12\sqrt{3}z^2(z^4+1)^2}{(z^4-2z^2\sqrt{3}-1)^3} \\ & \text{an octahedral function} \quad \frac{(z^8+14z^4+1)^3}{(z^{12}-33z^8-33z^4+1)^2} \\ & \text{an icosahedral function} \quad \frac{(z^{20}-228z^{15}+494z^{10}+228z^5+1)^3}{\{z^{30}+1+522z^5(z^{20}-1)-10005z^{10}(z^{10}+1)\}^{30}} \end{aligned} \right\} \end{aligned}$$

each of which is a rational function of z invariant under each finite polyhedral group.

$$4) \quad u(z) = z^m, \quad u(z) = e^z.$$

These can be obtained by considering the inverse function $u^{-1}(u)$ of $u(z)$ and then $u^{-1}(f(z))$. First $g(z) = u^{-1}(f(z))$ must be a uniform function of z , since $u^{-1}(u)$ has no other branch points than the values of $f(z)$, which $f(z)$ takes at all points in the same multiplicity for each value respectively with $u(z)$. (as shown in theorem 8, p. 223. Cf. (48)). By the same reasoning $z = g^{-1}(g) = f^{-1}(u(g))$ is uniform in each case from the assumption on $f(z)$. Hence $g(z)$ must be a linear function of the form $az + b$, if $u(z)$ is a transcendental function, and a general linear function $L_2(z)$ if $u(z)$ is a rational function.

Conversely if $f(z)$ is one of the above functions, then we can conclude that $z' = \varphi(z)$ are all linear functions.

For, if $f(z)$ is a rational function, then the proposition is evidently true. Now by the relations

$$L_1(u(L_2(z'))) = L_1(u(L_2(z)))$$

we have

$$u(L_2(z')) = u(L_2(z)).$$

Hence by putting $L_2(z) = Z$ we have

$$u(Z) = u(Z).$$

Now $u(Z)$ takes the same multiple values at all the equivalent points, and hence $Z = u^{-1}(u(Z))$ are all uniform for every point of Z .

For, by

$$u(Z) - \beta = u(Z) - \beta$$

and

$$A_{ij}(Z - Z_{ij})^{m_i} + \dots = A_{ij}(Z - Z_{ij})^{m_i} + \dots$$

we have

$$Z - Z_{ij} = e^{i\theta}(Z - Z_{ij}) + \dots$$

at every point of Z .

Thus $Z = \varphi(Z)$ being uniform in any finite part of the Z -plane, Z must be a linear function of Z by the theorems 11, 12 and 13.

Hence all the functions $z' = \varphi(z)$ become linear functions.

Similarly if all the functions $z' = \varphi(z)$ which satisfy (1) are finite (limited) valued, then the inverse functions $\varphi^{-1}(z')$ of $z' = \varphi(z)$ must be limited valued.

Hence they must be all algebraic functions of the form

$$z'^\nu + az^\nu + p(z', z) = 0,$$

where $p(z', z)$ has the form given before.

We can not assert that an algebraic function takes every value but every algebraic function $\Phi(z)$ which satisfies (1) takes every value except ∞ just each an equal number of times, since the highest term about z does not contain the variable z' .

It is evident that if all the functions $z' = \varphi(z)$ satisfying (1) are only algebraic functions (rational functions included) then they are infinite in number.

For, $w = f(z)$ must have an infinite number of roots for some values of w . Let $z_1, z_2, \dots, z_n, \dots$ be its roots, then in the function $f(z') - f(z)$ written in the product of factors $g_i(z', z)e^{p_i(z', z)}$ ⁽⁵¹⁾ there must appear an infinite number of $g_i(z', z)$, since every one of $g_i(z', z) = 0$ transforms only a finite number of z_i and the factors $e^{p_i(z', z)}$ can not give any transformations among z_i .

X

Let a finite⁽⁵²⁾ algebroidal (transcendental) function of degree ν be defined by the irreducible equation

$$g(u, z) = u^\nu + A_1(z)u^{\nu-1} + \dots + A_\nu(z) = 0 \quad (4)$$

where $A_i(z)$, ($i = 1, 2, \dots, \nu$) are integral functions, one of which is at least a transcendental integral function.

⁽⁵¹⁾ T. Shimizu: I. loc. cit. Sec. VII. p. 230. Cf. ⁽⁴⁸⁾.

⁽⁵²⁾ Sec. p. 238 ⁽²⁹⁾ and p. 249 ⁽⁴³⁾.

Among the functions $A_i(z)$, ($i = 1, 2, \dots, \nu$) there may or not exist some linear relations with constant coefficients.

Let μ be the number of linearly independent functions $A_i(z)$.

It is evident that the number $\nu - \mu$ of linear relations among $A_i(z)$ which are linearly independent can not surpass $\nu - 1$.

Then we have a theorem:

Theorem 17: *The values of a finite algebroidal function defined by (4), the number of linearly independent functions $A_i(z)$ being μ , cover at least one of the following $\mu + 1$ whole circumferences*

$$|u(z)| = \phi_i(r^{1+\delta}), \quad i = 0, 1, 2, \dots, \mu,$$

in a circle $|z| \leq r$ for an infinitely increasing sequence of values of r , where $\phi_0(r^{1+\delta}) = r^{1+\delta}$; and $\phi_i(r^{1+\delta})$ is some polynomial of $r^{1+\delta}$ with positive constant coefficients, and $r^{1+\delta} < \phi_1(r^{1+\delta}) < \phi_2(r^{1+\delta}) < \dots < \phi_{\mu-1}(r^{1+\delta}) < \phi_\mu(r^{1+\delta})$, δ being a positive number.

Now $A_i(z)$ being fixed we can choose a sequence of μ functions $B_i(z)$ ($i = 1, 2, \dots, \mu$) among them such that we have the relations

$$\begin{aligned} A_1(z) &= \alpha_{10} + \alpha_{11} B_1(z) + \dots + \alpha_{1\mu} B_\mu(z) \\ A_2(z) &= \alpha_{20} + \alpha_{21} B_1(z) + \dots + \alpha_{2\mu} B_\mu(z) \\ &\dots\dots\dots \\ A_\nu(z) &= \alpha_{\nu 0} + \alpha_{\nu 1} B_1(z) + \dots + \alpha_{\nu \mu} B_\mu(z) \end{aligned} \quad (5)$$

where $\alpha_{10}, \alpha_{11}, \dots$ and all fixed constants if $B_i(z)$ are fixed and $B_i(z)$ are all linearly independent.

For simplicity's sake we can choose $B_i(z)$ so that no one of $\alpha_{10}, \dots, \alpha_{\nu \mu}$ is zero, since $B_i(z)$ are linearly independent. Here we consider such fixed functions $B_i(z)$.

Substituting (5) in (4) we have

$$g(u, z) = p_0(u) + p_1(u) B_1(z) + \dots + p_\mu(u) B_\mu(z) \quad (6)$$

where

$$\begin{aligned} p_0(u) &= u^\nu + \alpha_{10} u^{\nu-1} + \dots + \alpha_{\nu 0} \\ p_1(u) &= \alpha_{11} u^{\nu-1} + \dots + \alpha_{\nu 1} \\ &\dots\dots\dots \\ p_\mu(u) &= \alpha_{1\mu} u^{\nu-1} + \dots + \alpha_{\nu \mu}. \end{aligned}$$

Consider for our purpose $\mu + 1$ complex values $u_1, u_2, \dots, u_{\mu+1}$, which are constants now (whose values $u_1(r), u_2(r), \dots, u_{\mu+1}(r)$ will be considered later as functions of r where $|z| = r$) with the following conditions:

1) First let $u_1, u_2, \dots, u_{\mu+1}$ be all larger than any zero points of $p_1(u) - \eta_1, p_2(u) - \eta_2, \dots, p_\mu(u) - \eta_\mu$, $\eta_1, \eta_2, \dots, \eta_\mu$ being arbitrary constants less than a fixed constant η in absolute value.

$A_1(z), A_2(z), \dots, A_\nu(z)$ and $B_1(z), B_2(z), \dots, B_\mu(z)$ being fixed, $p_1(u) - \eta_1 = 0, p_2(u) - \eta_2 = 0, \dots, p_\mu(u) - \eta_\mu = 0$ are all determinate equations except $\eta_1, \eta_2, \dots, \eta_\mu$.

Hence they have only zero points of limited moduli, thus if we will increase $|u_1|, |u_2|, \dots, |u_{\mu+1}|$ indefinitely in the later part of this section they will all satisfy the condition 1).

2) We consider $\mu + 2$ expressions

$$\begin{vmatrix} p_0(u_1) & p_1(u_1) & \dots & p_\mu(u_1) \\ \dots & \dots & \dots & \dots \\ p_0(u_\mu) & p_1(u_\mu) & \dots & p_\mu(u_\mu) \\ p_0(u_{\mu+1}) & p_1(u_{\mu+1}) & \dots & p_\mu(u_{\mu+1}) \end{vmatrix}, \begin{vmatrix} p_1(u_1) & \dots & p_\mu(u_1) \\ \dots & \dots & \dots \\ p_1(u_{\mu-1}) & \dots & p_\mu(u_{\mu-1}) \\ p_1(u_\mu) & \dots & p_\mu(u_\mu) \end{vmatrix},$$

$$\begin{vmatrix} p_1(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots \\ p_1(u_\mu) & \dots & p_\mu(u_\mu) \\ p_1(u_{\mu+1}) & \dots & p_\mu(u_{\mu+1}) \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} p_1(u_1) & \dots & p_\mu(u_1) \\ \dots & \dots & \dots \\ p_1(u_{j-1}) & \dots & p_\mu(u_{j-1}) \\ p_1(u_{j+1}) & \dots & p_\mu(u_{j+1}) \\ \dots & \dots & \dots \\ p_1(u_{\mu+1}) & \dots & p_\mu(u_{\mu+1}) \end{vmatrix} \quad (7)$$

$$j = 2, 3, \dots, \mu$$

where we consider $u_1, u_2, \dots, u_{\mu+1}$ as independent quantities.

We determine $u_1, u_2, \dots, u_{\mu+1}$ such that $|u_1| < |u_2| < \dots < |u_{\mu+1}|$ and the expressions (7) are all larger than a fixed number η in their absolute values.

The possibility of these determinations can be shown as follows.

Let $\eta_1, \eta_2, \dots, \eta_\mu$ be arbitrary complex numbers whose absolute values are less than η .

Now we take $|u_1|$ so large that we have for u_1

$$|p_1(u_1)| \geq \eta > 0,$$

where η is a given positive number.

u_1 can be determined for certain, for the coefficients $\alpha_{11}, \alpha_{21}, \dots, \alpha_{\nu 1} - \eta_1$ of the equation $p_1(u) - \eta_1 = 0$ being given except η_1 and $|\eta_1| \leq \eta$, and the degree of it less than ν , the roots of the equation $p_1(u) - \eta_1 = 0$

are all less than $1 + \frac{1}{|\alpha_{\lambda 1}|} \left(\sum_{i=\lambda}^{\nu-1} |\alpha_{i1}| + |\alpha_{\nu 1} - \eta_1| \right) \leq 1 + \frac{1}{|\alpha_{\lambda 1}|} \left(\sum_{i=\lambda}^{\nu} |\alpha_{i1}| + \eta \right)$

when η_1 varies throughout $|\eta_1| \leq \eta$, where $\alpha_{\lambda 1}$ is the first coefficient which does not vanish. This follows by a well-known theorem of algebra.

$$\text{Now for } |u_1| > 1 + \frac{1}{|\alpha_{\lambda 1}|} \left(\sum_{i=\lambda}^{\nu} |\alpha_{i1}| + \eta \right) \quad (7^\circ)$$

we have $|p_1(u_1)| \geq \eta$,

and a fortiori for $|u_2| > |u_1|$

$$|p_1(u_2)| \geq \eta.$$

Next consider the equation

$$\begin{vmatrix} p_1(u_1) & p_2(u_1) \\ p_1(u) & p_2(u) \end{vmatrix} = 0 \quad (8)$$

where u is an unknown quantity.

This equation (8) must contain u . For, otherwise, we would have

$$p_2(u) = k_1 p_1(u),$$

since $|p_1(u_1)| > \eta$, u_1 being considered as a constant, and $k_1 \neq \infty$.

Since the equation (8) is satisfied by $u = u_1$, it can not be an inconsistent equation.

Now k_1 being a finite constant, (if $k_1 \neq 0$) we would have that by (6) and (5), $A_i(z)$, ($i = 1, 2, \dots, \nu$) can be expressed linearly by the $\mu - 1$ functions

$$B_1(z) + k_1 B_2(z), B_3(z), \dots, B_\mu(z),$$

and hence there would exist $\nu - \mu + 1$ linearly independent relations among $A_i(z)$, ($i = 1, 2, \dots, \nu$) which is evidently impossible.

If $k_1 = 0$, (8) evidently contains u .

Since the equation (8) contains u and is of at most degree $\nu - 1$ we can determine u_2 so that $|u_2| > |u_1|$ and

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) \\ p_1(u_2) & p_2(u_2) \end{vmatrix} > \eta. \quad (9)$$

For, considering u_1 and u as independent variables let $q_1(u_1)$ be the first coefficient of the highest degree of (8) with respect to u which does not vanish.

If $q_1(u_1)$ contains u_1 , then we consider the equation $q_1(u_1) = 0$.

The coefficients of the polynomial $q_1(u_1)$ with respect to u_1 containing only $\alpha_{10}, \alpha_{11}, \dots, \alpha_{\nu\mu}$ we have by a theorem of algebra

$$|q_1(u_1)| > \eta$$

$$\text{if } |u_1| > 1 + \frac{1}{\beta_{\lambda}} \left(\sum |\beta_i| + \eta \right) \quad (8^\circ)$$

where β_i denote the coefficients of u_1 in $q_1(u_1) - \eta_2 = 0$, $\beta_{\lambda} \neq 0$, and β_i depending only on $\alpha_{10}, \alpha_{11}, \dots, \alpha_{\nu\mu}$, and are fixed values.

If $q_1(u_1)$ does not contain u_1 , then $|q_1(u_1)| > \gamma$, γ being a fixed constant, depending only on $\alpha_{10}, \alpha_{11}, \dots, \alpha_{\nu\mu}$.

Considering u_1 which satisfies (7°) and (8°) we can determine u_2 such that (9) holds good.

For the highest coefficient of u in the left hand side of (8) being larger than the minimum $\bar{\eta}$ of γ and η in absolute value, by applying algebra's theorem again we can show that the roots of the equation

$$\begin{vmatrix} p_1(u_1) & p_2(u_1) \\ p_1(u) & p_2(u) \end{vmatrix} - \eta_1 = 0, \quad |\eta_1| \leq \eta,$$

are all less than $1 + \frac{1}{\bar{\eta}} \left(\sum |\beta_i'| + \eta \right)$, where β_i' are the coefficients of the last equation with respect to u ,—which can be regarded as a function $t(u_1)$ of u_1 , where $t(u_1)$ can be taken as a polynomial of $|u_1|$ with positive coefficients depending only on $\alpha_{10}, \dots, \alpha_{\nu\mu}$ and η , and with at most degree $\nu - 1$ with respect to u_1 .

$$\text{Hence if } |u_2| > t(u_1) \quad (9^\circ)$$

then (9) must be satisfied.

Next consider the equation as regard to u

$$\begin{vmatrix} p_1(u_1) & p_2(u_1) & p_3(u_1) \\ p_1(u_2) & p_2(u_2) & p_3(u_2) \\ p_1(u) & p_2(u) & p_3(u) \end{vmatrix} = 0 \quad (10)$$

which must contain the term in which u appears.

For, otherwise, we would have by (9)

$$p_3(u) = k_1 p_1(u) + k_2 p_2(u).$$

And similarly as before we can proceed as follows.

Thus as before $A_i(z)$ would be represented linearly by $\mu - 1$

functions $B_i(z)$ which contradicts that the number of linearly independent relations among $A_i(z)$ must be $\nu - \mu$.

Now consider (10).

As before u_1, u_2, u as independent variables we denote the coefficient of the term of the highest degree with respect to u by $\{q_2(u_1)u_2^k + \dots\} \equiv q_{12}(u_1, u_2)$ ordered in descending order with respect to u_2 .

