7. An Elementary Proof of the Grace's Theorem on Double Sixers.

By Asajiro ICHIDA.

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Theorem: *If we have five lines a, b, c, d, e intersecting a given line p , then any four of them as a, b, c, d have another common intersecting line; then we get such five lines and by the theorem of double six, we know that these five lines are met by another line F . If we take six lines a, b, c, d, e, f meeting a given line p , then from each set of five we get a line like F and these six lines are all met by one and the same line.*

This theorem was proved first by J. H. Grace⁽¹⁾ and discussed from various points of view by T. L. Wren⁽²⁾, H. F. Baker⁽³⁾ and T. Kubota⁽⁴⁾. Here we will give an elementary proof in the same manner as A. C. Dixon proved the theorem of double six⁽⁵⁾.

If we denote another line intersecting c, d, e, f by (ab) and so on, then we have a common intersecting line A of $(ab), (ac), (ad), (ae), (af)$; a common intersecting line B of $(ab), (bc), (bd), (be), (bf)$; a common intersecting line C of $(ac), (bc), (cd), (ce), (cf)$; a common intersecting line D of $(ad), (bd), (cd), (de), (df)$; a common intersecting line E of $(ae), (be), (ce), (de), (ef)$; a common intersecting line F of $(af), (bf), (cf), (df), (ef)$. From the fact⁽⁶⁾ that there is a common intersecting line of A, B, C, D, E and F , we can prove the theorem by making use of the theorem of double six. Therefore it will suffice to prove the existence of the common intersecting line of A, B, C, D, E and F (Kubota's Lemma).

As a common intersecting line of the four lines C, D, E, F is (ab) , we will denote another common intersecting line by x .

(¹) J. H. Grace, *Circle, Spheres and Linear Complexes*, Trans. Camb. Phil. Soc., 16 (1898).

(²) T. L. Wren, *Proc. London Math. Soc.*, 15 (1916).

(³) H. F. Baker, *Proc. London Math. Soc.*, 15 (1916).

(⁴) T. Kubota, *Science Reports, Tôhoku Imp. University*, I, 6 (1917).

(⁵) A. C. Dixon, *Quarterly Journal of Math.*, 40 (1909).

(⁶) T. Kubota, *Lemma I*, loc. cit.

Since the lines x, D meet, the four points $xe, xf, D(cd), D(bd)$ lying on the four sides of the quadrilateral $e(cd), e(bd), f(bd), f(cd)$ (herein xe denotes the meet of the two lines x and e) are in a plane, and therefore

$$\frac{[e(cd) \cdot xe]}{[xe \cdot e(bd)]} \frac{[e(bd) \cdot D(bd)]}{[D(bd) \cdot f(bd)]} \frac{[f(bd) \cdot xf]}{[xf \cdot f(cd)]} \frac{[f(cd) \cdot D(cd)]}{[D(cd) \cdot e(cd)]} = 1,$$

where $[xe \cdot e(bd)]$ denotes the distance between the two points xe and $e(bd)$.

Similarly by the consideration of the lines x, c and the quadrilateral $e(cd), e(bc), f(bc), f(cd)$

$$\frac{[e(bc) \cdot xe]}{[xe \cdot e(cd)]} \frac{[e(cd) \cdot C(cd)]}{[C(cd) \cdot f(cd)]} \frac{[f(cd) \cdot xf]}{[xf \cdot f(bc)]} \frac{[f(bc) \cdot C(bc)]}{[C(bc) \cdot e(bc)]} = 1.$$

By the consideration of the lines x, B and the quadrilateral $e(bd), e(bc), f(bc), f(bd)$, and using the notation

$$\frac{[e(bd) \cdot xe]}{[xe \cdot e(bc)]} \frac{[e(bc) \cdot B(bc)]}{[B(bc) \cdot f(bc)]} \frac{[f(bc) \cdot xf]}{[xf \cdot f(bd)]} \frac{[f(bd) \cdot B(bd)]}{[B(bd) \cdot e(bd)]} = X,$$

we have as the condition for the intersection of x and B

$$X=1.$$

Again by the consideration of $(ad), D$ and the quadrilateral $e(cd), e(bd), f(bd), f(cd)$, we have

