Dynamics of a 4-Wheel Car*

By Shujiro KUNII** and Naoteru ODA***

1. Introduction

Dynamics of a 4-wheel car was already studied by Dr. Musashi,¹⁾ in Japan, but the study is imperfect because of the following points. That is, independent variables which specify rotatory motions are not correct, and only the vertical freedom of the springs is permitted²⁾. The writers treat more general case than the above mentioned, permitting transversal and lateral freedoms to the springs.

Theoretical treatment of vibration of 8-wheel bogie cars, more complicated problem than the case of 4-wheel cars, was studied by Matsudaira,³⁾ Nagashima and Oda.⁴⁾ In these papers, vertical vibration, transversal one, and rolling, pitching, and yawing are treated by separate vibration model. In this paper, the writers will clear up the problem in the case of 4-wheel cars.

Running tests of an 8-wheel bogie car indicate that there exists significant correlation between vertical vibration and transversal one, as shown by the vibration charts.⁵⁾ Character of the observation instrument and road bed condition must be considered. To clear up the dynamics of a 4-wheel car is to solve the problem above mentioned, at the same time.

In this paper, we treat the vibratory motion about a configuration of equilibrium of a 4-wheel car, which is supported by four equal springs, assuming the linearity of spring in all directions. The upper ends of the springs are fixed to the car body and the lower ends to the space, respectively. Calculating the kinetic and potential energies of the system, and determining the normal coordinates, the writers obtain the equations of motion by Lagrangian equations. Lastly, coupled oscillation of transversal one and rolling are discussed.

2. Analytical Representation of Motion

Let the centre of gravity of the car body O be the origin of a right-handed system of rectan-

gular axes $O-XYZ$ fixed to the body. OZ is directed vertically upwards, OX is directed longitudinally, and OY is perpendicular to both OX and OZ . Let $O-xyz$ be rectangular axes, fixed in space. $O-XYZ$ and $O-xyz$ coincide in a configuration of equilibrium.

The writers assume that the car body is a rigid body, and neglect its elastic vibrations.

Fig. 1. Representation of rotational motion of the car body.

Then the vibrating system has 6 degrees of freedom. Variables (x_0, y_0, z_0) which specify the translational motions and Euler's parameter (ξ, η, ζ, χ) ⁶⁾ which specify the rotational ones are used. Three of the Euler's parameters are independent, as we shall prove below.

Let the displacement be equivalent to a rotation through an angle ω about a line OR , and the Euler's parameters (ξ, η, ζ, χ) are defined as follows:

$$\begin{aligned}\xi &= \cos \alpha \sin \frac{\omega}{2}, \\ \eta &= \cos \beta \sin \frac{\omega}{2}, \\ \zeta &= \cos \gamma \sin \frac{\omega}{2}, \\ \chi &= \cos \frac{\omega}{2},\end{aligned}\quad (1)$$

* Read before at the 31th Session of the Japan Society for Applied Mechanics on 27th, May 1950. Received, July 10th, 1953.

** Kyoto University.

*** Steel Works, Shin-Fuso Metal Industries, Ltd. Osaka.

where α, β, γ are direction angles of the line OR .

These parameters satisfy the identical relation

$$\xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1, \quad (2)$$

therefore, three of the parameters are independent.

There exists the following relation between Eulerian angles θ, ϕ, φ and Euler's parameters:

$$\begin{aligned} \xi &= \sin \frac{\theta}{2} \cos \frac{1}{2}(\phi - \varphi), \\ \eta &= \sin \frac{\theta}{2} \sin \frac{1}{2}(\phi - \varphi), \\ \zeta &= \cos \frac{\theta}{2} \sin \frac{1}{2}(\phi + \varphi), \\ \chi &= \cos \frac{\theta}{2} \cos \frac{1}{2}(\phi + \varphi). \end{aligned} \quad (3)$$

We obtain for the direction-cosines of the two sets $O-XYZ$ and $O-xyz$ the following table:

	X	Y	Z
x	$\xi^2 - \eta^2 - \zeta^2 + \chi^2$	$2(\xi\eta - \zeta\chi)$	$2(\xi\zeta + \eta\chi)$
y	$2(\xi\eta + \zeta\chi)$	$-\xi^2 + \eta^2 - \zeta^2 + \chi^2$	$2(\eta\zeta - \xi\chi)$
z	$2(\xi\zeta - \eta\chi)$	$2(\eta\zeta + \xi\chi)$	$-\xi^2 - \eta^2 + \zeta^2 + \chi^2$

(4)

3. Kinetic Energy

Let $\omega_1, \omega_2, \omega_3$ be the components of angular velocity of the body about the moving axis, then we have

$$\begin{aligned} \omega_1 &= \dot{\phi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi, \\ \omega_2 &= \dot{\phi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi, \\ \omega_3 &= \dot{\phi} \cos \theta + \dot{\varphi}. \end{aligned} \quad (5)$$

Transforming (θ, ϕ, φ) to (ξ, η, ζ, χ) , we get

$$\begin{aligned} \omega_1 &= 2(\chi\dot{\xi} + \zeta\dot{\eta} - \eta\dot{\zeta} - \xi\dot{\chi}), \\ \omega_2 &= 2(-\zeta\dot{\xi} + \chi\dot{\eta} + \xi\dot{\zeta} - \eta\dot{\chi}), \\ \omega_3 &= 2(\eta\dot{\xi} - \xi\dot{\eta} + \chi\dot{\zeta} - \zeta\dot{\chi}). \end{aligned} \quad (6)$$

Since angle of rotation ω is a small quantity, we obtain from (1)

$$\xi = O(\omega), \quad \eta = O(\omega), \quad \zeta = O(\omega), \quad \chi = 1 + O(\omega^2).$$

Therefore, as a first approximation, we have

$$\omega_1 = 2\dot{\xi}, \quad \omega_2 = 2\dot{\eta}, \quad \omega_3 = 2\dot{\zeta}. \quad (7)$$

In general, OX, OY and OZ are considered to coincide with principal axis of inertia of the car body, so kinetic energy of the car body is given by the equation

$$T = \frac{1}{2}M(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2).$$

Or, by (7),

$$T = \frac{1}{2}M(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + 2(A\dot{\xi}^2 + B\dot{\eta}^2 + C\dot{\zeta}^2), \quad (8)$$

where M is mass of the car body, and A, B, C are principal moments of inertia of the car body.

4. Potential Energy

The car body is supported by four equal springs symmetrically to OX and OY . The upper ends of the springs are fixed to the car body and the lower ends are fixed in space. The linearity of the springs in all directions is assumed.

Let the coordinates of the upper ends with respect to $O-XYZ$ be (X_i, Y_i, Z_i) , and let the points be brought to (x_i, y_i, z_i) by a small rotation. For the direction-cosines of $O-XYZ$ and $O-xyz$, we have by (4), following table:

	X	Y	Z
x	1	-2ζ	$+2\eta$
y	$+2\zeta$	1	-2ξ
z	-2η	$+2\xi$	1

Hence, we have

$$\begin{aligned} x_i &= x_0 + X_i - 2\zeta Y_i + 2\eta Z_i, \\ y_i &= y_0 + 2\zeta X_i + Y_i - 2\xi Z_i, \\ z_i &= z_0 - 2\eta X_i + 2\xi Y_i + Z_i, \end{aligned} \quad (9)$$

Fig. 2. Scheme of the car body suspension.

where the values of (X_i, Y_i, Z_i) are shown in Fig. 2.

