Plastic Deformation of Light-metal Bars Strained with Combined Tension and Torsion

By Hisao Mii

1. Introduction

Although a few researches have been developed on the rupture or yielding of cast iron or mild steel respectively subjected to the combined loadings, few has been made on the plastic deformations of soft material in its strain-hardening region, especially on that of light metal.

The present paper deals with the plastic deformations of light-metal bars under the ideal simultaneous actions of loadings whose manners are being stated in the title. In this, the deformation of material in its strain-hardening region is assumed to obey the so-called "Maximum Shear Stress Theory," and the distributions of axial stress and shearing one in the cross section of bar, as well as the expressions of tensile force and twisting moment needed for the production of these stress distributions have been given, taking into account the reduction of the radius of bar accompanying the strong tension.

Though these expressions are of forms of definite integral as the tensile stress-strain relation of material being assumed to be spontaneous, they can be integrated numerically with ease provided any tensile stress-strain diagram is near at hands. Making use of the result of tension test of an American material 17 SO aluminium, several calculations have been carried out and indicated on the figures.

2. Plastic Deformation of Round Bar under Combined Tension and Torsion

There are three manners of loading in general in order to obtain the desired amount of plastic deformation of the bar under such a combination of external forces, namely (a) application of twisting moment of comparatively small amount to the bar already stretched with strong tensile force, (b) application of weak tensile force to the bar twisted previously with strong torsional moment, and (c) application of ideal simultaneous actions of tensile force and twisting moment from the start. Of course these differ in the loading career and we cannot make use of the same stress-strain relation. The fact that the path of deformation varies, though slightly, with the loading career is a subject to be discussed here, however this will be omitted for want of space and the readers may have to refer to the works, for instance, of Prof. W. Lode and Prof. Y. Nakagawa etc.

It has been assumed in the following for the simplicity's sake that the bar deforms under the loading manner (c) from the start.

The stress state in a bar deformed under the above-mentioned combined loadings is of rotational symmetry about its axis. Considering now large deformation, the elastic components of strain are assumed to be negligible against the plastic ones. Since no other forces such as internal or external pressures are being applied than the axial one, of the expression of general rotationally symmetrical state of deformation, the stresses $\sigma_r$ and $\sigma_\theta$ corresponding respectively to the strains $\varepsilon_r$ in radial and $\varepsilon_\theta$ in tangential directions at any point in the cross section diminish for the present and as for stresses the axial one $\sigma_z$ corresponding to the strain $\varepsilon_z$ and the shearing one $\tau$ corresponding to the shearing strain in torsion $\gamma$ acting tangentially to the circle in the plane perpendicular to the bar axis govern the circumstance and these are the functions of radial distance $r$ from the axis. In such a case three principal stresses at any point can be written as
Of the strains, the distribution of axial one $\varepsilon_z$ is not always uniform, in reality, at least near that part of bar in the deformation experienced usually in tension test at which so-called "local necking" occurs. However as it is too difficult to take this into account we will go on assuming an ideal uniformity of $\varepsilon_z$ along the axis. From this we have

$$\varepsilon_r = \varepsilon_0 = -\varepsilon_z/2 = \text{const}$$

by the integration of the approximate condition of incompressibility of plastic mass $\varepsilon_z + \varepsilon_r + \varepsilon_\tau = 0$, from which the three principal strains become for the present as follows

$$2\varepsilon_1 = \varepsilon_2/2 + \sqrt{9/4\cdot \varepsilon_z^2 + \gamma^2},$$
$$2\varepsilon_2 = -\varepsilon_1,$$
$$2\varepsilon_3 = \varepsilon_2/2 - \sqrt{9/4\cdot \varepsilon_z^2 + \gamma^2}.$$  

The maximum principal stress difference $\sigma$ and the maximum principal strain difference $\varepsilon'$ become respectively as follows

$$\sigma = \sigma_1 - \sigma_2 = \sqrt{\sigma_z^2 + 4\varepsilon_z^2},$$
$$\varepsilon' = \varepsilon_1 - \varepsilon_2 = \sqrt{9/4\cdot \varepsilon_z^2 + \gamma^2}.$$  

in which $\varepsilon'$ is nothing but the maximum principal shearing strain and is related to the normal strain $\varepsilon$ corresponding to $\sigma$ as

$$2\varepsilon' = 3\varepsilon,$$  

and moreover there is a functional relation

$$\sigma = f(\varepsilon') = f(3/2\cdot \varepsilon),$$  

and $\sigma$ and $\varepsilon$ are taken to be the true tensile stress and strain of the material in the plastic tension from the assumption of the maximum shear stress theory.