If $q_{12}(u_1, u_2)$ contains u_2 , then we consider $q_{12}(u_1, u_2) = 0$.

If $q_{12}(u_1, u_2)$ does not contain u_2 then we can determine u_1 such that $|q_{12}(u_1, u_2)| > \eta$. If any of u_1 and u_2 does not appear in it, then $q_{12}(u_1, u_2)$ must be a constant.

Similarly as before we can choose u_1 such that

$$|u_1| > 1 + \frac{1}{\beta} \sum |\beta| \quad (10^\circ)$$

and $|q_2(u_1)| > \eta$

and hence we can determine u_2 such that

$$|q_{12}(u_1, u_2)| > \eta.$$

For, again by the theorem of algebra the roots of the equations, as regard to u_2 ,

$$q_{12}(u_1, u_2) - \eta_1 = 0, \quad |\eta_1| < \eta \quad (11)$$

are all less than $1 + \frac{1}{\eta} \left(\sum |\gamma_i| \right)$ in absolute value, which is considered as a function $t_1(u_1)$ of u_1 , $t_1(u_1)$ being a polynomial of $|u_1|$ with at most degree $\nu - 1$ and, γ_i being the coefficients of (11).

$$\text{Hence for } |u_2| > t_1(u_1). \quad (11^\circ)$$

we have

$$|q_{12}(u_1, u_2)| > \eta. \quad (12)$$

Next we can take u_3 such that

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) & p_3(u_1) \\ p_1(u_2) & p_2(u_2) & p_3(u_2) \\ p_1(u_3) & p_2(u_3) & p_3(u_3) \end{vmatrix} > \eta. \quad (13)$$

For, again the roots of the equations (10) by (12) are all less than

$$1 + \frac{1}{\eta} \left(\sum |\gamma'_i| \right) \equiv s(u_1, u_2) < t_2(u_2) \quad (12^\circ)$$

where γ_i' are the coefficients of the term u of respective degrees and $t_2(u_2)$ is a polynomial of $|u_2|$ with some degree.

Thus for such $|u_3| > t_2(u_2)$ we have (13).

We take now u_1, u_2, u_3 such that (8°) (9°) (10°) (11°) hold good and then we have (13) which holds good.⁽⁵³⁾

Proceeding in these ways, that is, choosing $u_1, u_2, \dots, u_{\mu-1}$ over again in every time, such that they may satisfy all the conditions obtained up to that time we can show that for such $u_1, u_2, \dots, u_{\mu-1}$

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) & \dots & p_{\mu-1}(u_1) \\ p_1(u_2) & p_2(u_2) & \dots & p_{\mu-1}(u_2) \\ \dots & \dots & \dots & \dots \\ p_1(u_{\mu-1}) & p_2(u_{\mu-1}) & \dots & p_{\mu-1}(u_{\mu-1}) \end{vmatrix} > \eta. \quad (14)$$

We can prove it by mathematical induction.

In general assume that (14) holds good, then

$$\begin{vmatrix} p_1(u_1) & p_2(u_1) & \dots & p_\mu(u_1) \\ p_1(u_2) & p_2(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots & \dots \\ p_1(u_{\mu-1}) & p_2(u_{\mu-1}) & \dots & p_\mu(u_{\mu-1}) \\ p_1(u) & p_2(u) & \dots & p_\mu(u) \end{vmatrix} = 0 \quad (15)$$

must contain u in the left hand side of (15).

For, otherwise, we would have as before by (14)

$$p_\mu(u) = h_1 p_{\mu-1}(u) + h_2 p_{\mu-2}(u) + \dots + h_{\mu-1} p_1(u),$$

where $h_1, h_2, \dots, h_{\mu-1}$ are constants, depending only on $u_1, u_2, \dots, u_{\mu-1}$ and $\alpha_{10}, \dots, \alpha_{\nu\mu}$.

And $A_i(z)$ would be expressed by $\mu-1$ functions $B_1(z), B_2(z), \dots, B_{\mu-1}(z), h_1 B_1(z) + h_2 B_2(z) + \dots + h_{\mu-1} B_{\mu-1}(z)$, which contradicts the hypothesis.

Thus, (15) containing u , let $q_{12\dots\mu-1}(u_1, u_2, \dots, u_{\mu-1})$ be the coefficient of u of the highest degree with respect to u .

We order $q_{12\dots\mu-1}(u_1, u_2, \dots, u_{\mu-1})$ in a descending order with respect to $u_{\mu-1}$, and then with respect to $u_{\mu-2}, \dots$ and lastly with respect to u_1 .

⁽⁵³⁾ Since it takes many lines to discuss completely the general case in mathematical induction, we have proved up to the case $n=3$ precisely and have given only a proof somewhat simple in the general case, the reasoning being the same.

$$\begin{aligned}
& \left\{ \left\{ \left\{ \dots \left\{ q_{\mu-1}(u_1) u_2^k + \dots \right\} u_3^k + \dots \right\} u_4^k + \dots \right\} u_{\mu-1}^k + \dots \right. \\
& = q_{\mu-1, 2}(u_1, u_2) u_3^k + \dots \\
& = q_{\mu-2, 2, 3}(u_1, u_2, u_3) u_4^k + \dots \\
& \dots\dots\dots \\
& = q_{1, 2, 3, \dots, \mu-2}(u_1, u_2, \dots, u_{\mu-2}) u_{\mu-1}^k + \dots \\
& = q_{1, 2, 3, \dots, \mu-1}(u_1, u_2, \dots, u_{\mu-1})
\end{aligned} \tag{16}$$

If some or all of $u_1, u_2, \dots, u_{\mu-2}$ do not appear in (16), we can proceed as before without any condition on them.

Since the coefficients of u_1 in $q_{\mu-1}(u_1)$ consist of $\alpha_{10}, \alpha_{11}, \dots, \alpha_{v\mu}$, we can choose u_1 such that

$$\begin{array}{ll} |q_{\mu-1}(u_1)| > \eta \\ \text{for } |u_1| > P(\alpha) \end{array} \quad (13^o)$$

And then as before for

$$|u_2| > P_2(u_1) \quad (14^\circ)$$

$$|q_{\mu-2,2}(u_1, u_2)| > \eta$$

$$\begin{array}{ll}
\text{for} & |u_{\mu-2}| > P_{\mu-2}(u_{\mu-3}, u_{\mu-4}, \dots, u_2, u_1) \\
& |q_{2,2,3,\dots,\mu-2}(u_1, u_2, \dots, u_{\mu-2})| > \eta \\
\text{for} & |u_{\mu-1}| > P_{\mu-1}(u_{\mu-2}, u_{\mu-3}, \dots, u_2, u_1) \\
& |q_{1,2,\dots,\mu-1}(u_1, u_2, \dots, u_{\mu-1})| > \eta
\end{array} \quad (15^\circ)$$

Lastly for $|u_\mu| > P_\mu(u_{\mu-1}, u_{\mu-2}, \dots, u_2, u_1)$ we have (16°)

$$\left| \begin{array}{cccc} p_1(u_1) & p_2(u_1) & \cdots & p_\mu(u_1) \\ p_1(u_2) & p_2(u_2) & \cdots & p_\mu(u_2) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_{\mu-1}) & p_2(u_{\mu-1}) & \cdots & p_\mu(u_{\mu-1}) \\ p_1(u_\mu) & p_2(u_\mu) & \cdots & p_\mu(u_\mu) \end{array} \right| > \eta. \quad (17)$$

These follow perfectly similarly as before.

Here $P_1, P_2, \dots, P_\mu$ denote polynomials with positive coefficients depending only on η and $\alpha_{10}, \alpha_{11}, \dots, \alpha_{\nu\mu}$ and of degree finite with respect to the absolute values of arguments.

Now choose $u_1, u_2, \dots, u_\mu$ over again, such that they may satisfy all the conditions (7°) (8°).....(13°) (14°) (15°) (16°).

This is always possible, for by choosing u_1 such that u_1 may satisfy all the conditions imposed on itself, u_1 can be determined sufficiently large and satisfying all the conditions.

Next we take u_2 such that it may satisfy all the conditions by taking the largest $P_1(u_1)$ for $|u_2|$. Continuing in these ways we take next u_3 such that it may satisfy the conditions. And then $u_4, u_5, \dots$ and so on.

In these processes we must remark for the later part's sake that for sufficiently large values of $|u_1|$ we can take $|u_2|, |u_3|, \dots, |u_\mu|$ such that they are polynomials of finite degree with respect to $|u_1|$ with positive coefficients depending only on $\alpha_{10}, \alpha_{11}, \dots, \alpha_{1\mu}$ and η . Thus the polynomials $|u_2|, \dots, |u_{\mu+1}|$ of $|u_1|$ lastly determined (p. 266) satisfy all the conditions if the values of u_1 increases (varies) without altering the polynomials.

Now consider the equation

$$\begin{vmatrix} p_0(u_1) & p_1(u_1) & \dots & p_\mu(u_1) \\ p_0(u_2) & p_1(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots & \dots \\ p_0(u_\mu) & p_1(u_\mu) & \dots & p_\mu(u_\mu) \\ p_0(u) & p_1(u) & \dots & p_\mu(u) \end{vmatrix} = 0$$

which must contain the term u .

For, by (17) and by the property of $p_0(u)$, that is, $p_0(u)$ contains the term u^r of the highest degree with respect to u , and the coefficient of $p_0(u) > \eta$ in absolute value by (17).

Hence we can choose $u_{\mu+1}$ as before such that

$$\text{Absolute value of } \begin{vmatrix} p_0(u_1) & p_1(u_1) & \dots & p_\mu(u_1) \\ p_0(u_2) & p_1(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots & \dots \\ p_0(u_\mu) & p_1(u_\mu) & \dots & p_\mu(u_\mu) \\ p_0(u_{\mu+1}) & p_1(u_{\mu+1}) & \dots & p_\mu(u_{\mu+1}) \end{vmatrix} > \eta \quad (18)$$

for $|u_{\mu+1}| > P(u_1, u_2, \dots, u_\mu)$.

Now we shall show that we can take $u_1, u_2, \dots, u_{\mu+1}$ over again, such that all the conditions (7) may be satisfied.

First by $|u_{\mu+1}| > |u_\mu|$ we have a fortiori

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) & \cdots & p_\mu(u_1) \\ p_1(u_2) & p_2(u_2) & \cdots & p_\mu(u_2) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_{\mu-1}) & p_2(u_{\mu-1}) & \cdots & p_\mu(u_{\mu-1}) \\ p_1(u_{\mu+1}) & p_2(u_{\mu+1}) & \cdots & p_\mu(u_{\mu+1}) \end{vmatrix} > \eta. \quad (19)$$

Next by choosing $u_{\mu-1}$ over again, and then u_μ and further then $u_{\mu+1}$ we can determine them such that

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) & \cdots & p_\mu(u_1) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_{\mu-2}) & p_2(u_{\mu-2}) & \cdots & p_\mu(u_{\mu-2}) \\ p_1(u_\mu) & p_2(u_\mu) & \cdots & p_\mu(u_\mu) \\ p_1(u_{\mu+1}) & p_2(u_{\mu+1}) & \cdots & p_\mu(u_{\mu+1}) \end{vmatrix} > \eta. \quad (19^*)$$

In these processes we can choose such new $u_1, u_2, \dots, u_{\mu-1}, u_\mu, u_{\mu+1}$ that the relations (17), (18) and (19) remain true, since, taking large values of u_i 's, they still hold good. Thus we can continue these processes by choosing $u_{\mu-2}, u_{\mu-1}, u_\mu, u_{\mu+1}$ over again, then $u_{\mu-3}, u_{\mu-2}, \dots, u_{\mu+1}$, and then $u_{\mu-4}, u_{\mu-3}, \dots, u_{\mu+1}; \dots; u_2, u_3, \dots, u_{\mu+1}$ such that all

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) & \cdots & p_\mu(u_1) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_{\mu-3}) & p_2(u_{\mu-3}) & \cdots & p_\mu(u_{\mu-3}) \\ p_1(u_{\mu-1}) & p_2(u_{\mu-1}) & \cdots & p_\mu(u_{\mu-1}) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_{\mu+1}) & p_2(u_{\mu+1}) & \cdots & p_\mu(u_{\mu+1}) \end{vmatrix} > \eta \quad (19^{**})$$

$$\text{Absolute value of } \begin{vmatrix} p_1(u_1) & p_2(u_1) & \cdots & p_\mu(u_1) \\ p_1(u_3) & p_2(u_3) & \cdots & p_\mu(u_3) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_\mu) & p_2(u_\mu) & \cdots & p_\mu(u_\mu) \\ p_1(u_{\mu+1}) & p_2(u_{\mu+1}) & \cdots & p_\mu(u_{\mu+1}) \end{vmatrix} > \eta \quad (19^{**})$$

$$\text{Absolute value of } \begin{vmatrix} p_1(u_2) & p_2(u_2) & \cdots & p_\mu(u_2) \\ p_1(u_3) & p_2(u_3) & \cdots & p_\mu(u_3) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_\mu) & p_2(u_\mu) & \cdots & p_\mu(u_\mu) \\ p_1(u_{\mu+1}) & p_2(u_{\mu+1}) & \cdots & p_\mu(u_{\mu+1}) \end{vmatrix} > \eta \quad (19^{**})$$

may be satisfied.

Since by these choicing all the conditions (17), (18) (19) (19*), (19**), (19***) remain true, we can determine $u_1, u_2, \dots, u_{\mu+1}$ such that all the conditions (7) may be satisfied, and further we can choose them arbitrary if they satisfy the above condition only for their absolute values.

Now we can eliminate the functions $B_1(z), B_2(z), \dots, B_\mu(z)$ from the equations

$$g(u_i, z) = g_i(z), \quad (i = 1, 2, \dots, \mu + 1) \quad (20)$$

which is always possible, by

$$\begin{vmatrix} p_1(u_1) & \dots & p_\mu(u_1) \\ p_1(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots \\ p_1(u_\mu) & \dots & p_\mu(u_\mu) \end{vmatrix} \neq 0.$$

Finally we obtain

$$d_1 g_1(z) + \dots + d_{\mu+1} g_{\mu+1}(z) \equiv d_0 \quad (21)$$

$$\text{where } d_0 = \begin{vmatrix} p_0(u_1) & p_1(u_1) & \dots & p_\mu(u_1) \\ p_0(u_2) & p_1(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots & \dots \\ p_0(u_{\mu+1}) & p_1(u_{\mu+1}) & \dots & p_\mu(u_{\mu+1}) \end{vmatrix} \neq 0.$$

By the above choice $d_0, d_1, d_2, \dots, d_{\mu+1}$ which are the determinants (7) are all larger than η in their absolute values.

On the other hand $g_1(z), g_2(z), \dots, g_{\mu+1}(z)$ are all functions of z and not constants, for, otherwise, some of $B_i(z)$ would be linearly dependent, since $p_j(u) \neq 0$ for sufficiently large values of u .

Later on we must consider $d_0, d_1, \dots, d_{\mu+1}$ as functions of $u_1, \dots, u_{\mu+1}$. First d_0 is a polynomial of $\mu + 1$ variables $u_1, \dots, u_{\mu+1}$ of degree at most $\nu + \mu(\nu - 1)$. All the other functions $d_1, d_2, \dots, d_{\mu+1}$ are polynomials of $u_1, \dots, u_{\mu+1}$ of degree at most $\mu(\nu - 1)$.