Let k_1, k_2 and k_3 be the spring constants in the direction of Ox, Oy and Oz , respectively. Let $(x_{i0}, y_{i0}, -c)$ be coordinates of the upper ends of the springs with respect to $O-xyz$, and let z_{00} be the deflection of the springs by static load of the car body. Then, the potential energy V of the system is given by the equation

$$V = \frac{1}{2} k_1 \sum_{i=1}^4 (x_i - x_{i0})^2 + \frac{1}{2} k_2 \sum_{i=1}^4 (y_i - y_{i0})^2 + \frac{1}{2} k_3 \left\{ \sum_{i=1}^4 (z_{00} + z_i)^2 - 4z_{00}^2 \right\} - Mg z_0. \quad (10)$$

In a configuration of equilibrium, weight of the car body must balance to spring forces; therefore,

$$Mg = 4k_3 z_{00}. \quad (11)$$

From (10) and (11), the potential energy V is given by the equation

$$V = 2k_1(x_0^2 + 4b^2\zeta^2 + 4c^2\eta^2 - 4cx_0\eta) + 2k_2(y_0^2 + 4a^2\zeta^2 + 4c^2\xi^2 + 4cy_0\xi) + 2k_3(z_0^2 + 4a^2\eta^2 + 4b^2\xi^2). \quad (12)$$

5. Normal Coordinates

Kinetic and potential energies of the system are obtained in 3, 4; that is,

$$T = \frac{1}{2} M(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + 2(A\dot{\xi}^2 + B\dot{\eta}^2 + C\dot{\zeta}^2) \quad (8)$$

$$\frac{1}{2} V = k_1 x_0^2 + k_2 y_0^2 + k_3 z_0^2 + 4(k_2 c^2 + k_3 b^2) \xi^2 + 4(k_1 c^2 + k_3 a^2) \eta^2 + 4(k_1 b^2 + k_2 a^2) \zeta^2 - 4k_1 c x_0 \eta + 4k_2 c y_0 \xi. \quad (12')$$

Therefore, z_0 and ζ are normal coordinates, and T and V are expressed as homogeneous quadratic forms in which (x_0, η) and (y_0, ξ) are separable respectively. This means physically that the modal vibration in the directions of z_0 and ζ are separable; that is, vertical vibration and yawing of the car are separable to other modes.

On the other hand, x_0, η and y_0, ξ are coupled, respectively; that is, longitudinal vibration and pitching, and transversal vibration and rolling are coupled, respectively.

Therefore, we can analyse, as a first approximation, the vibration of such a four-wheel car as treated in this paper, considering separate vibration models of vertical vibration, yawing, transversal vibration and rolling, longitudinal vibration and pitching.

Other coordinates except z and ζ are not normal coordinates, and they can be normalized as follows:

By a transformation:

$$x' = \sqrt{M} x_0, \quad y' = \sqrt{M} y_0, \quad z' = \sqrt{M} z_0, \quad (13)$$

$$\xi' = 2\sqrt{A} \xi, \quad \eta' = 2\sqrt{B} \eta, \quad \zeta' = 2\sqrt{C} \zeta.$$

(8) and (12') change into the following expressions.

$$2T = \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 + \dot{\xi}'^2 + \dot{\eta}'^2 + \dot{\zeta}'^2, \\ \frac{1}{2} V = \frac{k_1}{M} x'^2 + \frac{k_2}{M} y'^2 + \frac{k_3}{M} z'^2 + \frac{1}{A} (k_2 c^2 + k_3 b^2) \xi'^2 + \frac{1}{B} (k_1 c^2 + k_3 a^2) \eta'^2 + \frac{1}{C} (k_1 b^2 + k_2 a^2) \zeta'^2 - \frac{2k_1 c}{\sqrt{MB}} x' \eta' + \frac{2k_2 c}{\sqrt{MB}} y' \xi'.$$

In the second step, by the linear transformation

$$\begin{pmatrix} x' \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} X \\ \theta \end{pmatrix}, \\ \begin{pmatrix} y' \\ \xi' \end{pmatrix} = \begin{pmatrix} \cos \mu & -\sin \mu \\ \sin \mu & \cos \mu \end{pmatrix} \begin{pmatrix} Y \\ \phi \end{pmatrix}, \quad (14)$$

the potential energy is normalized, in the case when the coefficients of product terms $X\theta$ and $Y\phi$ are made equal to zero.

