On Swinging of a Wire of Transmission Line with Suspension Insulators, I.

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Abstract.

The Author has made a theoretical study on the swinging motion in the vertical plane of the wire of over-head electric power transmission line. We take the case of several consecutive spans each suspended with suspension insulators, the two extreme ends being dead ended. In this Report I, the fundamental equations and characteristic equation of free oscillation together with a numerical example are given.

1. Introduction. There are many cases under which the wire of an over-head electric power transmission line makes oscillatory or swinging motions. Sudden flow of a heavy short circuit current through the wire, or sudden drop of coating of sleet from the wire causes the wire to swing in nearly the vertical plane. When there are consecutive several spans each suspended with suspension insulators, the two extreme ends being dead ended. In this Report I, the fundamental equations and characteristic equation of free oscillation together with a numerical example are given.

2. Fundamental Equations. Referring to Fig. 1, let us use the following notations;

\[ L = \text{span m, } l = \text{length of a string of suspension insulators m,} \]

\[ a = \text{angle of deflection of a string of suspension insulators rad.,} \]

\[ T = \text{horizontal tension of a wire kg,} \]

\[ x = \text{horizontal distance (abscissa) m,} \]

\[ E = \text{Young's modulus of the wire kg/cm}^2, \]

\[ w = \text{weight/m of the wire kg/m,} \]

\[ W = \text{weight of a span of wire kg,} \]

\[ S = \text{whole length of a span of wire m,} \]

\[ D = \text{sag of the wire m,} \]

\[ Q = \text{vertical load which hangs on the wire clamp kg,} \]

\[ Y = \text{ordinate (measured down-wards) of the wire at the position x m,} \]

\[ y = \text{value of the varied part of Y m,} \]

In order to make the analysis suitable to practical applications, the Author has made a simple assumption as to the relation between tension and deflection of the wire.

In order to specify to which span the quantities \(a_i, T_i, D_i\), etc. belong, we use suffixes and denote \(a_i, T_i\), etc. (see Fig. 1). For simplicity we assume in what follows that the successive spans have the same span length \(L\), but similar treatment can be made for the case in which span lengths \(L\) are not

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2. Department of Engineering, Keio-University.
4. V. Volterra J. P. Rea, Lecons sur la Composition et les fonctions permutable, 1924.
the same to each other. The suffix o is used to
denote the value at final, or stable state.

Using these notations, the equation of motion
of the wire is given by
\[ -w \frac{\partial^2 X_i}{\partial t^2} + T_i \frac{\partial Y_i}{\partial t} = -w. \] (1)

The relation between tension and elongation is taken
to be approximately given by
\[ \frac{T-T_0}{AE} = \frac{S-S_0}{L}. \] (2)

During the swinging motion, the span is increased
by an amount \( \Delta S \), where
\[ \Delta S = l \sin (\alpha_i - \alpha) \] (3)
so that the whole length \( S \) of the wire will be
given by
\[ S = \int_0^{L+\Delta S} \sqrt{1 + \left( \frac{\partial Y}{\partial x} \right)^2} \, dx \]
or, approximately, by
\[ S = L + \frac{1}{2} \int_0^L \left( \frac{\partial Y}{\partial x} \right)^2 \, dx + \Delta S \] (4)

Equation of force acting at a
suspension clamp is (Fig. 2)
\[ \frac{T_{i+1} - T_i}{1/2 (W_i + W_{i+1})} = \tan \alpha_i. \] (5)

At the initial state, which
coresponds to \( t=0 \), we sup-
pose that \( Y_i = Y_0, \alpha_i = 0, T_i = T_0 \), which means
that the transmission line is in equilibrium under
uniform tension. At \( t=0 \), the loading is suddenly
changed, and from \( t=0 \) onwards, the wire begin to
make swinging motion in the vertical plane. The
final state of equilibrium, if the change of loading
took place very gradually, will be denoted by \( Y_i = Y_0, \alpha_i = \alpha_0, T_i = T_0 \).