Again considering $d_0, d_1, d_2, \dots, d_{\mu+1}$ as constants we can show that there exists no linear homogeneous relation with constant coefficients among $g_1(z), g_2(z), \dots, g_{\mu+1}(z)$ such as

$$e_1 g_1(z) + e_2 g_2(z) + \dots + e_{\mu+1} g_{\mu+1}(z) \equiv 0.$$

For if there were some homogeneous linear relation

$$e_1' g_1(z) + e_2' g_2(z) + \dots + e_k' g_k(z) \equiv 0,$$

then we would have by (20) a linear relation with constant coefficients among $B_1(z), \dots, B_\mu(z)$,

$$\begin{aligned} & [e_1' p_1(u_1) + e_2' p_1(u_2) + \dots + e_k' p_1(u_k)] B_1(z) \\ & + [e_1' p_2(u_1) + e_2' p_2(u_2) + \dots + e_k' p_2(u_k)] B_2(z) \\ & \dots \dots \dots \\ & + [e_1' p_\mu(u_1) + e_2' p_\mu(u_2) + \dots + e_k' p_\mu(u_k)] B_\mu(z) \\ & + [e_1' p_0(u_1) + e_2' p_0(u_2) + \dots + e_k' p_0(u_k)] \equiv 0. \end{aligned} \quad (22)$$

$B_1(z), \dots, B_\mu(z)$ being linearly independent we must have from (22)

$$\begin{aligned} e_1' p_1(u_1) + \dots + e_k' p_1(u_k) &= 0 \\ \dots \dots \dots \\ e_1' p_\mu(u_1) + \dots + e_k' p_\mu(u_k) &= 0 \\ e_1' p_0(u_1) + \dots + e_k' p_0(u_k) &= 0. \end{aligned}$$

Now the determinant
$$\begin{vmatrix} p_0(u_1) & p_1(u_1) & \dots & p_\mu(u_1) \\ p_0(u_2) & p_1(u_2) & \dots & p_\mu(u_2) \\ \dots & \dots & \dots & \dots \\ p_0(u_{\mu+1}) & p_1(u_{\mu+1}) & \dots & p_\mu(u_{\mu+1}) \end{vmatrix} \neq 0 \quad \text{and } u_1,$$

$u_2, \dots, u_k$ being some of $u_1, u_2, \dots, u_{\mu+1}$, there exists at least one system

$$\begin{aligned} e_1' p_{\alpha_1}(u_1) + \dots + e_k' p_{\alpha_1}(u_k) &= 0 \\ e_1' p_{\alpha_2}(u_1) + \dots + e_k' p_{\alpha_2}(u_k) &= 0 \\ \dots \dots \dots \\ e_1' p_{\alpha_k}(u_1) + \dots + e_k' p_{\alpha_k}(u_k) &= 0 \end{aligned}$$

such that

$$\begin{vmatrix} p_{\alpha_1}(u_1) \dots p_{\alpha_1}(u_k) \\ p_{\alpha_2}(u_1) \dots p_{\alpha_2}(u_k) \\ \dots \dots \dots \\ p_{\alpha_k}(u_1) \dots p_{\alpha_k}(u_k) \end{vmatrix} \neq 0.$$

Hence we must have $e_1' = e_2' = \dots = e_k' = 0$, which shows that there exists no linear homogeneous relation between $g_1(z), \dots, g_{\mu+1}(z)$.

By dividing both sides of (21) by d_0 we have

$$F_1(z) + \dots + F_{\mu+1}(z) \equiv 1 \quad (23)$$

where $F_i(z) \equiv \frac{d_i}{d_0} g_i(z)$.

Considering r as a constant and regarding z as a variable, we proceed in the following to consider some functions of a variable z in a circle $|z| \leq \rho$ where ρ is an arbitrary constant.

Now we assume that the algebroidal function $u(z)$ does not take the values $u_1, u_2, \dots, u_{\mu+1}$.

Now $F_1(z), \dots, F_{\mu+1}(z)$ do not take the value 0 in $|z| \leq \rho$, and any of $F_1(z), \dots, F_{\mu+1}(z)$ satisfy no linear homogeneous relation with constant coefficients.

Hence we have

$$D(z) \equiv \begin{vmatrix} F_1(z) & F_2(z) & \dots & F_{\mu+1}(z) \\ \frac{dF_1(z)}{dz} & \frac{dF_2(z)}{dz} & \dots & \frac{dF_{\mu+1}(z)}{dz} \\ \dots & \dots & \dots & \dots \\ \frac{d^\mu F_1(z)}{dz^\mu} & \frac{d^\mu F_2(z)}{dz^\mu} & \dots & \frac{d^\mu F_{\mu+1}(z)}{dz^\mu} \end{vmatrix} \neq 0.$$

Differentiating μ times the identity (23) we obtain the system of relations

$$\begin{aligned} F_1(z) &+ F_2(z) + \dots + F_{\mu+1}(z) && \equiv 1 \\ \frac{F_1'(z)}{F_1(z)} F_1(z) &+ \frac{F_2'(z)}{F_2(z)} F_2(z) + \dots + \frac{F_{\mu+1}'(z)}{F_{\mu+1}(z)} F_{\mu+1}(z) && \equiv 0 \\ \dots & \dots & \dots & \dots \\ \frac{F_1^{(\mu)}(z)}{F_1(z)} F_1(z) &+ \frac{F_2^{(\mu)}(z)}{F_2(z)} F_2(z) + \dots + \frac{F_{\mu+1}^{(\mu)}(z)}{F_{\mu+1}(z)} F_{\mu+1}(z) && \equiv 0. \end{aligned} \quad (24)$$

Putting

$$\Delta(z) \equiv \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{F_1'}{F_1} & \frac{F_2'}{F_2} & \dots & \frac{F_{\mu+1}'}{F_{\mu+1}} \\ \dots & \dots & \dots & \dots \\ \frac{F_1^{(\mu)}}{F_1} & \frac{F_2^{(\mu)}}{F_2} & \dots & \frac{F_{\mu+1}^{(\mu)}}{F_{\mu+1}} \end{vmatrix} \equiv \frac{D(z)}{F_1 \dots F_{\mu+1}},$$

we obtain

$$F_i(z) = \frac{\Delta_i(z)}{\Delta(z)},$$

where $\Delta_i(z)$ denotes the minor determinant of $\Delta(z)$ with respect to the i -th element of the first line.

By the notations introduced by Nevanlinna⁽⁵⁴⁾ for a meromorphic function $f(z)$ in $|z| \leq \rho$ we have

$$m(\rho, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(\rho e^{i\theta})| d\theta$$

$$N(\rho, f) = \int_0^\rho \frac{n(t, f)}{t} dt,$$

where $n(t, f)$ denotes the number of poles of $f(z)$ in $|z| < t$.

By a well-known relations for positive number $a, b, \dots, c, d$,

$$\log^+(a \cdot b \cdot \dots \cdot d) \leq \log^+ a + \log^+ b + \dots + \log^+ d,$$

$$\log^+(\underbrace{a + b + \dots + d}_m) \leq \log^+ a + \log^+ b + \dots + \log^+ d + \log m,$$

we have

$$m(\rho, F_i) \leq m(\rho, \Delta_i) + m\left(\rho, \frac{1}{\Delta}\right). \quad (25)$$

Adding (25) for $\mu + 1$ functions F_i , ($i = 1, 2, \dots, \mu + 1$),

$$\sum_{i=1}^{\mu+1} m(\rho, F_i) \leq \sum_{i=1}^{\mu+1} m(\rho, \Delta_i) + (\mu + 1) m\left(\rho, \frac{1}{\Delta}\right).$$

Putting

$$\begin{aligned} \sum_{i=1}^{\mu+1} m(\rho, F_i) &= m(\rho), \\ m(\rho) &\leq \sum_{i=1}^{\mu+1} m(\rho, \Delta_i) + (\mu + 1) m\left(\rho, \frac{1}{\Delta}\right). \end{aligned} \quad (26)$$

Now $F_1, \dots, F_{\mu+1}$ and $D(z)$ are of course continuous functions of z , not identically zero, hence we can find a point β , which will be determined on the page 273, sufficiently near to the origin $z = 0$, at which

$$\Delta(\beta) \neq 0, F_1(\beta) \neq 0, \dots, F_{\mu+1}(\beta) \neq 0 \quad (27)$$

where $|\beta| = \rho_0 < \rho$.

⁽⁵⁴⁾ R. Nevanlinna: Theorie der meromorphen Funktionen. Acta mathematica **46** (1925) 1-99.

By a formula⁽⁵⁵⁾ used by Bloch and H. Cartan deduced from Jensen-Nevanlinna's formula for a meromorphic function $f(z)$ in $|z| \leq \rho$

$$\log |f(z)| \leq \frac{\rho + \rho_0}{\rho - \rho_0} m(\rho, f) - \frac{\rho - \rho_0}{\rho + \rho_0} m\left(\rho, \frac{1}{f}\right) - \sum_i \log \left| \frac{\rho^2 - \bar{a}_i z}{\rho(z - a_i)} \right| + \sum_j \log \left| \frac{\rho^2 - \bar{b}_j z}{\rho(z - b_j)} \right| \quad (28)$$

where a_i denote the zeros of $f(z)$, b_j the poles of $f(z)$ in $|z| < \rho$, we have, $\Delta(z)$ being holomorphic in $|z| \leq \rho$, for $|\beta| = \rho_0 < \rho$,

$$m\left(\rho, \frac{1}{\Delta}\right) \leq \left(\frac{\rho + \rho_0}{\rho - \rho_0}\right)^2 m(\rho, \Delta) + \left(\frac{\rho + \rho_0}{\rho - \rho_0}\right) \log^+ \frac{1}{|\Delta(\beta)|}. \quad (29)$$

Hence

$$m(\rho) < \sum_{i=1}^{\mu+1} m(\rho, \Delta_i) + (\mu + 1) \left\{ \left(\frac{\rho + \rho_0}{\rho - \rho_0}\right)^2 m(\rho, \Delta) + \left(\frac{\rho + \rho_0}{\rho - \rho_0}\right) \log^+ \frac{1}{|\Delta(\beta)|} \right\}. \quad (30)$$

Now for a meromorphic function $f(z)$

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{2\rho'}{(\rho' - \rho)^2} \left[m(\rho', f) + m\left(\rho', \frac{1}{f}\right) \right],$$

for $|z| = \rho < \rho'$.

In general

$$\left| \frac{f^{(n)}(z)}{f(z)} \right| \leq P_n \left[\rho', \frac{1}{\rho' - \rho}, m(\rho', f) + m\left(\rho', \frac{1}{f}\right) \right]$$

where P_n is a polynomial with respect to the three arguments, whose coefficients are positive constants depending only on n .

Now $\Delta(z)$ and $\Delta_i(z)$ being polynomials of $\frac{F'_i}{F_i}, \dots, \frac{F_i^{(\mu)}}{F_i}, (i = 1, 2, \dots, \mu + 1)$ we can apply the above formulae to

$$\left| \frac{F'_i}{F_i} \right|, \left| \frac{F''_i}{F_i} \right|, \dots, \left| \frac{F_i^{(\mu)}}{F_i} \right|.$$

⁽⁵⁵⁾ We follow here after Nevanlinna's deduction and also after H. Cartan's.

Nevanlinna: Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen. Acta mathematica **48** (1928) 367-391.

H. Cartan: Sur les systèmes de fonctions holomorphes a variétés linéaires lacunaires et leurs applications. Annales scientifiques de l'école normale supérieure, Ser. 3. **45** (1928) 255-346.

Here we consider of course ρ' as a constant, and z as a variable.

Thus

$$|\Delta_i(z)| \leq P_i \left[\rho', \frac{1}{\rho' - \rho}, m(\rho', F_i) + m\left(\rho', \frac{1}{F_i}\right), (i = 1, 2, \dots, \mu + 1) \right]$$

$$|\Delta(z)| \leq P \left[\rho', \frac{1}{\rho' - \rho}, m(\rho', F_i) + m\left(\rho', \frac{1}{F_i}\right), (i = 1, 2, \dots, \mu + 1) \right]$$

P_i and P denoting polynomials of $\mu + 1 + 2$ arguments whose coefficients are positive constants depending only on μ .

Thus we have

$$\log^+ |\Delta_i(z)| \leq P_i' \left[\log^+ \rho', \log^+ \frac{1}{\rho' - \rho}, \log^+ \left(m(\rho', F_i) + m\left(\rho', \frac{1}{F_i}\right) \right), \right. \\ \left. (i = 1, 2, \dots, \mu + 1) \right] \quad (31)$$

$$\log^+ |\Delta(z)| \leq P' \left[\log^+ \rho', \log^+ \frac{1}{\rho' - \rho}, \log^+ \left(m(\rho', F_i) + m\left(\rho', \frac{1}{F_i}\right) \right), \right. \\ \left. (i = 1, 2, \dots, \mu + 1) \right]$$

where P_i' and P' denote linear functions of $\mu + 1 + 2$ arguments, whose coefficients are positive constants depending only on μ .

Taking the mean values of the both sides of (31), and substituting them in (30) we have

$$m(\rho) < c_1 \sum_{i=1}^{\mu+1} \log^+ \left(m(\rho', F_i) + m\left(\rho', \frac{1}{F_i}\right) \right) + c_2 \log^+ \rho' + c_3 \log^+ \frac{1}{\rho' - \rho} \\ + c_4 + c_0 \left(\frac{\rho + \rho_0}{\rho - \rho_0} \right) \log^+ \frac{1}{|\Delta(\beta)|}. \quad (32)$$

where

$$c_i = c_i' + c_i'' \left(\frac{\rho + \rho_0}{\rho - \rho_0} \right)^2, \quad (i = 1, 2, 3, 4),$$

$c_0, c_1', c_1'', \dots, c_4', c_4''$ depending only on μ .

On the other hand by Nevanlinna's formula (28) we have, F_i being regular in $|z| < \rho'$

$$m\left(\rho', \frac{1}{F_i}\right) \leq \left(\frac{\rho' + \rho_0}{\rho' - \rho_0} \right)^2 m(\rho', F_i) + \left(\frac{\rho' + \rho_0}{\rho' - \rho_0} \right) \log^+ \frac{1}{|F_i(\beta)|}.$$

Hence substituting in (32) we have

$$\begin{aligned}
 m(\rho) \leq & c_5 \sum_{i=1}^{\mu+1} \log^+ m(\rho', F_i) + c_6 \log^+ \frac{\rho' + \rho_0}{\rho' - \rho_0} \\
 & + c_7 \sum_{i=1}^{\mu+1} \log^+ \left[\log^+ \frac{1}{|F_i(\beta)|} \right] + c_8 + c_2 \log^+ \rho' \\
 & + c_3 \log^+ \frac{1}{\rho' - \rho} + c_4 + c_0 \left(\frac{\rho + \rho_0}{\rho - \rho_0} \right) \log^+ \frac{1}{|\Delta(\beta)|}. \quad (33)
 \end{aligned}$$

where

$$c_i = c_i' + c_i'' \left(\frac{\rho + \rho_0}{\rho - \rho_0} \right)^2 \quad (i = 5, 6, 7, 8).$$

Now

$$F_i(z) = \frac{d_i}{d_0} g_i(z).$$

Hence

$$\log^+ \frac{1}{|F_i(\beta)|} \leq \log^+ \frac{1}{|d_i|} + \log^+ |d_0| + \log^+ \frac{1}{|g_i(\beta)|} + \log 3.$$

Since $g_i(\beta) = u_i^v + A_1(\beta)u_i^{v-1} + \dots + A_v(\beta)$ and $g_i(z)$ does not take the value zero in $|z| \leq \rho$, $g_i(\beta) \neq 0$ and $g_i(\beta)$ depends on β and u_i , and the absolute value of $g_i(\beta)$ increases indefinitely when $|u_i|$ increases.