In the next place, from the first and third terms of the well-known volume change relation of the theory of plasticity

$$\left(\varepsilon_1 - \varepsilon_2\right) / \left(\sigma_1 - \sigma_2\right) = \left(\varepsilon_2 - \varepsilon_3\right) / \left(\sigma_2 - \sigma_3\right) = \left(\varepsilon_3 - \varepsilon_1\right) / \left(\sigma_3 - \sigma_1\right),$$

we have

$$\left[3/2\cdot \varepsilon_1 + \sqrt{9/4\cdot \varepsilon_z^2 + \gamma^2}\right] / \left[\sigma_1 + \sqrt{\sigma_z^2 + 4\varepsilon_z^2}\right] = \sqrt{9/4\cdot \varepsilon_z^2 + \gamma^2}/\sigma,$$  

from which we obtain

$$\sigma = \sigma_1 / \left[\sqrt{1 + (2/3\cdot \gamma/\varepsilon_z)^2}\right],$$  

therefore from the first of eq. (4) we also have

$$\tau = \sigma_2 / \left[\sqrt{1 + (2/3\cdot \gamma/\varepsilon_z)^2}\right],$$  

and the second of the same equation can be written as

$$\varepsilon = \varepsilon_0\sqrt{1 + (2/3\cdot \gamma/\varepsilon_z)^2}.$$  

Now since the bar is considered to be pulled by strong tensile force, we have to take the reduction of its radius into account. It may be most reasonable to assume that the radial strain $\varepsilon_r$ being equal here to the effective one

$$\varepsilon_r = -\varepsilon_z/2 = \int_0^r dr/r,$$

from which we have

$$a = a_0 e^{-\varepsilon_z/2},$$  

where $a_0$ and $a$ are the original and new radii of the bar respectively. This shows that the more the bar is strongly pulled, the more its radius is reduced by the exponential law.

Since the shearing strain in torsion $\gamma$ is the function of radial position alone and is given by

$$\gamma = r\theta,$$  

provided $\theta$ is the specific angle of torsion, we have at the skin

$$\gamma_s = a\theta.$$  

Denoting further

$$\mu = \gamma_s / \varepsilon_s,$$  

we have

$$\gamma_s = \gamma_s / \varepsilon_s \gamma_s = \mu r / a.$$  

By this, eqs. (9) - (11) can be written as

$$\sigma = \sigma_0 / \left[\sqrt{1 + (2/3\cdot \mu r / a)^2}\right],$$
$$\tau = \sigma_2 / \left[\sqrt{1 + (2/3\cdot \mu r / a)^2}\right],$$
$$\varepsilon = a_0\sqrt{1 + (2/3\cdot \mu r / a)^2},$$
$$\sigma = f(\varepsilon).$$

Eq. (18) gives the distributions of axial stress and shearing one in torsion in the cross section of bar and we can calculate these for given values of $\mu$ and $\varepsilon_s$ the value of $\varepsilon$ at the skin. As is seen from eq. (18) the maximum of axial stress $\sigma$ occurs at $r=0$, center axis of the bar and decreases outward and the shearing stress $\tau$ is zero at the center.
axis and increases outward. In this case, furthermore, the angle $\varphi$ between the direction of tension and the maximum principal stress $\sigma_1$ becomes as

$$\varphi = \tan^{-1}\left(\left\{-\sigma_1 + \sqrt{\sigma_1^2 + 4\tau^2}\right\}/2\tau\right).$$

(19)

$\varphi$ diminishes for pure tension and $\varphi = \pi/4$ for pure torsion, and those for all other combinations lie between these two.

The tensile force $P$ and the twisting moment $M_d$ needed to produce the above stress distributions are given by

$$P = 2\pi \int_0^a \sigma r dr$$

$$= 2\pi \int_0^a \sigma \sqrt{1 + (2/3 \cdot \mu \cdot \tau/a)^2} r dr,$$

$$M_d = 2\pi \int_0^a \tau r^2 dr$$

$$= \pi \int_0^a \sigma \sqrt{1 - 1/[1 + (2/3 \cdot \mu \cdot \tau/a)^2]} r^2 dr.$$ (20)

These are to be simultaneous as for $\mu$ and $a$ and the integrations can be easily evaluated numerically.