$$\frac{[f(bd) \cdot D(bd)]}{[D(bd) \cdot e(bd)]} \frac{[e(bd) \cdot e(ad)]}{[e(ad) \cdot e(cd)]} \frac{[e(cd) \cdot D(cd)]}{[D(cd) \cdot f(cd)]} \frac{[f(cd) \cdot f(ad)]}{[f(ad) \cdot f(bd)]} = 1,$$

and by the consideration of $(ac), C$ and the quadrilateral $e(cd), e(bc), f(bc), f(cd)$

$$\frac{[f(cd) \cdot C(cd)]}{[C(cd) \cdot e(cd)]} \frac{[e(cd) \cdot e(ac)]}{[e(ac) \cdot e(bc)]} \frac{[e(bc) \cdot C(bc)]}{[C(bc) \cdot f(bc)]} \frac{[f(bc) \cdot f(ac)]}{[f(ac) \cdot f(cd)]} = 1,$$

and by the consideration of $(ab) B$ and the quadrilateral $e(bd), e(bc), f(bc), f(bd)$

$$\frac{[f(bc) \cdot B(bc)]}{[B(bc) \cdot e(bc)]} \frac{[e(bc) \cdot e(ab)]}{[e(ab) \cdot e(bd)]} \frac{[e(bd) \cdot B(bd)]}{[B(bd) \cdot f(bd)]} \frac{[f(bd) \cdot f(ab)]}{[f(ab) \cdot f(bc)]} = 1.$$

And by the consideration of $d, (bc)$ and the quadrilateral $e(ac), e(ab), f(ab), f(ac)$; $c, (bd)$ and the quadrilateral $e(ad), e(ab), f(ab), f(ad)$; $b, (cd)$ and the quadrilateral $e(ad), e(ac), f(ac), f(ad)$; p, d and the quadrilateral $e(ac), e(ab), f(ab), f(ac)$; p, c and the quadrilateral $e(ad), e(ab), f(ab), f(ad)$; p, b and the quadrilateral $e(ad), e(ac), f(ac), f(ad)$ we have

$$\frac{[e(ac) \cdot e(bc)]}{[e(bc) \cdot e(ab)]} \frac{[e(ab) \cdot d(ab)]}{[d(ab) \cdot f(ab)]} \frac{[f(ab) \cdot f(bc)]}{[f(bc) \cdot f(ac)]} \frac{[f(ac) \cdot d(ac)]}{[d(ac) \cdot e(ac)]} = 1,$$

$$\begin{aligned}
& \frac{[e(ab) \cdot e(bd)] [e(ad) \cdot e(ad)] [f(ad) \cdot f(bd)] [f(ab) \cdot e(ab)]}{[e(bd) \cdot e(ad)] [c(ad) \cdot f(ad)] [f(bd) \cdot f(ab)] [c(ab) \cdot e(ab)]} = 1, \\
& \frac{[e(ad) \cdot e(cd)] [e(ac) \cdot b(ac)] [f(ac) \cdot f(cd)] [f(ad) \cdot b(ad)]}{[e(cd) \cdot e(ac)] [b(ac) \cdot f(ac)] [f(cd) \cdot f(ad)] [b(ad) \cdot e(ad)]} = 1, \\
& \frac{[f(ab) \cdot d(ab)] [e(ab) \cdot pe] [e(ac) \cdot d(ac)] [f(ac) \cdot pf]}{[d(ab) \cdot e(ab)] [pe \cdot e(ac)] [d(ac) \cdot f(ac)] [pf \cdot f(ab)]} = 1, \\
& \frac{[f(ad) \cdot c(ad)] [e(ad) \cdot pe] [e(ab) \cdot c(ab)] [f(ab) \cdot pf]}{[c(ad) \cdot e(ad)] [pe \cdot e(ab)] [c(ab) \cdot f(ab)] [pf \cdot f(ad)]} = 1, \\
& \frac{[f(ac) \cdot b(ac)] [e(ac) \cdot pe] [e(ad) \cdot b(ad)] [f(ad) \cdot pf]}{[b(ac) \cdot e(ac)] [pe \cdot e(ad)] [b(ad) \cdot f(ad)] [pf \cdot f(ac)]} = 1.
\end{aligned}$$

Multiplying these twelve equations side by side we have

$$1 = X$$

which expresses the fact that B meets x . And it can similarly be proved that A also meets x . Q.E.D.

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