Lateral Deformation of a Ship of Uniform Skin Floating on Water; A Suggested Method of Solving an Integral Equation*

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Summary: The subject is treated as a problem of elastica. The resulting integral equation is solved by equating the coefficients of the same power of the independent variable on both sides of the equation, which are reduced in the form of power series.

In this paper, the ship is supposed to be very long compared with her lateral dimensions, in other words, the end effects are neglected. The cross section may be of any form.

As a first step, the section is supposed to be devoid of beams and pillars so that the problem is generalized to one of elastica under water pressure and forces and couples at the ends.

When beams and pillars are introduced, the corresponding forces and couples at both ends are obtained by the conditions that the ends of the curve remain fixed both before and after the deformation.

This enables to determine the forces and couples to be borne by beams and pillars.

For convenience, let us take the origin of the coordinate axes at the beam end, and put the section upside down as shown in Fig. 1.

Fig. 1.

Let s be the length measured along the curve from the origin O , l the length of the half girth of the section, θ the angle which the tangent at

a point P makes with the horizontal x . X, Y and M are the forces and the couple acting at the end B .

θ' and s' are the tangent angle and the length of the curve from the origin O of a point anywhere on the curve when s is temporarily considered as fixed.

Let Θ be the value of θ when the section is unstrained, or the value of the original ship's section. θ and Θ are both continuous functions of s . Then the equation of equilibrium is

$$EI \frac{d\theta}{ds} = - \int_s^l p' \left[\int_s^{s'} \cos(\theta' - \theta') ds' \right] ds' + M - X \int_s^l \sin \theta ds + Y \int_s^l \cos \theta ds + EI \frac{d\Theta}{ds};$$

$$p' = w \int_{s_0}^{s'} \sin \theta' ds',$$

$$\int_0^{s_0} \sin \theta ds = \text{the freeboard},$$

where p' is the water pressure at s' , and s_0 the value of s at the water line. E and I are the Young's modulus and the moment of inertia of the section of the curve, w the weight of water per unit volume.

Although in the present problem, the solution is far simpler with graphical method as we will see later in this paper, the following analytical method is here described, as it offers a simple method of solving an integral equation in general, in which the unknown function is developable in power series according to Taylor's theorem.

To solve the above equation, put $z = \frac{s}{l}$, and as the angle θ is one valued, finite and continuous function of z , it may be put in the form of a convergent series

$$\theta = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n + \dots$$

Also the angle Θ of the original section may

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be put in a similar form

$$\theta = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \dots + \alpha_n z^n + \dots,$$

where the constants α_s are known from the given section of ship.

Let the curve be vertical when $z=0$, and horizontal when $z=1$,

$$\cos \theta = 0, \quad \sin \theta = 1, \quad \text{when } s=0 \text{ or } z=0;$$

$$\cos \theta = 1, \quad \sin \theta = 0, \quad \text{when } s=l \text{ or } z=1.$$

Let

$$\cos \theta = c_1 z + c_2 z^2 + c_3 z^3 + \dots + c_n z^n + \dots$$

$$\sin \theta = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots + b_n z^n + \dots$$

The relation between c_s and b_s may be found by separately equating to zero the coefficients of different powers of z in the relation

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Thus we have successively

$$b_1 = 0, \quad b_2 = -\frac{c_1^2}{2}, \quad b_3 = -c_1 c_2,$$

$$b_4 = -\frac{1}{8} c_1^4 - c_1 c_3 - \frac{1}{2} c_2^2, \quad b_5 = -\frac{1}{2} c_1^3 c_2 - c_1 c_4$$

$$-c_2 c_3, \quad b_6 = -\frac{1}{16} c_1^6 - \frac{1}{2} c_1^3 c_3 - \frac{3}{4} c_1^2 c_2^2 - c_1 c_5$$