Making use of the results of tension test made on 17SO aluminium by Prof. C.W. MacGregor, some numerical calculations have been made on a bar of 1 cm in radius. In general, cases are found in two manners on the numerical computation of present problem in which (i) $\mu$ and $a$ satisfying eq. (20) are first obtained, for given $P$ and $M_d$, by making use of the last two of eq. (18), and thereafter $\sigma_1$ and $\tau$ are to be determined by the introduction of these two now obtained and (ii) the stress distributions are directly obtained from eq. (18) for given $\mu$ and $a$, and thereafter $P$ and $M_d$ required to produce these distributions are to be computed. Of these, several iterations are needed in (i), though no trouble takes place in (ii).

Fig. 1 shows examples of stress distributions for three combinations, in all of which $\varepsilon_a$, the value $\varepsilon$ at the skin, is equal to 0.12 on the $\sigma-\varepsilon$ diagram of the material. The pairs of $P$ and $M_d$, the new radius $a$ and the specific angle of torsion $\theta$ tabulated there are of those calculated from the manner (ii).

Of them, (a) is the case of the combination of the strongest tensile force and weakest twisting moment and (c) is the one of the weakest tensile force and strongest twisting moment. (b) is an example for moderate combinations.

Fig. 1 Stress and strain distributions when the deformation of skin came up to the same amount $\varepsilon_a=0.12$ and the corresponding values of pair $(P,M_d)$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\alpha$</th>
<th>$\theta$</th>
<th>$P$</th>
<th>$M_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cm</td>
<td>rad/cm</td>
<td>kg cm/kg</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a)</td>
<td>0.5</td>
<td>0.9446</td>
<td>0.0602</td>
<td>5555</td>
</tr>
<tr>
<td>(b)</td>
<td>3.0</td>
<td>0.9735</td>
<td>0.1651</td>
<td>3574</td>
</tr>
<tr>
<td>(c)</td>
<td>6.0</td>
<td>0.9855</td>
<td>0.1771</td>
<td>2246</td>
</tr>
</tbody>
</table>

Fig. 2 $P-M_d$ chart for constant $\varepsilon_a$.

Fig. 2 shows the combinations of $P$ and $M_d$ giving constant $\varepsilon_a$ for the same material in which (continued on page 24)
Hg あるいはそれ以下でも十分使える。
（3）音よりずつと遠い流れの境界層 非常な高速度では境界層が厚くなり、例えば平板境界層の排除厚は \( M = 7 \) では \( M = 1 \) の約 10 倍になる。このため、翼型表面の圧力分布における境界層の影響は無視できなくなる。平板についての実測値は理論とよく一致した。（4）非定常 流動高速度 HAVE 振動的な非定常流が発生し、圧縮機, 脱気倶楽部, 脱気倶楽部推進機等に利用される一方、空気圧入口や diffuser 等には障害を興える。たとえば Seippel の "comprex" 圧縮機では圧縮波と振動波がビストンの役割を演ずる。Ferri は軸対称の音より遠い空気圧入口に発生する振動の研究を行なないが、これが衝撃波と境界層の干渉に基づくことを明らかにした。また音より近い diffuser における衝撃流に由来する振動を観察するための基礎実験を行なっている。Huber は shock tube の中で、その流れが一次元理論で満足に説明されることを痛めた。(shock tube は、一本の管を音速までにし、異なる圧力の気体をつめたもので、ある時間後に壁を破ると高圧部から低圧部に衝撃波が伝わって行く。これを利用して高速気流を研究する装置である。Huber は高圧部の長さ 32 in, 低圧部 24 〜 72 in, 断面 3 × 3 in の管で圧力比 2:1 のものを使用している。)(今非功)

(Continued from page 18)

\( \mu \) is taken as parameter. By this hereafter we will be much saved from the troublesome iterations in the calculating manner (i).

4. Conclusions

The stress distributions and the pairs of external forces needed to produce them have been analyzed for a light-metal bar deformed largely by the combined loadings. The general features of phenomena have been made clear by the numerical examples made with the true tensile stress-strain diagram of 17SO aluminium. As no restriction is placed here in the form of stress-strain curve of material, the present results of analysis will be applied even for a material having continuous stress-strain curve such as alloy steel.

In conclusion the present author wishes to express his sincere thanks to Prof. C.W. MacGregor of M.I.T. who has had the kindness to permit him of the use herein of fig. 2 in his paper published on the cylinder problem.

References

1) Read before at the 27th Session of the Society held at Kyushu University on 10th, Dec. 1949.
2) Research Dept. K.K. Yasukawa Denki Seisa kusho
4) W. Lode, Forschungsarbeiten, Heft 303, 1928.