20. On analytical Transformations of a Reinhardt's Domain into itself.

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I. Introduction.

1. Let $x = x_1 + ix_2$, $y = y_1 + iy_2$, then Reinhardt's domain D in a four dimensional space (x_1, x_2, y_1, y_2) is defined as a domain, which is transformed into itself by the transformations; $x' = e^{i\alpha}x$, $y' = e^{i\beta}y$, where α, β are any real numbers. The origin is the center of D . An analytical transformation of a domain D into itself is a transformation of a form, $x' = f(x, y)$, $y' = g(x, y)$ which with its inverse transformation, transforms D into itself, where $f(x, y)$ and $g(x, y)$ are regular functions of x and y in D .

Reinhardt proved that if $x' = f(x, y)$, $y' = g(x, y)$ is an analytical transformation of a bounded Reinhardt's domain D into itself, the center being fixed, then $f(x, y)$ and $g(x, y)$ are homogeneous linear functions of x and y of special types.

In the following we denote $\mathfrak{G}$ the aggregate of all analytical transformations of D into itself so that the center is fixed.

Then Reinhardt's fundamental result is as follows:

Reinhardt's theorem⁽¹⁾. (i) If D is of the form: $A|x|^2 + B|y|^2 < 1$ ($A > 0, B > 0$), then $\mathfrak{G}$ consists of all transformations:

$$\left. \begin{aligned} x' &= ax + \frac{b}{\lambda}y \\ y' &= c\lambda x + dy \end{aligned} \right\} \quad (1)$$

where $\lambda = \sqrt{\frac{A}{B}}$ and $|a|^2 + |c|^2 = 1$, $|b|^2 + |d|^2 = 1$, $a\bar{b} + c\bar{d} = 0$ ⁽²⁾.

(ii) if D is a bounded Reinhardt's domain which is not of the above form, then $\mathfrak{G}$ consists of all transformations:

⁽¹⁾ Reinhardt; Ueber Abbildungen durch analytische Funktionen zweier Veränderlichen. (Math. Ann. **83** (1921) 211-255.)

⁽²⁾ In the following we denote the conjugate complex of z by $\bar{z}$.

$$\left. \begin{aligned} x' &= e^{i\alpha} x \\ y' &= e^{i\beta} y \end{aligned} \right\} \quad (2)$$

in general and in case when D is symmetric, $\mathfrak{G}$ contains outside (2) a transformation:

$$\left. \begin{aligned} x' &= \mu e^{i\alpha} y \\ y' &= \frac{1}{\mu} e^{i\beta} x \end{aligned} \right\} \quad (3)$$

where μ is a certain constant.

Reinhardt's original proof is geometrical and complicated. H. Cartan⁽³⁾ gave another proof which is group-theoretical, and the proof is based on the theorems that any closed sub-group of a Lie's group is also a Lie's group (E. Cartan—v. Neumann's theorem) and that there exists an invariant positive definite Hermitian form for any closed Lie's group (Weyl's theorem).

In this paper we will give a quite elementary proof of Reinhardt's theorem without any use of such deep theorems.

That $f(x, y)$ and $g(x, y)$ are homogeneous linear functions of x and y can be proved very simply⁽³⁾ so we will assume that $f(x, y)$ and $g(x, y)$ are linear. And H. Cartan⁽³⁾ proved that $\mathfrak{G}$ is a closed linear group, i. e. from a sequence $S_1, S_2, \dots, S_n, \dots$ of elements of $\mathfrak{G}$, we can select a partial sequence which converges to an element of $\mathfrak{G}$.

Hence we will assume in the following proof that

$\mathfrak{G}$ is a closed linear group.

Evidently $\mathfrak{G}$ contains transformations $S_{\alpha\beta}$; $x' = e^{i\alpha} x$, $y' = e^{i\beta} y$.

II. Some Lemmas.

2. Suppose $\mathfrak{G}$ contains a transformation;

$$S: \left. \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \right\} \quad (4)$$

which is different from $S_{\alpha\beta}$.

We transform S into a canonical form;

$$U = TST^{-1} = \begin{pmatrix} \lambda_1 & b_1 \\ 0 & \lambda_2 \end{pmatrix}$$

by a suitable non-singular matrix T .

(³) H. Cartan: Les fonctions de deux variables complexes et le problème de la représentation analytique. (Journ. de math. (1931), 1-114.)