Now

$$\begin{aligned}
 \Delta(x) &\equiv \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{F_1'}{F_1} & \frac{F_2'}{F_2} & \dots & \frac{F_{\mu+1}'}{F_{\mu+1}} \\ \dots & \dots & \dots & \dots \\ \frac{F_1^{(\mu)}}{F_1} & \frac{F_2^{(\mu)}}{F_2} & \dots & \frac{F_{\mu+1}^{(\mu)}}{F_{\mu+1}} \end{vmatrix} \equiv \begin{vmatrix} 1 & 1 & \dots & 1 \\ \frac{g_1'}{g_1} & \frac{g_2'}{g_2} & \dots & \frac{g_{\mu+1}'}{g_{\mu+1}} \\ \dots & \dots & \dots & \dots \\ \frac{g_1^{(\mu)}}{g_1} & \frac{g_2^{(\mu)}}{g_2} & \dots & \frac{g_{\mu+1}^{(\mu)}}{g_{\mu+1}} \end{vmatrix} \\
 &= \frac{1}{g_1 \dots g_{\mu+1}} \begin{vmatrix} g_1 & g_2 & \dots & g_{\mu+1} \\ g_1' & g_2' & \dots & g_{\mu+1}' \\ \dots & \dots & \dots & \dots \\ g_1^{(\mu)} & g_2^{(\mu)} & \dots & g_{\mu+1}^{(\mu)} \end{vmatrix}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Here } g_1 &= p_0(u_1) + p_1(u_1)B_1(z) + \dots + p_\mu(u_1)B_\mu(z), \dots, \\
 g_{\mu+1} &= p_0(u_{\mu+1}) + p_1(u_{\mu+1})B_1(z) + \dots + p_\mu(u_{\mu+1})B_\mu(z), \\
 g_1' &= p_1(u_1)B_1'(z) + \dots + p_\mu(u_1)B_\mu'(z), \dots, \\
 g_{\mu+1}' &= p_1(u_{\mu+1})B_1'(z) + \dots + p_\mu(u_{\mu+1})B_\mu'(z), \\
 &\dots \dots \dots \\
 g_1^{(\mu)} &= p_1(u_1)B_1^{(\mu)}(z) + \dots + p_\mu(u_1)B_\mu^{(\mu)}(z), \dots, \\
 g_{\mu+1}^{(\mu)} &= p_1(u_{\mu+1})B_1^{(\mu)}(z) + \dots + p_\mu(u_{\mu+1})B_\mu^{(\mu)}(z).
 \end{aligned}$$

Hence

$$\begin{aligned}
 g_1 \cdots g_{\mu+1} \Delta(x) &\equiv \begin{vmatrix} p_0(u_1) & p_1(u_1) & \cdots & p_\mu(u_1) \\ p_0(u_2) & p_1(u_2) & \cdots & p_\mu(u_2) \\ \cdots & \cdots & \cdots & \cdots \\ p_0(u_{\mu+1}) & p_1(u_{\mu+1}) & \cdots & p_\mu(u_{\mu+1}) \end{vmatrix} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ B_1(z) & B_1'(z) & \cdots & B_1^{(\mu)}(z) \\ B_2(z) & B_2'(z) & \cdots & B_2^{(\mu)}(z) \\ \cdots & \cdots & \cdots & \cdots \\ B_\mu(z) & B_\mu'(z) & \cdots & B_\mu^{(\mu)}(z) \end{vmatrix} \\
 &= d_0 \begin{vmatrix} B_1'(z) & \cdots & B_1^{(\mu)}(z) \\ \cdots & \cdots & \cdots \\ B_\mu'(z) & \cdots & B_\mu^{(\mu)}(z) \end{vmatrix}.
 \end{aligned}$$

Since $\Delta(z) \neq 0$ by hypothesis, we can determine now β so that

$$\begin{vmatrix} B_1'(\beta) & \cdots & B_1^{(\mu)}(\beta) \\ \cdots & \cdots & \cdots \\ B_\mu'(\beta) & \cdots & B_\mu^{(\mu)}(\beta) \end{vmatrix} = B_0 \neq 0,$$

where β is dependent only on $u(z)$, but not on $u_1, u_2, \dots, u_{\mu+1}$.

For such a value of β we have (if $\rho \geq \rho_0 + 1$, $\rho' \geq \rho_0 + 1$) $c_1 = c_1'''$, $c_2 = c_2'''$, $\dots$, $c_7 = c_7'''$, $c_8 = c_8'''$, where c_1''' , $\dots$, c_8''' depend only on μ and β .

Thus we obtain

$$\begin{aligned}
 m(\rho) &\leq c_9 \sum_{i=1}^{\mu+1} \log^+ m(\rho', F_i) + c_{10} \sum_{i=1}^{\mu+1} \log^+ \left[\log^+ \frac{1}{|F_i(\beta)|} \right] \\
 &+ c_{11} + c_{12} \log \rho' + c_{13} \log^+ \frac{1}{\rho' - \rho} + c_{14} \log^+ \frac{1}{|\Delta(\beta)|} \quad (34)
 \end{aligned}$$

$$\begin{aligned}
 &\leq c_{15} + c_{16} \log \rho' + c_{17} \log^+ \frac{1}{\rho' - \rho} + c_{18} \sum_{i=1}^{\mu+1} \log^+ m(\rho', F_i) \\
 &+ c_{19} \log^+ \frac{1}{|d_0|} + c_{20} \log^+ \frac{1}{|B_0|} + c_{21} \log^+ |g_1(\beta) \cdots g_{\mu+1}(\beta)| \\
 &+ c_{22} \sum_{i=1}^{\mu+1} \log^+ \log^+ \frac{1}{|d_i|} + c_{23} \sum_{i=1}^{\mu+1} \log^+ \log^+ |d_0| + c_{24} \sum_{i=1}^{\mu+1} \log^+ \log^+ \frac{1}{|g_i(\beta)|} \quad (35)
 \end{aligned}$$

where c depend only on μ and β , but not on $\rho, \rho', u_1, u_2, \dots, u_{\mu+1}$.

Putting

$$\Omega(u_1, \dots, u_{\mu+1}) = c_{19} \log^+ \frac{1}{|d_0|} + c_{20} \log^+ \frac{1}{|B_0|} + c_{21} \log^+ |g_1(\beta) \cdots g_{\mu+1}(\beta)|$$

$$+ c_{22} \sum_{i=1}^{\mu+1} \log \log^+ \frac{1}{|d_i|} + c_2 \log \log^+ |d_0| + c_{24} \sum_{i=1}^{\mu+1} \log \log^+ \frac{1}{|g_i(\beta)|}$$

we have, hereafter c 's denote some constants depending only on μ , ν and β ,

$$m(\rho) \leq \Omega(u_1, \dots, u_{\mu+1}) + c_{15} + c_{16} \log \rho' + c_{17} \log^+ \frac{1}{\rho' - \rho} + c_{18} \sum_{i=1}^{\mu+1} \log^+ m(\rho', F_i). \quad (36)$$

Now from the relations

$$F_i(z) = \frac{d_i}{d_0} g_i(z) \quad \text{and} \quad g_i(z) = \frac{d_0}{d_i} F_i(z)$$

we have

$$\log^+ |F_i(z)| \leq \log^+ \left| \frac{d_i}{d_0} \right| + \log^+ |g_i(z)| + \log 2. \quad (37)$$

$$\log^+ |g_i(z)| \leq \log^+ \left| \frac{d_0}{d_i} \right| + \log^+ |F_i(z)| + \log 2.$$

Hence putting

$$\overline{m}(\rho) = \sum_{i=1}^{\mu+1} m(\rho, g_i)$$

we have

$$\overline{m}(\rho) \leq c_{25} \sum_{i=1}^{\mu+1} \log^+ \left| \frac{d_0}{d_i} \right| + m(\rho) + c_{26}$$

and hence from (36) and (37)

$$\begin{aligned} \overline{m}(\rho) &\leq c_{25} \sum_{i=1}^{\mu+1} \log^+ \left| \frac{d_0}{d_i} \right| + c_{27} + c_{28} \sum_{i=1}^{\mu+1} \log^+ \left| \frac{d_i}{d_0} \right| + \Omega(u_1, \dots, u_{\mu+1}) \\ &\quad + c_{16} \log \rho' + c_{17} \log^+ \frac{1}{\rho' - \rho} + c_{18} \sum_{i=1}^{\mu+1} \log^+ m(\rho', g_i). \end{aligned} \quad (38)$$

Now by the relation

$$p_0(u_{\mu+1}) + p_1(u_{\mu+1})B_1(z) + \dots + p_\mu(u_{\mu+1})B_\mu(z) = g_{\mu+1}(z)$$

and the other μ relations

$$\begin{aligned} p_0(u_i) + p_1(u_i)B_1(z) + \dots + p_\mu(u_i)B_\mu(z) &= g_i(z), \\ i &= 1, 2, \dots, \mu, \end{aligned} \quad (39)$$

and their inverse relations

$$B_i(z) = \sum_{j=1}^{\mu} \frac{e_{ij}}{e_0} (g_j(z) - p_0(u_j)), \quad i = 1, 2, \dots, \mu, \quad (40)$$

which can always be determined, for, $e_0 \neq 0$ as we have given before, that is to say,

$$e_0 = d_1 = \begin{vmatrix} p_1(u_1) & p_2(u_1) & \cdots & p_\mu(u_1) \\ p_1(u_2) & p_2(u_2) & \cdots & p_\mu(u_2) \\ \cdots & \cdots & \cdots & \cdots \\ p_1(u_\mu) & p_2(u_\mu) & \cdots & p_\mu(u_\mu) \end{vmatrix} \neq 0,$$

we must have

$$\begin{aligned} \overline{m}(\rho) &\leq c_{38} \Omega(u_1, \dots, u_{\mu+1}) + c_{39} \Omega_1(u_1, \dots, u_{\mu+1}) \\ &+ \Omega_2(u_1, \dots, u_{\mu+1}) + c_{40} \log \rho' + c_{41} \log^+ \frac{1}{\rho' - \rho} + c_{42} \sum_{i=1}^{\mu} \log^+ m(\rho', B_i), \quad (41) \end{aligned}$$

where

$$\Omega_1(u_1, \dots, u_{\mu+1}) = c_{25} \sum_{i=1}^{\mu+1} \left\{ \log^+ |d_0| + \log^+ \left| \frac{1}{d_i} \right| \right\} + c_{28} \sum_{i=1}^{\mu+1} \left\{ \log^+ |d_i| + \log^+ \left| \frac{1}{d_0} \right| \right\} \quad (42)$$

$$\begin{aligned} \Omega_2(u_1, \dots, u_{\mu+1}) &= c_{34} + c_{35} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} \left\{ \log^+ |e_{ij}| + \log^+ \left| \frac{1}{e_0} \right| \right\} + c_{36} \sum_{j=1}^{\mu} \log^+ |p_0(u_j)| \\ &+ c_{37} \sum_{i=1}^{\mu+1} \sum_{j=0}^{\mu} \log^+ |p_j(u_i)|, \end{aligned}$$

$$\text{and } \overline{m}(\rho) = \sum_{i=1}^{\mu} m(\rho, B_i).$$

For, by (38) and (39) we have

$$m(\rho', g_i) \leq c_{29} + c_{30} \sum_{j=0}^{\mu} \log^+ |p_j(u_i)| + \sum_{j=1}^{\mu} m(\rho', B_j)$$

and by (40)

$$\begin{aligned} m(\rho, B_i) &\leq c_{31} + c_{32} \sum_{j=1}^{\mu} \log^+ \left| \frac{e_{ij}}{e_0} \right| + c_{33} \sum_{j=1}^{\mu} \log^+ |p_0(u_j)| \\ &+ \sum_{i=1}^{\mu} m(\rho, g_i). \end{aligned}$$

Putting

$$\begin{aligned} \varpi(u_1, \dots, u_{\mu+1}) &= c_{38} \Omega(u_1, \dots, u_{\mu+1}) \\ &+ c_{39} \Omega_1(u_1, \dots, u_{\mu+1}) + \Omega_2(u_1, \dots, u_{\mu+1}) \end{aligned}$$

and denoting by $M(\rho)$ the maximum of $m(\rho, B_i)$ for $i = 1, 2, \dots, \mu$, we have, by

$$M(\rho) \leq \overline{m}(\rho)$$

and

$$m(\rho', B_i) \leq M(\rho'),$$

$$M(\rho) \leq \varpi(u_1, \dots, u_{\mu+1}) + c_{40} \log \rho' + c_{41} \log^+ \frac{1}{\rho' - \rho} + c_{43} \log^+ M(\rho').$$

Since $m(\rho, B_i)$ increases when ρ increases, $M(\rho)$ is an increasing function of ρ .

Now taking

$$\rho' = \rho + \frac{1}{\log M(\rho)}$$

we have for such ρ as $\log M(\rho) > 1$

$$\begin{aligned} M(\rho) \leq \varpi(u_1, \dots, u_{\mu+1}) + c_{40} \log^+ \rho + c_{41} \log^+ \log M(\rho) \\ + c_{43} \log^+ M\left(\rho + \frac{1}{\log M(\rho)}\right). \end{aligned}$$

Now consider a fixed value r of ρ , such as $r > \rho_0 + 1$, and $M(r) > 1$. Then

$$M(r) \leq \varpi(u_1, \dots, u_{\mu+1}) + c_{40} \log r + c_{41} \log^+ M\left(r + \frac{1}{\log M(r)}\right). \quad (43)$$

Now we must consider $\varpi(u_1, \dots, u_{\mu+1})$.

For sufficiently large values of u_1 satisfying the conditions for which (7) holds good, we have

$$\text{all } |g_1(\beta)|, |g_2(\beta)|, \dots, |g_{\mu+1}(\beta)| > K,$$

and by (18) (21)

$$|d_0| > \eta$$

and by (19) (19*) ... (19**) and (21)

$$|d_i| > \eta$$

and

$$|e_0| = |d_1| > \eta.$$

where K is a fixed constant.

Hence

$$\Omega(u_1, \dots, u_{\mu+1}) < K_1 + K_2 \log^+ \log^+ |d_0| + K_3 \log^+ |g_1(\beta) \dots g_{\mu+1}(\beta)|.$$

Similarly

$$\Omega_1(u_1, \dots, u_{\mu+1}) < K_4 + \sum_{j=0}^{\mu} K'_j \log^+ |d_j|$$

and

$$\Omega_2(u_1, \dots, u_{\mu+1}) < K_5 + \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} K_{ij} |e_{ij}| + \sum_{i=1}^{\mu+1} \sum_{j=0}^{\mu} K'_{ij} \log^+ |p_j(u_i)|,$$

where

$$e_{ij} = (-1)^l \begin{vmatrix} p_1(u_1) & \dots & p_{i-1}(u_1) & p_{i+1}(u_1) & \dots & p_{\mu}(u_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_1(u_{j-1}) & \dots & p_{i-1}(u_{j-1}) & p_{i+1}(u_{j-1}) & \dots & p_{\mu}(u_{j-1}) \\ p_1(u_{j+1}) & \dots & p_{i-1}(u_{j+1}) & p_{i+1}(u_{j+1}) & \dots & p_{\mu}(u_{j+1}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_1(u_{\mu}) & \dots & p_{i-1}(u_{\mu}) & p_{i+1}(u_{\mu}) & \dots & p_{\mu}(u_{\mu}) \end{vmatrix},$$

l being properly chosen integers and $K_1, K_2, \dots, K'_j, \dots, K_{ij}, \dots, K'_{ij}, \dots$ are all fixed positive constants depending only on μ and β .

Now as we have shown before we can take u_i , ($i = 1, 2, \dots, \mu + 1$) such that they are some polynomials with respect to $|u_1|$, the coefficients being fixed positive constants.

Hence by choosing r so large that $r^{1+\delta}$, $\delta > 0$, is larger than the above fixed value $|u_1|$, we can apply the above argument.

Since for every large value of r we can take $r^{1+\delta}$, $\phi_1(r^{1+\delta}), \dots, \phi_{\mu}(r^{1+\delta})$ for $|u_1|, |u_2|, \dots, |u_{\mu+1}|$ respectively, $\phi_1, \phi_2, \dots, \phi_{\mu}$ being some polynomials, $d_0, d_1, \dots, d_{\mu}$, $e_{11}, e_{12}, \dots, e_{21}, e_{22}, \dots, e_{\mu\mu}$ and $p_0(u_1), \dots, p_0(u_{\mu+1})$, $p_1(u_1), \dots, p_{\mu}(u_{\mu+1})$, $g_1(\beta), \dots, g_{\mu+1}(\beta)$ are all polynomials with respect to $r^{1+\delta}$ for such $u_1, u_2, \dots, u_{\mu+1}$.

Hence $|d_0|, |d_i|, |e_{ij}|, |p_j(u_i)|, |g_i(\beta)|$ are all $= O(r^{\chi(\mu)})$

where $\chi(\mu)$ is dependent only on μ and ν but not on r .

Thus, if we assume that $u(z)$ does not take in $|z| \leq r$ some $\mu + 1$ values $u_1(r), u_2(r), \dots, u_{\mu+1}(r)$ whose absolute values are respectively equal to $r^{1+\delta}$, $\phi_1(r^{1+\delta}), \dots, \phi_{\mu}(r^{1+\delta})$ for sufficiently large values of r , then

$$M(r) \leq K'' \log r + c_{31} \log^+ M \left(r + \frac{1}{\log M(r)} \right), \quad (44)$$

for $\varpi(u_1, u_2, \dots, u_{\mu+1}) = O(\log r)$.