$$-c_2 c_4 - \frac{1}{2} c_3^2, \quad b_7 = -\frac{3}{8} c_1^5 c_2 - \frac{1}{2} c_1^3 c_4$$

$$-\frac{3}{2} c_1^2 c_2 c_3 - \frac{1}{2} c_1 c_2^3 - c_1 c_6 - c_2 c_5 - c_3 c_4, \dots = \dots,$$

and from the series for $\cos \theta$, or from the relation

$$\frac{d}{dz} \sin \theta = \cos \theta \frac{d\theta}{dz},$$

we get the values of c_s and b_s in terms of α_s as follows:

$$c_1 = -\alpha_1, \quad c_2 = -\alpha_2, \quad c_3 = \frac{1}{6} \alpha_1^3 - \alpha_3, \quad c_4 = \frac{1}{2} \alpha_1^2 \alpha_2$$

$$-\alpha_4, \quad c_5 = -\frac{1}{5!} \alpha_1^5 + \frac{1}{2} \alpha_1^2 \alpha_3 + \frac{1}{2} \alpha_1 \alpha_2^2 - \alpha_5,$$

$$c_6 = -\frac{1}{4!} \alpha_1^4 \alpha_2 + \frac{1}{2} \alpha_1^2 \alpha_4 + \alpha_1 \alpha_2 \alpha_3 + \frac{1}{6} \alpha_2^3 - \alpha_6,$$

$$c_7 = \frac{1}{7!} \alpha_1^7 - \frac{1}{4!} \alpha_1^4 \alpha_3 - \frac{1}{12} \alpha_1^3 \alpha_2^2 + \frac{1}{2} \alpha_1^2 \alpha_5 + \alpha_1 \alpha_2 \alpha_4$$

$$+ \frac{1}{2} \alpha_1 \alpha_3^2 + \frac{1}{2} \alpha_2^2 \alpha_3 - \alpha_7, \dots = \dots$$

$$b_2 = -\frac{\alpha_1^2}{2}, \quad b_3 = -\alpha_1 \alpha_2, \quad b_4 = \frac{1}{4!} \alpha_1^4 - \alpha_1 \alpha_3 - \frac{1}{2} \alpha_2^2,$$

$$b_5 = \frac{1}{6} \alpha_1^5 \alpha_2 - \alpha_2 \alpha_3 - \alpha_1 \alpha_4, \quad b_6 = -\frac{1}{6!} \alpha_1^6$$

$$+ \frac{1}{6} \alpha_1^3 \alpha_3 + \frac{1}{4} \alpha_1^2 \alpha_2^2 - \alpha_2 \alpha_4 - \alpha_1 \alpha_5 - \frac{1}{2} \alpha_3^2, \quad b_7 =$$

$$-\frac{1}{5!} \alpha_1^5 \alpha_2 + \frac{1}{6} \alpha_1^3 \alpha_4 + \frac{1}{2} \alpha_1^2 \alpha_2 \alpha_3 + \frac{1}{6} \alpha_1 \alpha_2^3 - \alpha_1 \alpha_6$$

$$- \alpha_2 \alpha_5 - \alpha_3 \alpha_4, \dots = \dots$$

The constants c_s and b_s being thus expressible in terms of α_s , the solution of the given equation is effected as follows:

Consider α_s as undetermined constants and perform the integration in the second member of the equation,

Then separately equate the coefficients of the same power of z on both members of the equation.

The system of equations thus obtained consists of one equation of absolute terms to determine the unknown moment M . This equation also gives $\alpha_1 = \alpha_1$, since the other terms must maintain equilibrium among themselves.

The other equations give the values of α_s in terms of α_1 successively. When $\alpha_1 = 0$, α_s is given in terms of α_2 .

When beams and pillars are introduced, the conditions

$$\int_0^1 \cos \theta \, dz = (\text{half breadth of ship}) \div l,$$

$$\int_0^1 \sin \theta \, dz = (\text{depth of ship}) \div l,$$

give X and Y . The corresponding value of M is given by the relation of absolute terms above mentioned.