Then

$$U = TS \ T^{-1} = \begin{pmatrix} \lambda_1 & b_1 \\ 0 & \lambda_2 \end{pmatrix} \quad (n=0, \pm 1, \pm 2, \dots). \quad (5)$$

Since $\mathfrak{G}' = T\mathfrak{G}T^{-1}$ is a closed group, we conclude that $|\lambda_1|=1, |\lambda_2|=1$ by making $n \rightarrow \infty$ or $n \rightarrow -\infty$ in (5). Since λ_1, λ_2 are the roots of the characteristic equation of S :

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

we have

$$|ad - bc| = |\lambda_1 \lambda_2| = 1 \quad (6)$$

$$|a+d| = |\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2| = 2. \quad (7)$$

Since $S_z = \begin{pmatrix} az & b \\ cz & d \end{pmatrix}$ ($|z|=1$) belongs to $\mathfrak{G}$, we get from (7), $|az+d| \leq 2$.

Choose z , such that $\arg az = \arg d$, then we get $|a| + |d| \leq 2$.

Hence we have

Lemma 1. *If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $\mathfrak{G}$ and λ_1, λ_2 are the roots of the characteristic equation of S , then*

$$(i) \quad |\lambda_1|=1, \quad |\lambda_2|=1$$

$$(ii) \quad |ad - bc| = 1$$

$$(iii) \quad |a| + |d| \leq 2.$$

3. Since the determinant of S_z is z times the determinant of S , by a suitable choice of z ($|z|=1$), we may assume in S ,

$$\Delta = ad - bc = 1. \quad (8)$$

Now the characteristic equation of S_z is

$$\lambda^2 - (az+d)\lambda + z = 0 \quad (9)$$

or rewriting

$$w = \frac{z}{\lambda} = \frac{\lambda - d}{a\lambda - 1}. \quad (10)$$

Consider w as a function of λ , then by Lemma 1, w lies on $|w|=1$ when λ moves on $|\lambda|=1$.

In (10), two cases occur according as w is a constant or not.

$$(i) \quad w = \frac{z}{\lambda} = \text{const.} = w_0 \quad (|w_0|=1).$$

Then $\frac{\lambda - d}{a\lambda - 1} = w_0$ for every λ on $|\lambda|=1$. Hence $w_0 a = 1, d = w_0$.

or $d = \bar{a}$, $|a| = 1$, $ad = 1$.

From $ad - bc = 1$, we get $bc = 0$. Hence $b = 0$ or $c = 0$.

If $c = 0$, then $S = \begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix}$.

Since $S_1 = \begin{pmatrix} a & bz \\ 0 & \bar{a}z \end{pmatrix}$ ($|z| = 1$) belongs to $\mathfrak{G}$, we choose z so that $\bar{a}z = a$, then $S_1 = \begin{pmatrix} a & b_1 \\ 0 & \bar{a} \end{pmatrix}$ ($b_1 = bz$) belongs to $\mathfrak{G}$. Now $S_1^n = \begin{pmatrix} a^n & na^{n-1}b_1 \\ 0 & \bar{a}^n \end{pmatrix}$, hence if $b \neq 0$, $|na^{n-1}b_1| \rightarrow \infty$ for $n \rightarrow \infty$, which contradicts that $\mathfrak{G}$ is closed, so that $b = 0$.

Hence if $w = \text{const.}$, S must be of the form: $S = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$ ($|a| = 1$).

We reach to the same result if we assume $b = 0$, instead of $c = 0$.

(ii) $w \neq \text{const.}$

In this case, $w = \frac{\lambda - d}{a\lambda - 1}$ moves on the unit circle, when λ moves on $|\lambda| = 1$, so that

$$|a| \neq 1, \quad d = \bar{a}.$$

By Lemma 1,

$$2|a| = |a| + |d| \leq 2 \quad \text{or} \quad |a| \leq 1.$$

Since the equality is excluded,

$$|a| < 1, \quad d = \bar{a}.$$

Lemma 2. If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ($ad - bc = 1$) belongs to $\mathfrak{G}$, then either

$$(i) \quad S = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \quad (|a| = 1)$$

or

$$(ii) \quad S = \begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix} \quad (|a| < 1, \quad bc \neq 0).$$

4. Next we will investigate the case (ii) and suppose that $abc \neq 0$. We make the change of variables;

$$T: \begin{cases} X = \lambda x \\ Y = y \end{cases} \quad (11)$$

By T , D is transformed into D' , which is evidently a bounded Reinhardt's domain, and is transformed into itself by a group $\mathfrak{G}' = T\mathfrak{G}T^{-1}$.