The conditions are satisfied by

$$\tan 2\lambda = \frac{2k_1 c}{\sqrt{MB}} \left\{ -\frac{k_1}{M} + \frac{1}{B} (k_1 c^2 + k_3 a^2) \right\}^{-1}, \\ \tan 2\mu = \frac{2k_2 c}{\sqrt{MA}} \left\{ \frac{k_2}{M} - \frac{1}{A} (k_2 c^2 + k_3 b^2) \right\}^{-1}. \quad (15)$$

Thus, the transformations (13) and (14) reduce kinetic and potential energies to following forms.

$$2T = \dot{X}^2 + \dot{Y}^2 + \dot{z}'^2 + \dot{\theta}^2 + \dot{\phi}^2 + \dot{\zeta}'^2, \quad (16)$$

$$\frac{1}{2} V = A' X^2 + B' Y^2 + C' z'^2 + E \theta^2 + F \phi^2 + G \zeta'^2.$$

where, A', B', C' and E, F, G are the constants:

$$A' = \frac{k_1}{M} \cos^2 \lambda - \frac{k_1 c}{\sqrt{MB}} \sin 2\lambda + \frac{1}{B} (k_1 c^2 + k_3 a^2) \sin^2 \lambda, \\ B' = \frac{k_2}{M} \cos^2 \mu + \frac{k_2 c}{\sqrt{MA}} \sin 2\mu + \frac{1}{A} (k_2 c^2 + k_3 b^2) \sin^2 \mu, \\ C' = \frac{k_3}{M}, \quad (17)$$

$$E = \frac{k_2}{M} \sin^2 \mu - \frac{k_2 c}{\sqrt{MA}} \sin 2\mu + \frac{1}{A} (k_2 c^2 + k_3 b^2) \cos^2 \mu,$$

$$F = \frac{k_1}{M} \sin^2 \lambda + \frac{k_1 c}{\sqrt{MA}} \sin 2\lambda + \frac{1}{B} (k_1 c^2 + k_3 a^2) \cos^2 \lambda,$$

$$G = \frac{1}{C} (k_1 b^2 + k_2 a^2).$$

6. Normal Vibrations

The normal vibrations of the system are obtained by substituting (16) into Lagrangian equations:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\xi}} \right) - \frac{\partial T}{\partial \xi} + \frac{\partial V}{\partial \xi} = 0.$$

Thus the equations of motion of the system are

$$\ddot{X} + 4A'X = 0, \text{ etc.} \quad (18)$$

Solving the equations, we have the normal vibrations:

$$\begin{aligned}
X &= p_x \cos(2\sqrt{A'}t + q_x), \\
Y &= p_y \cos(2\sqrt{B'}t + q_y), \\
z' &= p_z \cos(2\sqrt{C'}t + q_z), \\
\theta &= p_\theta \cos(2\sqrt{E}t + q_\theta), \\
\phi &= p_\phi \cos(2\sqrt{F}t + q_\phi), \\
\zeta &= p_\zeta \cos(2\sqrt{G}t + q_\zeta),
\end{aligned}$$

where, p , q , etc. are the constants determined by initial conditions.

Considering the transformations (13) and (14), we have

$$\begin{aligned}
x_0 &= \frac{1}{\sqrt{M}} \{p_x \cos \lambda \cos(2\sqrt{A'}t + q_x) \\
&\quad - p_\phi \sin \lambda \cos(2\sqrt{F}t + q_\phi)\}, \\
y_0 &= \frac{1}{\sqrt{M}} \{p_y \cos \mu \cos(2\sqrt{B'}t + q_y) \\
&\quad - p_\theta \sin \mu \cos(2\sqrt{E}t + q_\theta)\}, \\
z_0 &= \frac{1}{\sqrt{M}} p_z \cos(2\sqrt{C'}t + q_z), \\
\xi &= \frac{1}{2\sqrt{A}} \{p_x \sin \mu \cos(2\sqrt{B'}t + q_y) \\
&\quad + p_\theta \cos \mu \cos(2\sqrt{E}t + q_\theta)\}, \\
\eta &= \frac{1}{2\sqrt{B}} \{p_x \sin \lambda \cos(2\sqrt{A'}t + q_x) \\
&\quad + p_\phi \cos \lambda \cos(2\sqrt{F}t + q_\phi)\}, \\
\zeta &= \frac{1}{2\sqrt{C}} p_z \cos(2\sqrt{C'}t + q_z).
\end{aligned} \tag{20}$$

The fact that the vertical vibration, yawing, transversal one and rolling, and longitudinal one and pitching are separable, is represented clearly in the equations (20).