Now, since $\phi_{\mu}(r^{1+\delta}) > \phi_{\mu-1}(r^{1+\delta}) > \dots > r^{1+\delta}$, if the inequality (44) comes to a contradiction, then we can conclude

that the values of the function (1) must cover at least one of the $\mu + 1$ whole circumferences

$$|u(z)| = \phi_i(r^{1+\delta}), \quad i = 0, 1, 2, \dots, \mu, \text{ where } \phi_0(r^{1+\delta}) = r^{1+\delta},$$

in $|z| \leq r$.

For, fixing arbitrary μ values whose absolute values are respectively $\phi_i(r^{1+\delta})$, (i runs over any μ values among $0, 1, 2, \dots, \mu$), and assuming that the function (1) do not take such μ values in $|z| \leq r$, that is, that there exists a point on each of μ circumferences which is an exceptional value, the function (1) must take all the values lying on the circumference whose radius is the omitted value $\phi_i(r^{1+\delta})$ among $\mu + 1$ values.

Now since for every sufficiently large value of r the inequality (44) must hold good, consider such a sequence of infinitely increasing values of r belonging to ordinary values of r below mentioned.

Now since $M(r)$ increases monotonously, by Borel's theorem we have

$$M\left(r + \frac{1}{\log M(r)}\right) < M(r)^{1+\varepsilon}, \quad \varepsilon > 0, \quad (45)$$

except possibly for a set of intervals whose total length is finite.

We call "ordinary value" the set of intervals in which (45) holds good.

Thus we have for ordinary values of r

$$M(r) \leq O(\log r + \log M(r)). \quad (46)$$

Since $u(z)$ is a transcendental algebroidal function, there exists at least one of the functions $B_i(z)$, ($i = 1, 2, \dots, \mu$), which is a transcendental integral function.

By a well-known Nevanlinna's theorem we have

$$\lim_{r \rightarrow \infty} \frac{m(r, B_k(z))}{\log r} = \infty$$

for such a transcendental function $B_k(z)$.

Hence $M(r)$, which is larger than any $m(r, B_i)$ satisfies that

$$\lim_{r \rightarrow \infty} \frac{M(r)}{\log r} = \infty.$$

Since $M(r)$ increases indefinitely when r increases (indefinitely), the

inequality (46) thus is evidently impossible, and the theorem is completely proved.⁽⁵⁶⁾

XI

If a function $z' = \Phi(z)$ satisfies $f(\Phi(z)) = f(z)$, then we have

$$f(\Phi(\Phi(z))) = f(\Phi(z)) = f(z),$$

where $\Phi(\Phi(z))$ is defined by eliminating z_1 between $g_i(z_1, z) = 0$ and $g_i(z', z_1) = 0$, $\Phi(z)$ being defined by $g_i(z', z) = 0$, and $g_i(z', z)$ an integral function of z' and z .

So similarly

$$f(\Phi_n(z)) = f(z) = f(\Phi_{-n}(z)) \quad n = 1, 2, \dots$$

where $\Phi_n(z)$ and $\Phi_{-n}(z)$ are defined by eliminating $z_1, z_2, \dots, z_{n-1}$ from $g_i(z_1, z) = 0, g_i(z_2, z_1) = 0, \dots, g_i(z', z_{n-1}) = 0$, and $z_1, z_2, \dots, z_{n-1}$ from $g_i(z, z_1) = 0, g_i(z_1, z_2) = 0, \dots, g_i(z_{n-1}, z') = 0$ respectively.

The n -th iterated function $\Phi_n(z)$ may or may not be equal to $\Phi(z)$.

Now let $\Phi(z)$ and $\Psi(z)$ be any two functions satisfying (1), then the functions $\Phi(\Psi(z))$ and $\Psi(\Phi(z))$ must satisfy (1), and hence the

⁽⁵⁶⁾ Selberg has extended Nevanlinna's theorems for an algebroidal function.

H. L. Selberg: Über die Wertverteilung der algebroiden Funktionen: Math. Zeit.

31 (1930) 709-728.

But his method is not easy to be applied for our purpose. For, we must evaluate $|u(z, a_1, \dots, a_\mu, \infty)|$ in his notation as a function of $|a_1| \dots |a_\mu|$ which plays an essential part in our proof. The other parts of his method are perfectly adaptable for our purpose.

By the relation $A_i(z) = \sum_{j=0}^u p_j(u) B_j(z)$, ($B_0(z) \equiv 1$), and the relations given by Valiron, H. Cartan, and Selberg

$$\frac{1}{\nu} m(r, A_j) - O(1) \leq m(r, u) \leq \frac{1}{\nu} \sum_{j=0}^{\nu} m(r, A_j) + O(1),$$

where $m(r, u) = \frac{1}{2\pi} \sum \int_{\Gamma_\nu}^+ \log |u(re^{i\theta})| d\theta$ in Selberg's notation, we can obtain a relation from (41) in this present paper

$$m(r, u) \leq \varpi(r) + c + c' \log r + c'' \log^+ \left(r + \frac{1}{m(r, u)}, u \right).$$

Thus we can prove that the values of the function (1) covers at least one of the circumferences $|u(z)| = \phi_i(m(r, u)^k)$, ($i = 0, 1, 2, \dots, \mu$) for any large but fixed k in $|z| \leq r$.

This is more precise than the theorem given above if the order of an algebroidal function is positive.

functions satisfying (1) have a group-property with respect to the iteration.

Since there exists in general an infinite number of functions $\Phi(z)$, $\Psi(z)$, in the decomposition of $f(z') = f(z)$ into factors, they are classified into cycles with respect to the iteration of independent functions, that is, the functions which are not iterated functions of any other functions.

Evidently the iteration forms a discontinuous group as the transformation of equivalent points for $f(z)$.

The fixed points of the iterated functions $\Phi_n(z)$, that is, the points α for which $\alpha = \Phi_n(\alpha)$ at least for a branch of $\Phi_n(z)$ are contained in a point set for which $f'(z) = 0$.

For, by

$$\frac{f(z') - f(z)}{z' - z} = g(z', z) \equiv 0 \quad (47)$$

$\Phi_n(z)$ must satisfy $g(\Phi_n(z), z) \equiv 0$.

This comes from the fact that $\Phi(z)$, which is defined by a factor $g_i(z', z) = 0$ of $g(z', z)$, gives, as the second iterated function, $\Phi_2(z)$ which takes at a point some of the equivalent points of $f(z)$, satisfying $g(z', z) \equiv 0$, and coincides in a certain domain with a function defined by some factor of $g(z', z)$.

Hence $\Phi_2(z)$ must satisfy $g(\Phi_2(z), z) \equiv 0$.

And similarly $\Phi_n(z)$ satisfies the same relation.

Now (47) being an identity with respect to z' and z , let z' tend to z , then the left hand side of (47) becomes $f'(z)$, and the right hand side becomes $g(z, z)$.

Hence any point α , for which $\alpha = \Phi_n(\alpha)$ must satisfy

$$g(\Phi_n(\alpha), \alpha) = f'(\alpha) = 0.$$

As examples: There exists only a finite number of fixed points for the iterated functions of $\Phi(z)$, satisfying $e^{P(z')} = e^{P(z)}$, where $P(z)$ is a polynomial, since $f'(z) = 0$ has only a finite number of roots.

The iterated functions have no fixed point for any function $\Phi(z)$, satisfying $\int_0^{z'} e^{P(z')} dz' = \int_0^z e^{P(z)} dz$, since $f'(z)$ has no zero.

The functions $z' - \left(\frac{\pi}{2} \pm n\pi\right) = -\left(z - \left(\frac{\pi}{2} \pm n\pi\right)\right)$ $n = 1, 2, \dots$

satisfying $\sin z' = \sin z$ have an infinite number of fixed points $\frac{\pi}{2} \pm n\pi$, $n = 0, 1, 2, \dots$.

Now in order to consider some properties of algebraic functions satisfying (1) we shall consider the iteration of finite algebraic functions.

The theory of the iteration of rational functions is developed by Fatou, Julia and others recently.⁽⁵⁷⁾ On the other hand that of algebraic functions is little known. Since algebraic functions are many valued, the behaviours of the iteration even in a vicinity of a fixed point is not the same with those of the iteration of a uniform function.

Let $z_1 = \Phi(z)$ be an algebraic function of z defined by

$$P(z_1, z) = 0,$$

where $P(z_1, z)$ is a polynomial with respect to z_1 and z , and

$$b_1(z) = \alpha_{10} + \alpha_{11}(z - a)^{\lambda_1} + \dots$$

$$\dots$$

$$b_\nu(z) = \alpha_{\nu 0} + \alpha_{\nu 1}(z - a)^{\lambda_\nu} + \dots$$

be its ν -branches at a point a , where λ_i are all rational and positive.

We call as before such a point a (∞ included) at which at least one of the ν -branches has the expansion

$$b_\mu(z) = a + \alpha_{\mu 1}(z - a)^{\lambda_\mu} + \dots$$

a fixed point. We can take this point as the origin.

In a sufficiently small vicinity V_z of a fixed point 0 such that the function has no other branch points than the origin, we have the expansion of the form

$$z_1 = b(z) = \alpha_1 z^\lambda + \dots \tag{48}$$

for a branch $b(z)$ of $\Phi(z)$ which is zero at the origin.

We consider the iteration of $\Phi(z), \Phi_2(z), \dots, \Phi_n(z), \dots, \Phi_{-1}(z), \dots, \Phi_{-n}(z), \dots$ defined as before as in the general case.

We shall show that the branch points of the iterated functions $\Phi_n(z)$ are the antecedents, up to the rank $n - 1$ included, of the branch points of $\Phi(z)$, and those of $\Phi_{-n}(z)$ are the consequents of those of $\Phi_{-1}(z)$.

⁽⁵⁷⁾ Fatou: Sur les équations fonctionnelles I. II. III. Bull. Soc. Math. France **47-48** (1919-1920). Julia: Sur l'itérations rationnelles. Jour. d. math. pur. et. appl. Ser. 8. **1** (1918) 47-245.

For, let $P(z_1, z) = 0$ be the equation which defines $\Phi(z)$, and $c_1, c_2, \dots, c_k$ be the branch points of $z_1 = \Phi(z)$.

Now denote by $z_1 = \Phi(z)$, $z_2 = \Phi_2(z)$, $\dots$ the iterated functions.

By $P(z_j, z_{j-1}) = 0$, $z_j = \Phi_j(z)$ is defined if we put $z_{j-1} = \Phi_{j-1}(z)$ in it.

Hence the branch points of $z_j = \Phi_j(z)$ which do not belong to the branch points of $\Phi_{j-1}(z)$ belong to the roots of the equation $\Phi_{j-1}(z) = c_i$ $i = 1, 2, \dots, k$.

Putting $j = n, n-1, n-2, \dots, 3, 2, 1$ (here we put $z_1 = \Phi(z)$, $z_0 = z$) we can at once see that the branch points of $\Phi_n(z)$ belong to the roots of $\Phi_{n-1}(z) = c_i$, $\Phi_{n-2}(z) = c_i, \dots, \Phi_2(z) = c_i$, $\Phi(z) = c_i$, or $z = c_i$. ($i = 1, 2, \dots, k$).

The roots of the above equations being the antecedents of the branch points of $\Phi(z)$, up to the $n-1$ -th rank included, the former part of the theorem is proved.

The latter part can be similarly proved.

Hence if the branch points of the n -th iterated function $b_n(z)$ of the element $b(z)$ in V_z accumulate in an infinite number to the origin, then they are the antecedents of the branch points of $\Phi(z)$.

Throughout the following cases we exclude the case in which the branch points of $b_n(z)$ or of $b_{-n}(z)$ accumulate to the origin.

(I) If $0 < \lambda < 1$ in (48). Then we consider the inverse function of the branch $b(z)$ in V_z of the origin, whose value is zero at 0.

We choose a vicinity V_{z_1} of the origin for the inverse function such that it does not contain other branch points than the origin and it consists of a domain which is common to all the branches of the inverse function of the branch $b(z)$ in V_z , which can be obtained by the finiteness of $b(z)$.

Thus in V_{z_1} we have

$$z = b_{-1}(z_1) = \beta_1 z_1^\mu + \dots \quad \text{and } \mu > 1.$$

We can reduce this case (I) to the case (II) by considering $b_{-1}(z_1)$ instead of $b(z)$ within V_{z_1} .

(II) $\lambda > 1$. We consider the value of the differential coefficient of $b(z)$ at the origin.

If the branch points of $b_n(z)$ do not accumulate to the origin, then there exists a sufficiently small vicinity V_{z_0} of the origin within which no branch point of $b_n(z)$ appears, and we have the expansion

$$z_n = b_n(z) = \alpha^{1+\lambda+\dots+\lambda^{n-1}} z^{\lambda^n} + \dots \quad \text{in } V_{z_0}$$

for $n = 1, 2, \dots$.

Now in V_{z_0} within V_z we have

$$\left| \frac{b(z)}{z} \right| < \eta, \quad \eta < 1,$$

for any values of the branches of $b(z)$ in V_{z_0} .

Thus within V_{z_0} we have

$$\left| \frac{b_n(z)}{z} \right| < \eta^n.$$

Thus there exists in this case an infinite number of consequents of a point sufficiently near to the origin in V_{z_0} accumulating to 0.

(III) If $\lambda = 1$, we have to distinguish several cases.

Now let V_z be a vicinity of the origin in which all $b_n(z)$, $n = 1, 2, \dots$ have no branch point other than the origin, then we have

$$z_n = b_n(z) = \alpha_n z + \dots \quad \text{in } V_z.$$

(IIIa) $|\alpha_1| > 1$. We can reduce this case to the case (IIIb) by considering the inverse function of $b(z)$.

(IIIb) $|\alpha_1| < 1$. In this case we have in a sufficiently small vicinity of the origin

$$\left| \frac{b(z)}{z} \right| < \eta, \quad \eta < 1.$$

Thus we have again under the assumption that there exists no branch point other than the origin in V_{z_0}

$$\left| \frac{b_n(z)}{z} \right| < \eta^n.$$

Hence in the cases (IIIa) and (IIIb) an infinite number of consequents or antecedents of a point in some vicinity of the origin converges to the origin.

(IIIc) $|\alpha_1| = 1$ and $\alpha_1 = e^{i\theta\pi}$, θ being an irrational number.

We consider the inverse function

$$z = b_{-1}(z_1) = e^{-i\theta\pi} z_1 + \dots$$

within V_{z_1} , and assume that there exists a vicinity V_{z_0} in which $z_n = b_n(z)$ and $z_{-n} = b_{-n}(z)$, $n = 1, 2, \dots$ have no branch point other than the origin in V_{z_0} .

We will first give a lemma which will be used.

Lemma. If $z_1 = \alpha z + \dots$, $\alpha \neq 0$, is p -valued within a vicinity V_{z_0} , then the inverse function $z = \frac{1}{\alpha} z_1 + \dots$ is also p -valued within some vicinity V_{z_1} under the condition that there exists no other branch point than the origin for $z_1 = \alpha z + \dots$ and for its inverse function within it.

For, let

$$z_1 = \alpha z + \beta z^\lambda + \dots, \quad \lambda > 1,$$

be p -valued in V_{z_0} .

Then by putting

$$Z_1 = z_1^{\frac{1}{p}}, \quad Z = z^{\frac{1}{p}},$$

we have

$$z_1 = Z^p \{ \alpha + \beta Z^{p\lambda - p} + \dots \}.$$

In a small vicinity $|Z_1| < \delta$,

$$Z = Z_1 \{ a + b Z_1^\mu + \dots \},$$

Thus

$$Z^p = Z_1^p \{ c' + d' Z_1^{\nu'} + \dots \}.$$

There must exist in $\{ \dots \}$ at least a term $z_1^{\frac{\nu}{p}}$, which is p -valued, or else the inverse function $z_1 = \alpha z + \dots$ would be less than p -valued.

Hence

$$z = c' z_1 + d' z_1^\eta + \dots, \quad \eta > 1.$$

Therefore z is a p -valued function of z_1 in V_{z_1} .

Now consider the function (48), that is,

$$z_1 = e^{i\theta\pi} z + bz^\gamma + \dots, \quad \gamma > 1.$$

Assume that (48) is m -valued in a vicinity of the origin.