By (11), $S = \begin{pmatrix} a, b \\ c, \bar{a} \end{pmatrix}$ becomes

$$S_1: \left. \begin{aligned} X' &= aX + \lambda bY \\ Y' &= \frac{cX}{\lambda} + \bar{a}Y \end{aligned} \right\} \quad (a\bar{a} - bc = 1, \quad |a| < 1, \quad bc \neq 0) \quad (12)$$

We will show that a real λ can be chosen so that

$$|X'|^2 + |Y'|^2 = |X|^2 + |Y|^2$$

or

$$a\bar{a} + \frac{c\bar{c}}{\lambda^2} = 1, \quad a\bar{a} + \lambda^2 b\bar{b} = 1, \quad \bar{b}\lambda + \frac{c}{\lambda} = 0. \quad (13)$$

In fact, we get $\lambda^2 = \frac{|c|^2}{1 - |a|^2} (> 0)$ from the first equation of (13), and this λ satisfy the remaining equations of (13).

Hence for such a λ , S_1 becomes a unitary matrix with a determinant 1. If we make correspond to $S = \begin{pmatrix} a, b \\ c, \bar{a} \end{pmatrix}$ of $\mathfrak{G}'$, a transformation

$\sigma: z' = \frac{a+bz}{c+\bar{a}z}$ on the z -plane or on the Riemann's sphere K , then we get

a group g' of transformations of K . If S is a unitary transformation with determinant unity, then σ is a rotation of K about its diameter. Since $\mathfrak{G}'$ contains $S_{\alpha\beta}$, g' contains $S_\theta: z' = ze^{i\theta} (0 \leq \theta \leq 2\pi)$ or all rotations about the diameter through the north pole. And g' contains another rotation σ_1 which corresponds to S_1 of $\mathfrak{G}'$, as we have seen above. By a geometrical consideration, we can easily see that all rotations of K can be obtained by a suitable combination of S_θ and σ_1 . Hence $\mathfrak{G}'$ contains all unitary transformations with determinant unity.

Lemma 3. If $S = \begin{pmatrix} a, b \\ c, \bar{a} \end{pmatrix} (a\bar{a} - bc = 1, abc \neq 0)$ belongs to $\mathfrak{G}$, we can find a real λ , such that $\mathfrak{G}' = \begin{pmatrix} \lambda, 0 \\ 0, 1 \end{pmatrix} \mathfrak{G} \begin{pmatrix} \lambda, 0 \\ 0, 1 \end{pmatrix}^{-1}$ contains all unitary transformations with determinant unity.

5. We prove the following lemmas only for completeness.

Lemma 4. If $|X'|^2 + |Y'|^2 = |X|^2 + |Y|^2$, there exists a unitary transformation with determinant unity, which transforms (X, Y) to (X', Y') .

Proof. Let

$$\left. \begin{aligned} X' &= aX + bY \\ Y' &= -\bar{b}X + \bar{a}Y \end{aligned} \right\} \quad (|a|^2 + |b|^2 = 1)$$

be such a transformation, then

$$\bar{Y}' = a\bar{Y} - b\bar{X}.$$

Solving for a, b , we get

$$a = \frac{\bar{X}X' + Y\bar{Y}'}{|X|^2 + |Y|^2}, \quad b = \frac{\bar{Y}X' - X\bar{Y}'}{|X|^2 + |Y|^2}$$

and these a, b satisfy the condition $|a|^2 + |b|^2 = 1$.

Lemma 5. *If*

$$\left. \begin{aligned} X' &= aX + bY \\ Y' &= cX + dY \end{aligned} \right\} \quad (14)$$

transforms $|X|^2 + |Y|^2 < R$ into itself, then

$$|a|^2 + |c|^2 = 1, \quad |b|^2 + |d|^2 = 1, \quad a\bar{b} + c\bar{d} = 0.$$

Proof. Since (14) transforms the boundary $|X|^2 + |Y|^2 = R$ into itself,

$$|X'|^2 + |Y'|^2 = R \quad \text{if} \quad |X|^2 + |Y|^2 = R. \quad (15)$$

Now

$$\begin{aligned} |X'|^2 + |Y'|^2 &= |X|^2(a\bar{a} + c\bar{c}) + |Y|^2(b\bar{b} + d\bar{d}) \\ &\quad + X\bar{Y}(a\bar{b} + c\bar{d}) + \bar{X}Y(\bar{a}b + \bar{c}d). \end{aligned} \quad (16)$$