3. Treatment of Linear Equation for \( y_i \). During the swinging motion, we put
\[ Y_i = Y_0 + \psi_i. \] (6)

Suppose that the angle \( \alpha_i \) of inclination of insulator
string is sufficiently small so that we can take \( \alpha_i \)
instead of \( \tan \alpha_i \).

Substituting the value (6) and combining the
equations (1) to (5) we have, after neglecting \( y_i^2 \)
in comparison with \( Y_i^2 \),
\[ \frac{\partial^2 \psi_i}{\partial t^2} + \frac{\partial^2 \psi_i}{\partial x^2} + \frac{w}{w_0^2} \left( \frac{w}{w_0} \right) \int_0^L \psi_i \, dx \]
\[ + \frac{l}{L} (\alpha_i - \alpha_i) = 0, \] (7)

where
\[ \psi_i = \sqrt{\frac{g T_0}{w_0}}. \] (8)

In order to solve these system of linear differential
equations, we put
\[ \psi_i = \Sigma \gamma_i \cos \omega_i t, \]
\[ \alpha_i = \Sigma \gamma_i \sin \omega_i t. \] (9)

where \( \gamma_i \) are functions of \( x \). Thus the solution
for \( \psi_i \) consist of aggregate of harmonic oscillations
of angular frequencies \( \omega_i \). In what follows we shall
find each separate component oscillations. We may
omit the suffix \( n \) without causing any confusion.

Then we have
\[ \frac{\omega_{i,n}^2 \gamma_i}{v_i^2} + \frac{d^2 \gamma_i}{dx^2} + \frac{w_0 A E}{T_0^2} \int_0^L \gamma_i \, dx \]
\[ + \frac{l}{L} (\alpha_i - \alpha_i) = 0. \] (10)
The initial condition is satisfied by putting
\[ \sum \gamma_i = \psi_i = 0, \]
\[ \sum \alpha_i = 0. \]

The solution of the eq. (10) which satisfy the
boundary condition that at \( x=0 \) or \( L, \gamma_i = 0 \) is
given by
\[ \gamma_i = \frac{1}{\Delta} \left[ \sin \left( \frac{\omega_i}{v_i} x + \varphi_i \right) - \sin \left( \frac{\omega_i}{v_i} \varphi_i \right) \right], \]
where
\[ B_i = \frac{w_0 A E L}{T_0^2} (\alpha_i - \alpha_i), \]
\[ C_i = \left( \frac{w_0 A E}{T_0 L} \right)^2, \]
\[ a_i = \left( \frac{w_0 A E}{T_0 L} \right) \cos \left( \frac{\omega_i}{v_i} \frac{L}{2} \right) + 2 \left( \frac{\omega_i}{v_i} \right), \]
\[ \sin \left( \frac{\omega_i}{v_i} \frac{L}{2} \right), \]
\[ \Delta_i = \left( \frac{w_0 A E}{T_0 L} \right)^2 \cos \left( \frac{\omega_i}{v_i} \frac{L}{2} \right) + 2 \left( \frac{\omega_i}{v_i} \right), \]
\[ \sin \left( \frac{\omega_i}{v_i} \frac{L}{2} \right), \]
\[ E_i = \frac{1}{\Delta_i} \left[ \frac{v_i}{\omega_i} \sin \left( \frac{\omega_i}{v_i} \frac{L}{2} \right) - \frac{L}{2} \cos \left( \frac{\omega_i}{v_i} \frac{L}{2} \right) \right]. \] (11)

4. Characteristic Equation for the
Swinging Motion in the Vertical Plane.

Putting the above solution (9) (11) into the equation
(7) we have
\[ \alpha_{i+1} = \mu \left[ -F_1 (E_i (\alpha_{i+1} - \alpha_i) - E_i (\alpha_i - \alpha_i)) \right. \]
\[ + \frac{l}{L} \left( (\alpha_{i+1} + \alpha_i) - (\alpha_i - \alpha_i) \right), \] (12)

where
\[ F_1 = \frac{w_0}{T_0 L}, \quad \mu = \frac{AE}{1/2 (W_i + W_{i+1})}. \]
In this eq. we have to take $i=1,2,3, \ldots$ with $a_i=0$ for $i=6$ or for $i=1$.