Since we can transform the two variables z_1, z by the same uniform function $z_1 = s(y_1)$, $z = s(y)$ into the two variables y_1, y in the process of iteration, the results being the same, we put

$$z_1 = y_1^m, \quad \text{and} \quad z = y^m,$$

and we have

$$y_1^m = e^{i\theta\pi} y^m + by^{m'} + \dots, \quad m' > m \text{ and } m' \text{ being integers.}$$

Hence we have m functions in a vicinity of the origin

$$y_{1j} = e^{\frac{i\pi j}{m} + \frac{i2\pi j}{m}} y(1 + b'y^{m''} + \dots)$$

$$= e^{\frac{i(\theta+2)\pi j}{m}} y(1 + h_j(y)),$$

$j = 1, 2, \dots, m$, and $h_j(y)$ being regular and $h_j(0) = 0$, which are regular in a vicinity of the origin.

These m functions $y_{11}, y_{12}, \dots, y_{1m}$ can be used for our iteration.

Similarly we have m^2 functions

$$y_{2jk} = e^{i\left\{\frac{(\theta+2)\pi}{m}j + \frac{(\theta+2)\pi}{m}k\right\}} y\{1 + h_{jk}(y)\},$$

when the iteration is performed twice, and so on.

Now consider the case (IIIc).

Since z_1 is in general a many valued function, the iteration of $z_1 = e^{i\theta\pi}z + \dots$ is equivalent to that of the system of functions

$$y_{1j} = e^{i\frac{(\theta+2)\pi}{m}j} y + \dots \quad \text{for each } j = 1, 2, \dots, m.$$

Thus we obtain in general infinitely many different systems of iterated functions in the vicinity of the origin.

Consider any one of the systems of the iteration $y_{1j_1}(y_{1j_2} \dots (y_{1j_k}(y)) \dots)$, $j_1, j_2, \dots$ being some one of $1, 2, \dots, m$.

To this system there corresponds the iteration

$$y_1 = e^{i\theta_1\pi}y + \dots$$

$$y_2 = e^{i\theta_2\pi}y + \dots$$

$$\dots$$

$$y_n = e^{i\theta_n\pi}y + \dots$$

$$\dots$$

all $\theta_1, \theta_2, \dots, \theta_n, \dots$ being rational numbers different from each other.

Thus we have a family of functions $u_n = t_n(y) = \frac{y_n}{e^{i\theta_n\pi}} = y + \dots$ which are mono-valued (schlicht) within some vicinity V_y of the origin.

Hence we can apply Julia's research on this family of functions.

All the functions u_n have the derivative 1 at the origin and any of them do not take its values in V_y more than once.

Hence we can apply Koebe's theorem and obtain that when y describes the curve V , the boundary of V_y , the point u_n describes a closed curve u_n , whose shortest distance from the origin is larger than a fixed

constant $d > \frac{\sqrt{2}-1}{4}\rho$, ρ being the radius of a circle contained in V_y .

Similarly the upper limit of $|u_n|$, which is independent of n , can be assured from the theorem on the mono-valued functions.

Hence we can conclude that the origin is within a region U which is simply connected, and interior to the common part of all U_n , $n = 1, 2, \dots$. U can be transformed bi-univoquely into itself by $u_n = t_n(y)$ and by the branch of $y = t_n^{-1}(u_n)$ which is zero at the origin.

Now U can be mapped conformally on the interior of a circle with the center at the origin by $y = s_{-1}(v)$.

The transformed relation $v_n = q(v)$ between v_n and v , where v_n corresponds to u_n will be the rotation $v_n = ve^{i\theta_n\pi}$, $n = 1, 2, \dots$ which, conserving all circles with the center at the origin, conserves an infinite number of analytic curves bounding and containing U .

We can find in the vicinity of the origin a regular function $v = s(y)$ of y such that the relation between v_n and v satisfies

$$v_n = e^{i\theta_n\pi}v.$$

Thus $s(y)$ which is regular at the origin satisfies the functional equation of Schröder $s(t_n(y)) = e^{i\theta_n\pi}s(y)$.

Inversely if it has a solution which is regular at the origin, all the points y near to the origin have all its consequents on the analytic curve which is derived from $|v| = \text{const.}$ by the conformal mapping $v = s(y)$, $s(0) = 0$, and on this curve the consequents lie everywhere dense like as the consequents of v by $v_n = e^{i\theta_n\pi}v$ on $|v| = \text{const.}$

For a specially chosen system of iteration we have $y_1 = e^{i\theta\pi}y + \dots$, $\dots$, $y_n = e^{in\theta\pi}y + \dots$ and the function $v_n = ve^{in\theta\pi}$. At least for such a system the consequents lie everywhere dense on a closed curve lying round the origin.

For each transformation $v_n = ve^{i\theta_n\pi}$ we have, by the transformation $y = s_{-1}(v)$, and then by $y^m = z$ and $y_1^m = z_1$, a sequence of transformed consequents by the iteration $b(z)$.

Since m is fixed, we have an everywhere dense set of consequents lying on some curves round the origin by the iteration of the function $b(z)$.

Thus in this case (IIIc) either the consequents of a point converges to the origin, or there exists an infinite number of consequents of a point having a closed curve as a limiting set.

(III_d) $|\alpha_1| = 1$ and $\alpha_1 = e^{i\theta\pi}$ where θ is a rational number, $\theta = \frac{q}{p}$.

Then we must distinguish two cases.

(A) For some iterated functions, which are of the following form

$$z_l = z + \dots \text{ or } z_{-l} = z + \dots,$$

we have

$$z_l \neq z \quad \text{or} \quad z_{-l} \neq z.$$

(B) For all iterated functions, which are of the following form

$$z_l = z + \dots \quad \text{and} \quad z_{-l} = z + \dots$$

we have

$$z_l \equiv z \quad \text{and} \quad z_{-l} \equiv z.$$

Since $z_j = e^{\frac{qj}{p}\pi} z + \dots$ we must have $z_l = z + \dots$ at least for the p_j -th iterated function of z_j .

If $z_l = z + az^\gamma + \dots$ and $z_l \neq z$ then we can apply the argument as in the case (III_c) and we must have the antecedents of a branch point of z_1 converging to the origin. For, otherwise, the functions

$$z_{nl} = z + na z^\gamma + \dots,$$

being all mono-valued and having the definite derivative at the origin, are limited in absolute values, and hence they form a normal family in some domain within V_z . Hence we can choose a sequence of functions which converges to a regular function. This is evidently impossible.

After Julia and Fatou it can be shown that the antecedents and consequents of a point within V_z accumulate to the origin.

For some iterated functions combined by taking from $z_{j_1}, z_{j_2}, \dots$ in another manner, we have $z_{j_1+j_2+\dots+j_k} = e^{\frac{qk}{pk}\pi} z + \dots$.

Here we may or may not have a function $z_l = z + \dots$.

In the former case the above argument holds good. In the latter case there occur an infinite number of times the functions of the form $z_l = e^{i\alpha\pi} z + \dots$ where α is the same for these functions.

Now consider the case (B).

If $z_l \equiv z$, then there exists only a finite number of different functions in the vicinity of the origin. Let $z_1, z_2, \dots, z_l$ be such functions, then $z_1, z_2, \dots, z_l$ form a cyclic group.

For such functions a vicinity of the origin in the z -plane can be divided into a finite number of parts, each of which is transformed into another by some $z_l = e^{i\theta\pi} z + \dots$.

Thus in order that there may exist only a finite number of consequents and antecedents of a point in a vicinity of a point on the z -plane

it is necessary that the functions in the iteration either have no fixed point other than ∞ , or have only fixed points of the property (III_d) (B).

Rausenberger has shown⁽⁵⁸⁾ that if there appears the function $z_1 \equiv z$ among the iterated functions of an algebraic function $z_1 = \Phi(z)$ in a vicinity of a fixed point, then the algebraic function must be of the form

$$z_1 = \psi^{-1} e^{i\theta\pi} \psi(z),$$

where ψ is an algebraic function and θ is a rational number and further $z_1 = \psi^{-1} e^{i\theta\pi} \psi(z)$ is the function defined by eliminating x between $P(z_1, x) = 0$ and $P\left(z, \frac{x}{e^{i\theta\pi}}\right) = 0$, P being a polynomial of the two variables.

Hence in order that there may exist only a finite number of consequents and antecedents of a point in any limited part of the plane for the iteration of a finite algebraic function it is necessary that either the function $\Phi(z)$ can be written in the form $z_1 = \psi^{-1} e^{i\theta\pi} \psi(z)$, or no iterated functions $\Phi_1(z)$, $\Phi_2(z)$, ..., $\Phi_n(z)$, ..., $\Phi_{-1}(z)$, $\Phi_{-2}(z)$, ..., $\Phi_{-n}(z)$, ... should have any fixed point other than ∞ .

We shall illustrate several cases by the following examples.

The function defined by $P(z_1) = P(z) + c$, $P(z)$ being a polynomial of z , and c a constant, has no fixed point (except ∞) for any iterated functions. An irreducible factor of $P(z_1) = P(z)$ defines an algebraic function which generates only a finite number of iterated functions, and hence it forms a finite discontinuous group.

The function defined by $\sqrt[m]{z_1} = \sqrt[m]{z} + 1$ shows us that though there exists only a finite number of consequents and antecedents about a fixed point $\bar{z} = \frac{1}{(1-\omega)^m}$ in the iteration of the element $z_1 - \bar{z} = e^{i\frac{2\pi}{m}}(z - \bar{z}) + \dots$ there appears an infinite number of consequents and antecedents in the whole z -plane. The function which is defined by $\sqrt{z_1} = \sqrt{z} + 1$ (or $\sqrt{z_1} - \frac{1}{2} = -\left(-\sqrt{z} - \frac{1}{2}\right)$, written in the form $\psi^{-1}(-\psi(z))$), forms a properly discontinuous group, while that by $\sqrt[5]{z_1} = \sqrt[5]{z} + 1$ does not form a discontinuous group.⁽⁵⁹⁾

⁽⁵⁸⁾ Rausenberger: Theorie der allgemeinen Periodicität. Math. Annalen, **18** (1881) 385-386.

⁽⁵⁹⁾ See. p. 294.

For the iteration of a finite algebraic function which forms a finite group in the whole plane we have the theorem that in order that there may appear only a finite number of different functions in the iteration of a finite algebraic function $z_1 = \Phi(z)$ when iterated in any manner, it is necessary and sufficient that $z_1 = \Phi(z)$ satisfies

$$P(z_1) = P(z),$$

where $P(z)$ is a polynomial of z .

If there exists only a finite number of iterated functions when iterated in any manner, then it is evident that there exists only a finite number of consequents and antecedents of a point, that is, $z_0, \Phi(z_0), \Phi_2(z_0), \dots, \Phi_m(z_0)$, for, $\Phi_i(z_j) = \text{some of } \Phi_k(z_0)$ by the finiteness of the number of different functions.

First it must be necessary,⁽⁶⁰⁾ for, let $z_{01}, z_{02}, \dots, z_{0k}$ be all the consequents and antecedents of a point z_0 by $\Phi_1(z), \Phi_2(z), \dots, \Phi_m(z)$ defined by $P_1(z_1, z) = 0, P_2(z_1, z) = 0, \dots, P_m(z_1, z) = 0$ respectively.

∞ being a fixed point for all the branches of $\Phi(z)$ we can take the coefficients of the highest degree of z_1 in $P_i(z_1, z) = 0$ as 1.

Let $P_1(z_1, z) = 0, \dots, P_m(z_1, z) = 0$ be all and all different functions which appear in the iteration, and choose z_0 such that the following equation has all simple roots.

$$\text{Then} \quad (z_1 - z_0)P_1(z_1, z_0) \cdots P_m(z_1, z_0) = 0 \quad (49)$$

expresses all $z_{01}, z_{02}, \dots, z_{0k}$.

Now consider one of $z_{01}, z_{02}, \dots, z_{0k}$, for example z_{01} , instead of z_0 , and

$$(z_1 - z_{01})P_1(z_1, z_{01}) \cdots P_m(z_1, z_{01}) = 0. \quad (50)$$

Let c be any root of (49), then, by

$$c = \Phi_i(z_0) = \Phi_i(\Phi^{-1}(z_{01})) = \Phi_j(z_{01}),$$

c must be a root of (50), and the degree of (49) and (50) are equal.

Then the roots of the equation (50) represent all the consequents and antecedents of z_{01} , and hence the two equations are equivalent.

Hence, if, putting (49) and (50) in the forms,

$$z_1^{k+1} + a_1(z_0)z_1^k + \cdots + a_{k+1}(z_0) = 0, \quad (51)$$

$$z_1^{k+1} + a_1(z_{01})z_1^k + \cdots + a_{k+1}(z_{01}) = 0, \quad (52)$$

(60) For this proof I owe much to Mr. Shinji Nakashima.

then these two equations have the same roots.

Hence

$$\begin{aligned} a_1(z_{01}) &= a_1(z_0) \\ a_2(z_{01}) &= a_2(z_0) \\ &\dots\dots\dots \\ a_{k+1}(z_{01}) &= a_{k+1}(z_0). \end{aligned}$$

Thus z_{01} as a function of a variable z_0 , putting z_1 and z instead of z_{01} and z_0 respectively, must satisfy one of the relations $a_i(z_1) = a_i(z)$, where $a_i(z)$ are polynomials of z .

Now not all $a_i(z)$ can be constants, for, otherwise all the coefficients of (51) become constants, and (51) defines merely a constant and not a function.

That $P(z_1) = P(z)$ is a sufficient condition follows immediately by considering the iteration

$$\begin{aligned} P(z_1) &= P(z), \\ P(z_2) &= P(z_1). \end{aligned}$$

For, there exists only a finite number of different functions when iterated in any manner.

Further we can show that if all the points of the z -plane have only a finite (not necessarily limited) number of consequents and antecedents of a point in the iteration of a finite algebraic function, then there exists only a finite number of iterated functions (and conversely).

To show this we shall first prove that there exists an infinite number of points all of whose consequents and antecedents are limited in number.

For, if there exists only a finite number of points whose consequents and antecedents are n , ($n = 1, 2, \dots$), then these points form a denumerable infinite set, and there would be an infinite number of points whose consequents and antecedents are infinite.

Thus we can suppose that the consequents and antecedents of $a_1, a_2, \dots, a_n, \dots$ are all n in number respectively.

Suppose that there exists an infinite number of different functions $\Phi_1(z), \Phi_2(z), \dots$, then we can choose l functions among them, l being a sufficiently large number, and consider

$$P_1(z_1, z) \cdot P_2(z_1, z) \cdot \dots \cdot P_l(z_1, z) = 0 \quad (53)$$

where $P_i(z_1, z) = 0$ is the equation which defines $\Phi_i(z)$.

This defines algebraic functions, $\neq 0$, and $\Phi_i(z)$ can not be equal to $\Phi_j(z)$, $i \neq j$, at an infinite number of points. For, assuming the contrary, a polynomial would vanish at an infinite number of points.

Since there exists only a finite number of z , at which the number of different values z_1 is less than the degree of (53), the number of the consequents and antecedents of $a_1, a_2, \dots, a_n, \dots$ become larger than n , which is impossible.

Hence there exists only a finite number of different functions in the iteration under the above conditions.

Thus we have the theorem that in order that the iteration of a finite algebraic function forms a finite discontinuous group, it is necessary and sufficient that the function should be defined by a factor of $P(z_1) = P(z)$, where $P(z)$ is a polynomial of z .

XII

We shall consider in this section some properties of algebraic functions or transcendental functions $z' = \Phi(z)$ which satisfy

$$f(z') = f(z). \quad (1)$$

First it is evident that:

In order that an algebraic function $\Phi(z)$ may satisfy (1) it is necessary and sufficient that $\Phi(z)$ should be defined by a factor $g_i(z', z) = 0$ in the decomposition into the factor-varieties $\prod_{i=1}^{\infty} (g_i(z', z))^{\alpha_i} = 0$ of $f(z') - f(z) = 0^{(61)}$ for some meromorphic function $f(z)$, where $g_i(z', z)$ is a polynomial with respect to z' and z .

Next an algebraic function which satisfies (1) must be of the form

$$z'^m - az^m + p(z', z) = 0,$$

where $|a| = 1$ and $p(z', z)$ is a polynomial with respect to z' and z of degree not larger than m with respect to z' and z , and less than m with respect to each of z' and z .