An application of this analytical treatment will be found in my paper on the strength of a studied chain cable, which is now in the hands of this society.

To illustrate the graphical method of solution, take the section as in Fig. 2.

The bending moment at a point z is, when we start from the origin O ,

$$B.M. = - \int_0^z p' \left(\int_{z'}^z \cos(\theta' - \theta'') \, dz'' \right) dz' \\ + C'y + D'x,$$

Fig. 2.

where C' and D' are the horizontal and vertical reactions of water pressure at the origin, and θ' is the angle corresponding to z'' , a current point on the curve.

Transform the origin to (O, h) , where h is the freeboard. Take l for the unit of length and wl^2 for the unit of force.

Then writing the moment in terms of the coordinates x and y , we have

$$B.M. = -x \int_0^y y' dx' + \int_0^x x' y' dx'$$

$$= -\frac{1}{6} y^3 + C'(y+h) + D'x.$$

Fig. 3.

Fig. 4.

The bending moments for the section with beams only and for the section with both beams and pillars are shown in Fig. 3.

To calculate the lateral deformation, let us take for simplicity sake $EI=1$. Then the bending moment twice integrated with respect to z gives the deflection u .

The first integral curve $\frac{du}{dz}$ does not terminate with zero value generally. Join the end point of the curve $\frac{du}{dz}$ with the origin O with a straight line.

The value of the intercepted part between this straight line and the curve, and its integral curve which shows the deflection u is shown in Fig. 4.

The deflection in the case of the section with pillars is too small to be shown together in this figure.

The meaning of this process is to impose the condition that the curve of deflection shall have its tangent horizontal at the end point $z=1$.

At the same time, it means to lower or raise the horizontal axis of the bending moment curve correspondingly. The remaining bending moment at $z=1$ is the actual couple acting at that point.

By this means, the amount of the couples at $z=0$ and $z=1$ are simultaneously revealed.

The corrected base line is inscribed in Fig. 3.

The moments acting at the origin and at the terminal point, namely, at the end of the beam and at the keel, are found to be -0.023 and 0.035 for the section without pillars, and -0.0093 and -0.0071 for the section with pillars.

On the Dislocation, the Group of Holonomy and the Theory of Yielding*

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By the critical study on the theory of incompatibility and dislocation of the elastic body based on Riemannian geometry, the author is led to the analytical expression of the yielding phenomena. This paper presents the fundamental ideas.

§1. Elasticity and Riemannian Geometry.

If each material point of a body is denoted by rectangular cartesian coordinates (x_B^1, x_B^2, x_B^3) fixed to the point in the ordinary Euclidean space of three dimensions, the distance ds_B of two adjacent points x_B^i and $x_B^i + dx_B^i$ is given by the quadratic differential form

$$(ds^2 =) ds_B^2 = (dx_B^1)^2 + (dx_B^2)^2 + (dx_B^3)^2 \quad (1)$$

This method of description referred to a point of space may be called "Eulerian". If we apply on the contrary a system of the so-called

Lagrangian co-ordinates x_L^λ ($\lambda=1, 2, 3$) fixed to each material point, the metric (1.1) may be written

$$ds_B^2 = g_{B\lambda\mu} dx_L^\lambda dx_L^\mu \quad (2)$$

We have sometimes, however, to consider the non-Euclidean, especially Riemannian metric for the state free of any strain. We may call such a state the "free state". But it cannot necessarily be realized and the strain cannot necessarily be taken away from the elastic body, so far as it is constrained in the ordinary space which is considered Euclidean. The stresses which accompany the remaining strains in such cases after the removal of the external forces are the so-called residual stresses.

The Eulerian metric of the elastic body even with the residual stresses is Euclidean so far as it is the real space. Taking away of the stress and strain means the swelling of the elastic

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