The system of equations (12) for amplitudes of oscillation $a_i$, are in form of a system of difference equations. Eliminating $a_i$s from these equations there results a determinant equation which is the characteristic equation for a component oscillation. There are an infinite number of solutions of this characteristic equation, which gives values of angular frequencies $\omega_1, \omega_2, \ldots$. For practical purpose, only some lowest of them will be required.

As an application of this theory, let us take the case in which there are consecutive five spans, and let us assume that the state of equilibrium and swinging motion all occur symmetrically about the central point of the middle span. In such a case we have

\[
\begin{align*}
   a_5 &= u_5[-F_1(E_5(a_5-a_3)-E_4(a_5-a_2)) + \frac{1}{L}(a_5-2a_3+a_1)], \\
   a_4 &= u_4[-F_3(E_3(a_4-a_2)-E_2(a_4-a_3)) + \frac{1}{L}(a_4-2a_3+a_2)], \\
   a_3 &= u_3[-F_3(E_3(a_3-a_2)-E_2(a_3-a_1)) + \frac{1}{L}(a_3-2a_2+a_1)], \\
   a_2 &= u_2[-F_2(E_2(a_2-a_1)-E_1(a_2-a_1))] + \frac{1}{L}(a_2-2a_1+a_0). \\
\end{align*}
\]

Putting the condition of symmetry viz., $E_5=E_1$, $E_4=E_2$, $u_5=u_1$, $u_4=u_2$, $u_3=u_2$, $a_5=a_2$, $a_3=a_4$ into there system of equations, and eliminating the unknown amplitudes $a_1 \cdots a_5$ from them, we have

\[
\begin{align*}
   \left[ \frac{1}{u_4} + 2\frac{L}{L} - F_3(E_4+E_3) \right] \cdot \left[ \frac{1}{u_3} + 3\frac{L}{L} - F_3(E_3+E_2) \right] = \left[ \frac{1}{L} - F_3(E_4+E_3) \right] \cdot \left[ \frac{1}{L} - F_3(E_3+E_2) \right].
\end{align*}
\]

Finding the value $\omega$ of angular freq. of this equation, the natural frequency of the swinging motion is determined.

5. Numerical Example. The Author has made numerical calculations of the above mentioned theoretical formula. Only an example will be reported here. We take the case of consecutive five spans, as shown in Fig. 3, where initially each span is loaded with the same intensity $3\omega_0(\text{kg/m})$ of vertical loads and is in equilibrium. At the instant $t=0$, the load $3\omega_0$ of No. 3 span is supposed to be suddenly changed into $\omega_0$ (probably due to sudden fall of sleet coating). It is required the subsequent swinging motion of the wire.

![Fig. 3](image)

Previous to consideration of the case of sudden fall, let us take the case in which the fall occurs very gradually. The angles $a_i$ of inclination of insulator string is supposed to take the following values under which the new configuration of equilibrium is established. $a_0=0$, $a_5=10^\circ$, $a_4=20^\circ$, $a_3=-20^\circ$, $a_2=-10^\circ$. We shall take the span $L=194\text{ m}$, weight of wire $\omega_0=1.00\text{ kg/m}$ and final value of tension $T_3=657\text{ kg}$. According to (4), the corresponding values of $T_4$ and $T_5$ which equilibrate with the above mentioned value of $T_3$ is found to be $T_4=4.06\omega_0L$, $T_5=4.58\omega_0L$.