For, otherwise either the function $\Phi(z)$ or $\Phi^{-1}(z')$ would have some pole in the finite part of the z - or z' -plane, or map the circle $|z| = r$ or

⁽⁶¹⁾ T. Shimizu: loc. cit. (On the fundamental domains etc. I) This Journal **8**, 230, line 7-line 15.

$|z'| = r$ on some closed curve lying outside $|z'| = r$ or $|z| = r$ respectively by the relation between the last coefficient and the product of all the roots of the equation.

Hence in any case as in the proofs of theorems 11 and 12 we can easily show that $f(z') = f(z)$ is impossible.

I have given in the section XI some necessary conditions in order that there may exist only a finite number of consequents and antecedents of a point in any limited part of the plane for the iteration of a finite algebraic function (which has no polar singularity).

On the other hand the consequents and antecedents of a point being equivalent points of the point, any function, satisfying (1) for a meromorphic function $f(z)$, can not have an infinite number of different consequents and antecedents in a vicinity of a fixed point other than ∞ .

Hence we have

Theorem 18: *In order that a finite algebraic function may satisfy (1), it is necessary that either the function $\Phi(z)$ can be written in the form*

$$\psi^{-1} e^{i\theta\pi} \psi(z),$$

where $\psi(z)$ is an algebraic function of z and θ is a rational number, or no iterated functions $\Phi(z), \Phi_2(z), \Phi_3(z), \dots, \Phi_{-1}(z), \Phi_{-2}(z), \dots$ should have any fixed point other than ∞ .

This condition is not sufficient, for as we shall show in the following the function $\Phi(z)$ defined by $\sqrt[5]{z'} = \sqrt[5]{z} + 1$ which may be written in the form $\sqrt[5]{z'} - \alpha = \omega(\sqrt[5]{z} - \alpha)$, $\omega^5 = 1$, $\alpha(1 + \omega) = 1$; can not satisfy $f(z') = f(z)$ for any meromorphic function $f(z)$.

An algebraic function whose iteration generates only a finite number of consequents and antecedents in a vicinity of a fixed point may form a finite group—it generates only a finite number of iterated functions in the whole plane—or not.

The function $\Phi(z)$ defined by $P(z') = P(z)$, $P(z)$ being a polynomial, belongs to the first case and the function defined by $\sqrt{P(z')} = \sqrt{P(z)} + c$, or $\sqrt[5]{P(z')} = \sqrt[5]{P(z)} + c$ belongs to the latter case, in which there appears an infinite number of different iterated functions. The function $\Phi(z)$ defined by $\sqrt{P(z')} = \sqrt{P(z)} + c$ however satisfies (1) for $f(z) = \cos\left\{\frac{2\pi}{c}\sqrt{P(z)}\right\}$ and the iteration of $\Phi(z)$ forms a discontinuous group and all the fixed points are of periodical property.

The function $\Phi(z)$ defined by $P(z') = P(z) + c$ has no fixed point except ∞ for any iterated functions, and satisfies (1) for $f(z) = e^{\frac{2\pi i}{c}P(z)}$ forming also an infinite discontinuous group.

By the theorem in the section XI we have at once that in order that a finite algebraic function which forms a finite group may satisfy (1) it is necessary and sufficient that the function $\Phi(z)$ should be defined by a factor of $P(z') = P(z)$, where $P(z)$ is a polynomial of z .

It will be not without interest to remark that there exists a finite algebraic function $\Phi(z)$ which never satisfies $P_1(z') = P_2(z)$, $P_1(z)$ and $P_2(z)$ being polynomials—that is, any polynomial of z' can not be uniform—but satisfies $f(z') = f(z)$ for an integral transcendental function.⁽⁶²⁾

Consider the functions $\Phi(z)$ defined by $\sqrt[m]{z'} = \sqrt[m]{z} + c$ (or $z' = (\sqrt[m]{z} + c)^m$), $m \geq 2$ and m being integers, these functions can not satisfy any of the relations $P_1(z') = P_2(z)$.

For, consider z'^k , $k = 1, 2, \dots, p$, then by expanding the right hand sides of $z'^k = (\sqrt[m]{z} + c)^{mk}$ we have the terms in them

$$p_{1,k} z^{\frac{1}{m}}, p_{2,k} z^{\frac{2}{m}}, \dots, p_{m-1,k} z^{\frac{m-1}{m}}, p_{m,k},$$

where $p_{1,k}, \dots, p_{m-1,k}$, $p_{m,k}$ are polynomials of z and $p_{m-1,k}$ is of degree $k-1$.

Hence for any polynomials of z' with constant coefficients the term $z^{\frac{m-1}{m}}$ can not vanish and any polynomial of z' can not be uniform, which proves the enunciation.

On the other hand we have meromorphic functions $f_m(z)$ for which

$$f_m(z'_m) = f_m(z),$$

where z'_m denotes the function defined by $\sqrt[m]{z'_m} = \sqrt[m]{z} + c$, m being equal to one of 2, 3, 4, 6 respectively.

For z'_2 we have the integral function $f_2(z) = \cos\left\{\frac{2\pi}{c}\sqrt{z}\right\}$ for which

$$\cos\left\{\frac{2\pi}{c}\sqrt{z'_2}\right\} = \cos\left\{\frac{2\pi}{c}\sqrt{z} + 2\pi\right\} = \cos\left\{\frac{2\pi}{c}\sqrt{z}\right\}.$$

For z'_3 we have as a required function $f_3(z)$ the following function $H(z)$. $H(z) = q(\sqrt[3]{z}) + q(\omega\sqrt[3]{z}) + q(\omega^2\sqrt[3]{z})$, ω being the primitive root of

(62) This shows some difference between the structure of the Riemann surface of an integral transcendental function and that of a polynomial.

$x^3 = 1$, where $q(z)$ is a doubly periodic function with periods c and ωc .

For, $H(z)$, being symmetric with respect to $\sqrt[3]{z}$, $\omega\sqrt[3]{z}$ and $\omega^2\sqrt[3]{z}$, and $\omega^2 = -(1 + \omega)$, is uniform and hence a meromorphic function, and

$$H(z_3') = q(\sqrt[3]{z} + c) + q(\omega\sqrt[3]{z} + \omega c) + q(\omega^2\sqrt[3]{z} + \omega^2 c) = H(z).$$

For z_4' we have a function $f_4(z) = H(z)$ which is required

$$H(z) = q(\sqrt[4]{z}) + q(-\sqrt[4]{z}) + q(i\sqrt[4]{z}) + q(-i\sqrt[4]{z}),$$

where $q(z)$ is a doubly periodic function with periods c and ic . Since $-c$ and $-ic$ are periods, $H(z)$ is a meromorphic function and $H(z') = H(z)$.

For z_6' we have a required function $f_6(z)$ denoting by $H(z)$,

$H(z) = q(\sqrt[6]{z}) + q(\omega\sqrt[6]{z}) + q(\omega^2\sqrt[6]{z}) + q(\omega^3\sqrt[6]{z}) + q(\omega^4\sqrt[6]{z}) + q(\omega^5\sqrt[6]{z})$, ω being the primitive root of $x^6 = 1$, where $q(z)$ is a doubly periodic function with periods ωc and $\omega^2 c$.

$H(z)$ is a meromorphic function and satisfies $H(z') = H(z)$, since

$$\omega^3 c = -c = (\omega^2 + \omega)c, \quad \omega^4 c = -\omega c, \quad \text{and} \quad \omega^5 c = -\omega^2 c. \quad (63)$$

Now we shall show that the functions $\Phi(z)$ defined by $\sqrt[m]{z'} = \sqrt[m]{z} + c$, where $m = 5$ or $m \geq 7$ can not satisfy (1) for a meromorphic function $f(z)$.

To show this we shall show that if the function $\Phi(z)$ satisfies $f(z') = f(z)$, then $f(z)$ must have three or more independent periods.

Let ω be the primitive root of $x^m = 1$, $m = 5$ or $m \geq 7$, then there exist at most $m - \varphi(m)$ linearly independent relations with rational coefficients among $1, \omega, \omega^2, \dots, \omega^{m-1}$, $\varphi(m)$ being Euler's function.

For, if there exist more than $m - 1 - (\varphi(m) - 1) = m - \varphi(m)$ relations

$$n_{0,i} + n_{1,i}\omega + \dots + n_{m-1,i}\omega^{m-1} = 0, \quad i = 1, 2, \dots$$

with integral coefficients $n_{0,i}, n_{1,i}, \dots, n_{m-1,i}$, $i = 1, 2, \dots$, among $1, \omega, \dots, \omega^{m-1}$, then ω would satisfy an algebraic equation with degree less than $\varphi(m) - 1$ which is evidently contradictory.

Thus the number of independent ones among $1, \omega, \omega^2, \dots, \omega^{m-1}$ is at least $\varphi(m)$.

Now let $m = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ be the decomposition of m into the prime factors $p_1 < p_2 < \dots < p_k$, then $\varphi(m) = p_1^{m_1-1}(p_1 - 1) \dots$

(63) In the same way $\sqrt[m]{P(z')} = \sqrt[m]{P(z)} + c$, $m = 2, 3, 4, 6$ satisfies $H(p(z')) = H(p(z))$.

Hence if $p_1 > 3$, or $p_1 \geq 3$ and $m_1 \geq 2$, or $p_1 = 2$ and $m_1 \geq 3$, then $\varphi(m) \geq 3$. Thus we can at once deduce that in order that $\varphi(m) \leq 2$, we must have $m = 2, 3, 4$ or 6 .

Now if the functions $\Phi(z)$ defined by $\sqrt[m]{z'} = \sqrt[m]{z} + c$, $m = 5$ or $m \geq 7$, satisfy (1), then we would have

$$f(z') = f((\sqrt[m]{z} + c)^m) = f(z) = f((\sqrt[m]{z})^m).$$

This is true for any values of $\sqrt[m]{z}$, that is, for $\sqrt[m]{z}$, $\omega \sqrt[m]{z}$, $\omega^2 \sqrt[m]{z}$, $\dots$, $\omega^{m-1} \sqrt[m]{z}$.

Now $f(z)$ is a uniform function $g(Z)$ of Z , putting by Z some value of $\sqrt[m]{z}$.

$g(Z + \omega^k c) = f((Z + \omega^k c)^m) = f((\omega^k Z + \omega^k c)^m) = f((Z + c)^m) = g(Z)$, $k = 1, 2, \dots, m-1$, hence $\omega^k c$, $k = 1, 2, \dots, m-1$ are all periods of the function $g(Z)$.

If $m = 5$ or $m \geq 7$, then the number of independent periods among $c, \omega c, \dots, \omega^{m-1} c$ is larger than or equal to 3.

Hence $g(Z)$ can not be uniform which is contradictory.

There is another proof, being essentially the same with the former.

By $f(z') = f(z)$, the consequents and antecedents of the iteration of the function $\Phi(z)$ are equivalent points of $f(z)$.

Now by putting $\sqrt[m]{z'} = Z$, $\sqrt[m]{z} = Z$, we have, from $\sqrt[m]{z'} = \sqrt[m]{z} + c$,

$$Z' = Z + c, \quad Z' = \omega Z + c, \dots, \quad Z' = \omega^{m-1} Z + c,$$

whose composition is equivalent to the iteration of $\Phi(z)$.

Thus we have

$$Z' - \alpha_j = \omega^j (Z - \alpha_j), \quad \text{where} \quad \alpha_j = \frac{c}{1 - \omega^j}.$$

In order that they may form a discontinuous group, α_j should lie on some lattice points generated by some primitive transformations

$$Z' - \alpha_j = \omega^j (Z - \alpha_j).$$

Hence, unless $m = 2, 3, 4, 6$, the consequents and antecedents of a point must lie everywhere dense, and $\Phi(z)$ can not satisfy $f(z') = f(z)$.

Now we shall consider the functions $z' = \varphi(z)$ defined by (1) when they are infinitely many valued. Let us denote as before by $z' = \Phi(z)$ an irreducible factor of $z' = \varphi(z)$.

Then by the theorem 10 if $\Phi(z)$ is not an algebraic function, $\Phi(z)$ is infinitely many valued in any domain.

Further we have as before that in order that an infinitely many valued function $\Phi(z)$ may satisfy (1) for a meromorphic function $f(z)$, it is necessary and sufficient that $\Phi(z)$ should be defined by $g_i(z', z) = 0$, $g_i(z', z)$ being a factor of the decomposition into the factor-varieties of $f(z') = f(z)$, and an integral function with respect to both z' and z .

Further the mapped image of $|z| \leq r$ by $z' = \Phi(z)$ can not contain $|z'| \leq r$ within it and can not be contained within $|z'| \leq r$.

This follows by comparing the maximum moduli of both sides of (1).

Theorem 19: *The singularities of the function $\Phi(z)$ come from those of the inverse function $z = f^{-1}(w)$ of $w = f(z)$, and all the singular points (the accessible boundary points of the Riemann surface) of $z' = \Phi(z)$ which lie in the finite part of the z -plane are of transcendental determinate character (algebraic branch points included), that is, the *Häufungsbereich* at the singular point consists of only one point.*

Only the ∞ may be a complicated singularity for $\Phi(z)$.

Proof: By Iversen's researches⁽⁶⁴⁾ the prolongation of the elements of the inverse function of a meromorphic function is completely studied.

Let $\bar{w}_1, \bar{w}_2, \dots$ be the set of singular points of $f^{-1}(w)$ whose projections on the w -plane are $w_1, w_2, \dots$ respectively.

Further let $z_{1i}, z_{2i}, \dots$ be the points for which $f(z_{ji}) = w_i$.

These $z_{1i}, z_{2i}, \dots$ exist for certain except possibly for two values of w by Picard's theorem.

When z tends to a point α , not belonging to z_{ji} but corresponding to a point at which all the elements of $f^{-1}(w)$ are regular, then $f(z)$ tends to a definite value β , and $\Phi(z)$ is regular at α , for it is defined by some branch of $f^{-1}(w)$ which is regular at β and in which w , considered as the function $f(z)$, is regular at α .

On the other hand all the branches of $f^{-1}(w)$ are regular at β , hence all the branches of $\Phi(z)$ are regular at α .

If z tends to a point α , not belonging to z_{ji} , but corresponding to an algebraic branch point of at least one of the branches of $f^{-1}(w)$, then the branches of $\Phi(z)$ are regular or algebraic at α .

Since $f(z)$ can not have only multiple roots except for four values of w by another Picard's theorem, algebraic branch points appear in general for at least one of the functions $\varphi(z)$ which satisfy (1).

⁽⁶⁴⁾ F. Iversen: *Recherches sur les fonctions inverse des fonctions méromorphes*. Helsingfors, 1914.

When z tends to a point α , belonging to z_{ji} , then $f(z)$ tends to w_i . On the other hand some branches of $f^{-1}(w)$ tend to ∞ by Hurwitz-Iversen's theorem when w tends to w_i .

Hence some of the functions $\varphi(z)$ satisfying (1) and corresponding to such branches of $f^{-1}(w)$ tends to ∞ , and z_{ji} is a transcendental singular point of determinate character for such a function $\Phi(z)$.

There may exist some branches of $\Phi(z)$ which are regular or algebraic at z_{ji} .

But if we consider all the branches of $\Phi(z)$, the Häufungsbereich of $\Phi(z)$ at z_{ji} consists of a discrete point set having only a limiting point ∞ .

The above proof is based on the theorems of Iversen. Without those we can prove it similarly as he has done.

If z tends to ∞ , then $f(z)$ may or may not tend to any value, and $\Phi(z)$ is not in general determinate at ∞ .

If z_{ji} be an isolated singular point of $\Phi(z)$, then it is a directly critical, indirectly critical or directly and indirectly critical point after Iversen's classification of isolated singular points.

In order that all the irreducible factors $\Phi_i(z)$, $i = 1, 2, \dots$ of $\varphi(z)$ satisfying (1) may be regular at α , it is necessary and sufficient that all the branches of $f^{-1}(w)$ should be regular at $f(\alpha)$, or $f(\alpha)$ may be singular points of $f^{-1}(w)$ but $f(z) = f(\alpha)$ should have no root, or $f(\alpha)$ may be algebraic branch points of some branches of $f^{-1}(w)$ but $f(z) = f(\alpha)$ should have no root or only integral multiple roots of the order of branch points.