7. Coupled Vibration of Transversal One and Rolling

We shall now trace the loci of instantaneous centre of rotation, i.e. the space centrode, of transversal vibration and rolling. Results obtained above prove that we can treat this problem in the yz plane. Coordinates of any part of the body are given by (9),

$$\begin{aligned}
y &= y_0 + Y - 2\xi Z, \\
z &= 2\xi Y + Z.
\end{aligned} \tag{21}$$

If $(\delta y, \delta z)$ are arbitrary infinitesimal increments of coordinates of centre of gravity, we have

$$\begin{aligned}
\delta y &= \delta Y_0 - 2Z\delta\xi, \\
\delta z &= 2Y\delta\xi.
\end{aligned}$$

Considering the relations (21), i.e.,

$$Y = y - y_0 + 2\xi z, \quad Z = z - 2\xi(y - y_0),$$

we have

$$\begin{aligned}
\delta y &= \delta y_0 - 2\{z - 2\xi(y - y_0)\}\delta\xi, \\
\delta z &= 2(y - y_0 + 2\xi z)\delta\xi.
\end{aligned}$$

If the displacements are given only by a rotation about a point $(\bar{y}, \bar{z})$; that is, $\delta y = 0$, $\delta z = 0$ with respect to this point,

$$\begin{aligned}
0 &= \delta y_0 - 2\{\bar{z} - 2\xi(\bar{y} - y_0)\}\delta\xi, \\
0 &= 2(\bar{y} - y_0 + 2\xi\bar{z})\delta\xi.
\end{aligned}$$

From the above relation we have

$$\begin{aligned}
\bar{y} &= y_0 - \xi \frac{\delta y_0}{\delta\xi}, \\
\bar{z} &= \frac{1}{2} \frac{\delta y_0}{\delta\xi}.
\end{aligned}$$

δy_0 , $\delta\xi$ are the displacement during δt , and as a

limit case, $\lim_{\delta t \rightarrow 0} \frac{\delta y_0}{\delta t} = \dot{y}_0$, $\lim_{\delta t \rightarrow 0} \frac{\delta\xi}{\delta t} = \dot{\xi}$; thus, we

have the equations of the space centrode:

$$\begin{aligned}
\bar{y} &= y_0 - \xi \frac{\dot{y}_0}{\dot{\xi}}, \\
\bar{z} &= \frac{1}{2} \frac{\dot{y}_0}{\dot{\xi}}.
\end{aligned} \tag{22}$$

Here, we must notice that $\bar{y}$ and y_0 are the quantity of the same order, and $\bar{z}$ is not an infinitesimal quantity. Therefore there exists no fixed centre of rotation.

But in the case when the system vibrates in a normal mode, there exists a fixed centre of rotation. For example, if $P_\theta = 0$, we have from (20),

$$\begin{aligned}
y_0 &= \frac{p_y}{\sqrt{M}} \cos \mu \cos(2\sqrt{B'}t + q_y), \\
\xi &= \frac{p_y}{2\sqrt{A}} \sin \mu \cos(2\sqrt{B'}t + q_y).
\end{aligned}$$

So, the equations of the space centrode become,

$$\begin{aligned}
\bar{y} &= y_0 - \xi \frac{\dot{y}_0}{\dot{\xi}} = y_0 - y_0 = 0, \\
\bar{z} &= \frac{1}{2} \frac{\dot{y}_0}{\dot{\xi}} = \frac{\sqrt{A}}{\sqrt{M}} \cot \mu = \text{const.}
\end{aligned}$$

8. Conclusion

We study small oscillations of a 4-wheel car, the body of which is supported by four equal springs, assuming that the springs are linear in all directions. We treat the small oscillations of the system which has 6-degrees of freedom, using Euler's parameters as the variables which specify the rotational motions.