Put $(X, Y) = (\sqrt{R}, 0)$ and $(0, \sqrt{R})$ in (16), we get by (15),

$$a\bar{a} + c\bar{c} = 1, \quad b\bar{b} + d\bar{d} = 1,$$

so that on $|X|^2 + |Y|^2 = R$,

$$X\bar{Y}(a\bar{b} + c\bar{d}) + \bar{X}Y(\bar{a}b + \bar{c}d) = 0.$$

Putting $(X, Y) = \left(\sqrt{\frac{R}{2}}, \sqrt{\frac{R}{2}}\right)$ and $\left(\sqrt{\frac{R}{2}}, i\sqrt{\frac{R}{2}}\right)$ in (16), we have

$$a\bar{b} + c\bar{d} = 0, \quad \text{q. e. d.}$$

III. Proof of Reinhardt's Theorem.

6. Suppose $\mathfrak{G}$ contains $S = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$ ($a\bar{a} - bc = 1$, $abc \neq 0$), then by

Lemma 3, $\mathfrak{G}' = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \mathfrak{G} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ contains all unitary transformations with determinant unity. Let D be transformed into D' by $X = \lambda x$, $Y = y$, then we will show that D' coincides with $|X|^2 + |Y|^2 < R$.

To prove this, let the upper limit of $|X|^2 + |Y|^2$ in D' be R , then there exists a point (X_0, Y_0) in D' , such that $|X_0|^2 + |Y_0|^2 = R - \varepsilon$ ($\varepsilon > 0$) and a continuous curve L ; $X = X(t)$, $Y = Y(t)$ ($0 \leq t \leq 1$) which connects the origin to (X_0, Y_0) in D' .

Through every point $(X(t), Y(t))$ on L , there passes a sphere;

$$|X|^2 + |Y|^2 = |X(t)|^2 + |Y(t)|^2. \quad (17)$$

By Lemma 4, there exists a unitary transformation with determinant unity which transforms $(X(t), Y(t))$ to any (X, Y) on this sphere and since such a transformation belongs to $\mathfrak{G}'$, we see that the whole sphere (17) and hence $|X|^2 + |Y|^2 \leq R - \varepsilon$ belongs to D' . Since ε is arbitrary, $|X|^2 + |Y|^2 < R$ belongs to D' . But R is the upper limit of $|X|^2 + |Y|^2$ in D' , so that D' coincides with $|X|^2 + |Y|^2 < R$.

By Lemma 5, all linear transformations which transforms $|X|^2 + |Y|^2 < R$ into itself are known, i. e.,

$$\left. \begin{aligned} X' &= aX + bY \\ Y' &= cX + dY \end{aligned} \right\}$$

where $|a|^2 + |c|^2 = 1$, $|b|^2 + |d|^2 = 1$, $a\bar{b} + c\bar{d} = 0$.

Returning to D , D is of the form; $\lambda^2 |x|^2 + |y|^2 < R$ and is transformed into itself by the transformations,

$$\left. \begin{aligned} x' &= ax + \frac{b}{\lambda}y \\ y' &= c\lambda x + dy \end{aligned} \right\}$$

$$|a|^2 + |c|^2 = 1, \quad |b|^2 + |d|^2 = 1, \quad a\bar{b} + c\bar{d} = 0.$$

Hence the first part of the theorem is proved.

To prove the second part, suppose $\mathfrak{G}$ contains transformations with determinant unity only of the form; $S = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$ ($a\bar{a} - bc = 1, abc = 0$), then

by Lemma 2, such a S must be of the form; $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ ($|a|^2 = 1$) or

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} (bc = -1).$$

If we take off the limitation that the determinant is unity, $\mathfrak{G}$ contains only transformations of the form; $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}$ or $\begin{pmatrix} 0 & \mu e^{i\alpha} \\ \frac{1}{\mu} e^{i\beta} & 0 \end{pmatrix}$.

And in the latter transformation, μ can take only one certain value. For if two transformations $\begin{pmatrix} 0 & \mu_1 e^{i\alpha_1} \\ \frac{1}{\mu_1} e^{i\beta_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mu_2 e^{i\alpha_2} \\ \frac{1}{\mu_2} e^{i\beta_2} & 0 \end{pmatrix}$ exist, then

their product $\begin{pmatrix} \frac{\mu_1}{\mu_2} e^{i(\alpha_1+\beta_2)}, & 0 \\ 0, & \frac{\mu_2}{\mu_1} e^{i(\alpha_2+\beta_1)} \end{pmatrix}$ belongs to $\mathfrak{G}$, so that by Lemma 1

$$\left| \frac{\mu_1}{\mu_2} \right| = 1 \text{ or } |\mu_1| = |\mu_2|.$$

Hence the second part is proved.

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