Theorem 20: *If $z' = \Phi(z)$, satisfying (1), is infinitely many valued, then the inverse function $z = \Phi^{-1}(z')$ is also infinitely many valued.*

For, since $z' = \Phi(z)$ is transcendental, $z = \Phi^{-1}(z')$ can not be a rational or an algebraic function of z' .

On the other hand if $z' = \Phi(z)$ satisfies (1), then by the symmetrical property of $f(z') = f(z)$, $z = \Phi^{-1}(z')$ must satisfy (1).

Hence by the theorem 10 the theorem is proved.

Theorem 21: *If $\Phi(z)$, satisfying (1), is finite valued, then ∞ is a pole or an algebraic pole.*

For, $\Phi(z)$ being an algebraic function, and having no pole in the finite part of the plane, the theorem is evident.

Theorem 22: *If $\Phi(z)$, satisfying (1), is infinitely many valued, then ∞ is an essential singular point of complete indetermination, if we consider*

all the branches of $\Phi(z)$, that is, the *Häufungsbereich* of $\Phi(z)$ at ∞ for all branches of $\Phi(z)$ is the whole z' -plane.

Proof: Let v be an arbitrary value, and $C(v, \varepsilon)$ be an arbitrarily small circle $|z' - v| < \varepsilon$ with the center v .

On the other hand by the application of Gronwall-Hahn's theorem on the function of two variable, $z' = \Phi(z)$ is defined by

$$g(z', z) = g_0(z) + g_1(z)z' + \cdots + g_n(z)z'^n + \cdots = 0,$$

where $g(z', z)$ is an integral function with respect to z' and z , and $g_0(z), g_1(z), \cdots, g_n(z), \cdots$ all integral functions of z .⁽⁶⁵⁾

Then for an infinite sequence of decreasing positive numbers ε_i ($i = 1, 2, \cdots$), $\varepsilon_i \rightarrow 0$ for $i \rightarrow \infty$, there exist at least n_i points $P_{i,1}, P_{i,2}, \cdots, P_{i,n_i}$ such as

$$\Phi(P_{i,m}) = \beta_i, \quad m = 1, 2, \cdots, n_i,$$

where β_i is a properly chosen value within a circle $C(v, \varepsilon_i)$, and n_i tends to ∞ when i tends to ∞ .

For, if there exist at most n points P such that

$$\Phi(P) = \beta$$

for all values β lying in a circle $C(v, \varepsilon)$, then the function $\Phi^{-1}(z')$ would be finite valued within $C(v, \varepsilon)$ as a function of z' .

This is impossible by the theorem 10 and the theorem 20.

Since $z' = \Phi(z)$ is defined by $g(z', z) = 0$, the limiting point of $P_{i,1}, P_{i,2}, \cdots, P_{i,n_i}$, $i = 1, 2, \cdots$ is only the point ∞ .

Thus there exists an infinite number of points P_{i,n_i} at which $\Phi(P_{i,n_i}) = \beta_i$ in any vicinity of ∞ when considered for all branches, and hence the *Häufungsbereich* of $\Phi(z)$ at the point ∞ contains v .

v being arbitrary, the theorem is proved.

Since $\Phi(z)$ has no continuum which lies outside the part covered by the Riemann surface of $\Phi(z)$, ∞ is an accessible point. Similarly $z' = \infty$ is also an accessible point for the Riemann surface of $z = \Phi^{-1}(z')$.

Hence perhaps it will be true that we can determine a path of infinite determination for any function $\Phi(z)$, that is, there will exist at least a simple continuous curve running from a point to ∞ along which $\Phi(z)$ tends to ∞ when z tends to ∞ .

We can prove, however, that

⁽⁶⁵⁾ T. Shimizu: I. loc. cit. p. 230. Cf. ⁽⁶¹⁾

Theorem 23: *Any finite exceptional value for $z' = \Phi(z)$ is an asymptotic value, that is, if $\Phi(z)$ does not take the value w_0 in the finite part, then there exists a path running to ∞ along which $\Phi(z)$ tends to w_0 .*

Proof: If z' tends to w_0 , then by $f(z') = f(z)$, $f(z)$ tends to $f(w_0)$, which is a determinate value since $w_0 \neq \infty$.

By $f(w_0) = f(z)$ we have the following possible cases,

- 1) z tends to w_0 or oscillates.
- 2) z tends to a value β_1 for which $f(\beta_1) = f(w_0)$ and $w_0 = \Phi(\beta_1)$.
- 3) z tends to a value β_1' for which $f(\beta_1') = f(w_0)$ but $w_0 \neq \Phi(\beta_1')$, that is, β_1' does not belong to the function $\Phi(z)$, but to another function $z' = \Phi_i(z)$, which is also a factor of $z' = \varphi(z)$ defined by (1).
- 4) z tends to ∞ .

Now $z \rightarrow w_0$ can not occur unless $z' \equiv z$, and further z can not oscillate, for limiting point sets of z' for such a case must contain a value other than w_0 .

Now z can not tend to β_1' , for, otherwise, by the discontinuity of equivalent points for which $f(z) = f(w_0)$, z' must tend to a value other than w_0 unless β_1' belongs to β_1 .

Lastly z can not tend to β_1 , for, otherwise, $\Phi(z)$ must take the value w_0 or tend to ∞ . These come from the fact that, if $f(w_0)$ be a regular or an algebraic point for the function $f^{-1}(w)$, then z' must take w_0 , for the part containing w_0 corresponds to the part β_1 by $f(w_0) = f(\beta_1)$. If $f(w_0)$ be a singular point of $f^{-1}(w)$, then z' must tend to ∞ , for z' must have only a transcendental critical point of determinate character.

Hence we must have only the case that z tends to ∞ , and w_0 is an asymptotic value.

Another proof of the theorem: If z' does not take w_0 , then w_0 is a boundary point of the Riemann surface of $z = \Phi^{-1}(z')$. Since there is no continuum not covered by the Riemann surface of $z = \Phi^{-1}(z')$, and the boundary point is a transcendental critical point of determinate character, z tends to ∞ , if z' tends to w_0 . Hence w_0 is an asymptotic value.

We can give some more precise theorem for some special case.

Theorem 24: *Let $f(z)$ be a meromorphic (not rational) function. If $f(z') = f(z)$ is decomposed into only two functions $z' = z$ and $z' = \Phi(z)$. Then the values which $z' = \Phi(z)$ takes at most only at n different points (multiplicity disregarded) can not exceed $2n + 2$ (except ∞)—the exceptional values therefore can not exceed two.*

∞ may be considered as an exceptional value, since it corresponds to the boundary of the Riemann surface of $z' = \Phi(z)$.

Proof: By Picard's theorem there exist at most two values α, β (∞ included) for which $f(z) = \alpha$ and $f(z) = \beta$ have both only a finite number of roots.

If c be a value for which $f(c) = \gamma$, $\gamma \neq \alpha$, $\gamma \neq \beta$, then $f(z) = f(c)$ has an infinite number of roots $z_1, z_2, \dots, z_n, \dots$.

Hence, if $\gamma \neq \infty$, by $\frac{f(c) - f(z_i)}{c - z_i} = \frac{g(c, z_i)}{f_2(c)f_2(z_i)} = 0$, where $f(z) = \frac{f_1(z)}{f_2(z)}$ as before, $f_1(z)$ and $f_2(z)$ having no common factors, we have $g(c, z_i) = 0$, since $f_2(c) \neq \infty$, $f_2(z_i) \neq \infty$.

If $\gamma = \infty$, by

$$\frac{\frac{1}{f(c)} - \frac{1}{f(z_i)}}{c - z_i} = \frac{g(c, z_i)}{f_1(c)f_1(z_i)} = 0$$

we have $g(c, z_i) = 0$.

Thus from the hypothesis we see that $z' = \Phi(z)$ has an infinite number of points z_i for which $\Phi(z_i) = c$.

Now let $a_0, a_1, \dots, a_m$ be the values for which $f(a_i) = \alpha$, $i = 0, 1, \dots, m$ and $b_0, b_1, \dots, b_n$ be the values for which $f(b_i) = \beta$, $i = 0, 1, \dots, n$, and assume that there is no other value for which $f = \alpha$ or $f = \beta$.

Let d be a value for which $g(a_j, d) = 0$ for some a_j among a_i , ($i = 0, 1, \dots, m$), and $d \neq a_i$, ($i = 0, 1, \dots, m$).

If $\alpha \neq \infty$, $f(d) \neq \infty$, then by $\frac{f(a_j) - f(d)}{a_j - d} = \frac{g(a_j, d)}{f_2(a_j)f_2(d)} = 0$

we have $f(a_j) = f(d)$, if $g(a_j, d) = 0$, for $f_2(a_i) \neq 0$, $f_2(d) \neq 0$.

This is contradictory.

If $\alpha = \infty$, $f(d) = \infty$, then we can not have $d \neq a_i$.

If $\alpha = \infty$ or $f(d) = \infty$, then considering

$$\frac{\frac{1}{f(a_j) - k} - \frac{1}{f(d) - k}}{a_j - d} = \frac{g(a_j, d)}{\{f_1(a_j) - kf_2(a_j)\}\{f_1(d) - kf_2(d)\}}$$

and choosing such as $k \neq 0$ when $\alpha = 0$, $f(d) = \infty$, or $\alpha = \infty$, $f(\alpha) = 0$,

otherwise $k = 0$, we have, if $g(a_j, d) = 0$, $\frac{1}{f(a_j) - k} = \frac{1}{f(d) - k}$.

For if $\alpha = \infty$, $f(d) = 0$ then $f_1(a_j) \neq 0$, $f_2(d) \neq 0$, if $f(d) = \infty$, $\alpha = 0$, then $f_2(a_j) \neq 0$, $f_1(d) \neq 0$, if $\alpha = \infty$, $f(d) \neq 0$, or $\alpha \neq 0$, $f(d) = \infty$, then $f_1(a_j) \neq 0$, $f_1(d) \neq 0$.

Hence we have $f(a_j) = f(d)$ which is contradictory to $d \neq a_i$.

So similarly for β .

Thus $\Phi(z)$ can not take any of a_i at more than m points, and any of b_i at more than n points.

Hence if $n \geq m$, then there exist at most $2n + 2$ values which $\Phi(z)$ takes at most at n points.

Now the function $z' = \Phi(z)$ defined by $\frac{z' - 1}{z'} e^{z'} = \frac{z - 1}{z} e^z$, can not take the values 1 and 0. For $\Phi(z)$ can not take 0 (and 1) at a point other than 0 (and 1 respectively), since $\frac{z - 1}{z} e^z = \infty$ (and 0) has only one root $z = 0$ (and 1).

On the other hand we can show that $\Phi(1) \neq 1$ and $\Phi(0) \neq 0$, for, from

$$\frac{(1 - z') \left(1 + z' + \frac{z'^2}{2} + \cdots + \frac{z'^n}{n!} + \cdots \right)}{z'} = \frac{(1 - z) \left(1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots \right)}{z}$$

we have

$$g(z', z) = \left\{ 1 + \frac{zz'}{2} + \cdots + \frac{(n-1)}{n!} \{ z'^{n-1} + z'^{n-2}z + \cdots + z'z^{n-2} + z^{n-1} \} zz' + \cdots \right\}.$$

That $g(z', z) = 0$ is irreducible can be shown by perfectly similar way as that given on the page 234 of the section VII⁽⁶⁶⁾.

Thus $\Phi(z)$ has two exceptional values 0 and 1.

Similarly the function $z' = \Phi(z)$ defined by

$$\frac{(z' - 2)(z' - 4) \cdots (z' - 2n)}{(z' - 1)(z' - 3) \cdots (z' - (2n - 1))} e^{z'} = \frac{(z - 2)(z - 4) \cdots (z - 2n)}{(z - 1)(z - 3) \cdots (z - (2n - 1))} e^z$$

has $2n + 2$ values which $\Phi(z)$ takes only at n points.

Remark: I have assumed in most places the function $f(z)$ to be a transcendental meromorphic function but not a rational function.

(66) T. Shimizu. I. loc. cit. Cf. (48).

If we consider the group for a rational function $R(z)$ defined by

$$R(z') = R(z),$$

then it is simple and has not so much difficulty about it.

We shall give here only some simple properties of it.

Putting

$$R(z) = \frac{R_1(z)}{R_2(z)},$$

where $R_1(z)$ and $R_2(z)$ are polynomials of z , without common factors, we have

$$\begin{aligned} R(z') - R(z) &= \frac{R_1(z')R_2(z) - R_1(z)R_2(z')}{R_2(z)R_2(z')} \\ &= \frac{(z' - z)P_1(z', z) \cdots P_m(z', z)}{R_2(z)R_2(z')}, \end{aligned}$$

where $P_1(z', z), \dots, P_m(z', z)$ are polynomials with respect to z' and z .

In this case the function $z' = \Phi(z)$ defined by $P_i(z', z) = 0$ may have some poles in the finite part, and $\Phi(z)$ is either a rational function (including polynomials) or an algebraic function.

As an example $z' = \Phi(z) = \frac{\alpha z}{z - \alpha}$ has a pole α , which is a factor of

$$\frac{z'^2}{z' - \alpha} - \frac{z^2}{z - \alpha} = \frac{(z' - z)\{z'z - (z' + z)\alpha\}}{(z' - \alpha)(z - \alpha)}.$$

This fact can not appear for the case for which $f(z)$ is a transcendental function.

The functions $\Phi(z)$ defined by a rational function evidently forms a finite discontinuous group as the transformation of equivalent points or as the iteration of them.

Such an algebraic function $\Phi(z)$ has some remarkable properties, and I shall discuss some of them in future.

Remark: Some of the properties given in this paper and in the previous one (On the fundamental domains and the groups for meromorphic functions I. loc. cit.) hold good for more general classes of functions, for instance, for a uniform function having only isolated essential singular points.

Remark to the section **VI** of Part **I**.

On the pages 224 (from the line 1 to the line 5 of the next page) and 226 (from the line 5 to the line 3 of the next page) we have shown that $u^{-1}(f(z)) = g(z)$ must be a linear function for each case.

But we can prove them in a simpler way as follows.

First $g(z) = u^{-1}(f(z))$ must be a meromorphic function as shown there.

Perfectly similarly we can show that $z = f^{-1}(u(g))$ is also a uniform function of g by expanding into a power series of g , since by the assumption f^{-1} has no other branch point than the values of $u(g)$, which $u(g)$ takes at all points in the same multiplicity for each value.

Hence $g(z)$ must be a linear function of z .

(Cf. the proof of theorem 16, page 255).

The proofs on the pages 224 and 226 show us that $f(z)$ must be of the forms given there under less conditions.

To these proofs it is more desirable to add the following:

(At the end of the line 18, page 224) For, if there exist only singular points of indirect character without algebraic branch points, then any accessible singular points would be of direct character.

(Between the line 26 and the line 27, page 226) Now it is evident that $\beta \neq 0, \neq 1, \neq \infty$, and so the system of points for which $g_1(z) - \beta = 0, g_2(z) - \beta = 0, \dots, g_k(z) - \beta = 0$ are all different from each other by the property of $u(z)$.

Errata: We must read $\frac{az+b}{cz+d}$ for $az+b$.

in the line	20	on the page	177
„	23	„	177
„	37	„	220
„	5	„	227
„	7	„	227
„	27	„	227

(Cf. Pages 252-255 Section IX, these $az+b$ are replaced by $L_2(z)$ there).

We must read theorem **9** for theorem **7** on the page 229.

Remark to the section **X** of Part **II**.

In the theorem 17 we can replace “for an infinitely increasing sequence” by “for all sufficiently large values of r except possibly for a set of intervals whose total length is finite” which is sharper than the former.

By the theorem 17 we can not conclude that the number of exceptional values of $u(z)$ is at most μ , for, there may exist some values for which $g_i(z)$ reduces to a constant.

Since there exist at most $\nu - 1$ values for which $g_i(z)$ becomes a constant or a polynomial, all $g_i(z)$ are transcendental functions if we consider sufficiently large values of $|u_i|$.

Hence we can also prove the theorem 17 by showing that the relation (38) is contradictory. It is sufficient for this to denote by $M(\rho)$ the maximum value of $m(\rho, g_i)$, omitting 26 lines from the line 16 of the page 274 to the line 19 of the page 275, and to replace B_i by g_i on the pages 276-278.

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