Results obtained: (1) Vertical vibration and

yawing are completely separable.

(2) Transversal vibration and rolling are coupled, and longitudinal vibration and pitching are coupled.

(3) In general, we cannot assume the instantaneous centre of coupled vibratory rotation of transversal vibration and rolling as a fixed point.

Note

- (1) K. Musashi: On the Natural Vibration of a four-wheel car. Bulletin of the Railway Technical Laboratory. Vol. 17, No. 10 (1929)
- (2) Horr's paper on the vibration of a locomotive is not a general case. He, also, permitted only one freedom of springs in the vertical direction. cf. W. Horr: Technische Schwingungslehre Springer, Berlin (1922), S. 321.
- (3) T. Matsudaira: Natural Frequencies of Passenger and Tram Cars. Bulletin of the Railway Technical Laboratory. Vol. 6, No. 2 (1949)
- (4) K. Nagashima: A Study of the Period of Free Vibration of the Eight-Wheel Car. J. Soc. Mech. Eng. Jap., Vol. 37, No. 205 (1934) p. 290.
K. Nagashima, N. Oda: 2nd Report, Trans. Soc. Mech. Eng. Jap., Vol. 15, No. 51, (1950) p. 80
- (5) N. Oda: On the Correlation between Vertical Vibration and Transversal one of a Running Railway Car. Read before at the 27th annual meeting of the Japan Society of Mechanical Engineers. (1950)
- (6) Whittaker: Analytical Dynamics (1937) p. 8.

国井・小田氏の "Dynamics of a 4-wheel car" に対する紙上討論

難解だと考えられている Euler のパラメーターが工学上の問題に応用されている点は非常に面白いと思います。けれどもこのような線型の問題を取扱う場合は、工学者としてはもっと簡単な方法を好むのではないのでしょうか。つまり微小回転の範囲では、回転角はベクトル和の法則に従いますから、最初から x, y, z 軸まわりの微小回転角 α, β, γ で回転を指定することが判り易くまた簡単ではないかと思ひます。この時各点 (x_i, y_i, z_i) の変位は平行移動 (x_0, y_0, z_0) および各軸まわりの回転の合成として $(x_0 + \beta z_i - \gamma y_i, y_0 + \gamma x_i - \alpha z_i, z_0 + \alpha y_i - \beta x_i)$ で表されます。また御説の結論を導くためだけであつたとしても簡単な推論も可能ではないかと思ひます。すなわちまず連成振動の組別は次のようにすることができます。全系を車体とバネ系に分け、車体が $(x_0, y_0, z_0, \alpha, \beta, \gamma)$ の変位をする時に、バネ系から受ける力およびモーメントを $(F_x, F_y, F_z, M_x, M_y, M_z)$ と表せば、それは一般に、

$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} = - \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \quad (1)$$

$$\text{たとえば } -F_x = k_{11}x_0 + k_{12}y_0 + k_{13}z_0 + k_{14}\alpha + k_{15}\beta + k_{16}\gamma \quad (2)$$

で与られます。このバネ係数 k_{ij} は系の対称性を考慮する時多くのものが 0 となり、上式中の点線で囲んだものだけが残ります。これは車体をたとえば x_0 の変位のみをさせた場合にバネ系からどんな反力を受けるかという普通の考察からも導かれますが、また次のように証明することもできます。すなわち全系は xz 平面および yz 平面について対称でありますから、 x 軸、 y 軸をそれぞれ単独にまたは同時に反転しても事情は全く同一で、したがって k_{ij} は同一値をとります。ただし右手系、左手系に限らず回転およびモーメント、たとえば α および M_x は $y \rightarrow z$ の方向を正にとります。他も同様。所が一方ある変位——力の状態に対し、たとえば y 軸を反転すれば、同状態を表す数値が $(x_0, y_0, z_0, \alpha, \beta, \gamma) \rightarrow (x_0, -y_0, z_0, -\alpha, \beta, -\gamma)$; $(F_x, F_y, F_z, M_x, M_y, M_z) \rightarrow (F_x, -F_y, F_z, -M_x, M_y, -M_z)$ と変化します。前述のバネ係数値の不変性を考慮に入れば、この状態は

$$-F_x = k_{11}x_0 - k_{12}y_0 + k_{13}z_0 - k_{14}\alpha + k_{15}\beta - k_{16}\gamma \text{ 等 } (3)$$