Now, turning to the case of sudden fall, we must solve the numerical equation (13), where we have

\[
\begin{align*}
   \mu_3 &= 0.295, \quad \mu_4 = 0.740, \quad \mu_5 = 0.656, \\
   \lambda_3 &= 1520, \quad \lambda_4 = 1260, \quad \lambda_5 = 1120, \\
   C_3L^2 &= \mu_3^3\lambda_3 = 132, \quad C_4L^2 = \mu_3^3\lambda_4 = 690, \\
   C_5L^2 &= \mu_5^3\lambda_5 = 480.
\end{align*}
\]

If we put for shortness

\[
\begin{align*}
   \xi_3 &= \frac{\omega}{\sqrt{\frac{L}{2}}}, \\
   \xi_4 &= \frac{\omega}{\sqrt{\frac{L}{2}}} = 1.58\xi_3, \\
   \xi_5 &= \frac{\omega}{\sqrt{\frac{L}{2}}} = 1.49\xi_3,
\end{align*}
\]

and the characteristic equation (13) becomes:

\[
\begin{align*}
   \left[ 2 - \mu_1 \left( \frac{\mu_3\lambda_3}{4 \xi_3^3 \cos \xi_3^2} + C_4L^2(\sin \xi_3^2 - \xi_3^2 \cos \xi_3^2) \right) \right] \\
   - \mu_2 \left( \frac{\mu_3\lambda_3}{4 \xi_3^3 \cos \xi_3^2} - C_4L^2(\sin \xi_3^2 + \xi_3^2 \cos \xi_3^2) \right) \\
   \left[ 3 - \mu_3 \left( \frac{\mu_3\lambda_3}{4 \xi_3^3 \cos \xi_3^2} + C_4L^2(\sin \xi_3^2 - \xi_3^2 \cos \xi_3^2) \right) \right] \\
   - 2 \mu_4 \left( \frac{\mu_3\lambda_3}{4 \xi_3^3 \cos \xi_3^2} + C_4L^2(\sin \xi_3^2 + \xi_3^2 \cos \xi_3^2) \right) \\
   - 1 - \mu_5 \left( \frac{\mu_3\lambda_3}{4 \xi_3^3 \cos \xi_3^2} + C_4L^2(\sin \xi_3^2 - \xi_3^2 \cos \xi_3^2) \right) \\
   \cdot \left[ 1 - \mu_5 \left( \frac{\mu_3\lambda_3}{4 \xi_3^3 \cos \xi_3^2} + C_4L^2(\sin \xi_3^2 - \xi_3^2 \cos \xi_3^2) \right) \right] = 0
\end{align*}
\]

or,

\[
\begin{align*}
   2 - \frac{690(\sin \xi_3^2 - \xi_3^2 \cos \xi_3^2)}{4 \xi_3^3 \cos \xi_3^2 + 690(\sin \xi_3^2 - \xi_3^2 \cos \xi_3^2)} = 0
\end{align*}
\]
The root of this equation can be obtained by graphical method. In Fig. 4, the curve A shows the graph of the value of left-hand side of the numerical equation (14), the values of $\xi$ being taken as abscissa. From the curve we infer that the lowest root of the equation (14) is given by $\xi=143^{\circ}=2.50$ rad.

Corresponding to this, we find,

$$\omega = \xi v_3 \cdot \frac{2}{L} = 2\xi \sqrt{\frac{gT_3}{L^2}} = 2.07,$$

so that the period of swinging motion will be $2\pi/2.07 = 3.04$ sec. If the above mentioned numerical example, we change the value of $T_3$ from 657 into $657 \times 1.5 = 986$ kg, while the other data remain quite the same, the curve A in Fig. 4 becomes changed into the curve B of the same figure. So that we have $\xi=112^{\circ}=1.95$ rad. Hence

$$\omega = 1.95 \times 2 \times \sqrt{0.171 \times 1.50} = 1.98$$

and the period of swinging motion will be about 3.17 seconds.

Having thus determined the natural frequency of swinging motion, the amplitude can be determined by using the initial condition as mentioned in the previous article. The further argument will be given in the Report II of this study.