とも表される。同様に x, y 軸を同時に反転すれば $(x_0, y, z_0, \alpha, \beta, \gamma) \rightarrow (-x_0, -y_0, z_0, -\alpha, -\beta, \gamma)$; $(F_x, F_y, F_z, M_x, M_y, M_z) \rightarrow (-F_x, -F_y, F_z, -M_x, -M_y, M_z)$ と変化し

$$-F_x = k_{11}x_0 + k_{12}y_0 - k_{13}z_0 + k_{14}\alpha + k_{15}\beta - k_{16}\gamma \quad (4)$$

等とも表される。(2), (3), (4) から

$$k_{12} = k_{13} = k_{14} = k_{16} = 0.$$

となります。他も同様。力 $(F_x, F_y, F_z, M_x, M_y, M_z)$ は運動 $(M\ddot{x}_0, M\ddot{y}_0, M\ddot{z}_0, A\ddot{\alpha}, B\ddot{\beta}, C\ddot{\gamma})$ を生じさせますから、結局

$$M\ddot{z}_0 = -k_{33}z_0, \quad C\ddot{\gamma} = -k_{66}\gamma$$

$$\begin{bmatrix} M\ddot{x}_0 \\ B\ddot{\beta} \end{bmatrix} = - \begin{bmatrix} k_{11} & k_{15} \\ k_{51} & k_{55} \end{bmatrix} \begin{bmatrix} x_0 \\ \beta \end{bmatrix},$$

$$\begin{bmatrix} M\ddot{y}_0 \\ A\ddot{\alpha} \end{bmatrix} = - \begin{bmatrix} k_{22} & k_{24} \\ k_{42} & k_{44} \end{bmatrix} \begin{bmatrix} y_0 \\ \alpha \end{bmatrix}$$

が得られます。勿論実際の k の値を出すには御説の如きエネルギー的考察が最も簡単明快であります。

つぎに節点の問題ですが、節点はやはり規準振動に

ついでのみ定義するのが適当であると思います。有限自由度の場合一般振動の節点は考えられません。何故ならどんなに規準振動を組合せても不動点ができるならば、その点は実は固定点であったことになり、これを節点と呼ぶのは trivial ではないかと思われるからです。この場合始めの一般座標の選び方が独立でなく自由度が一つ減少することになります。現在の問題の (y, α) の連成振動では二つの規準振動の節点は一致せず（一致すれば上述のように y と α の間には束縛がある）、この二つの振動が混在する時は、各節点まわり

の回転角の比は、一般に振動数或いは初位相の差によって時々刻々に変化し、それに応じて瞬間回転中心は z 軸上を一般に $-\infty$ から $+\infty$ まですを往復する筈です。したがって (3) の結論は当然のことのように思われます。ただ振動数の近似計算によく用いられるいわゆる Rayleigh の方法では (y, α) に束縛を加えて一つの振動形態を仮定しますからこの場合は不動点が存在することになりますが、勿論これは近似計算の手段であります。（大島信徳）

書 評

推進機関の空気力学 D. Küchemann & J. Weber: *Aerodynamics of Propulsion*. Mc. Graw-Hill Book Co., Inc., New York, 1953 年初版, ix+340 頁, 価格 9 ドル。筆者たちは戦時中にゲッテンゲンの空気力学研究所で研究し、ドイツが降伏した後、英国軍需省 (Ministry of Supply) の要求と勧めでその成果をまとめたのが本書である。航空機の進歩とともに、機体と原動機の双方に関連する空気力学の分野が開けて、ここにいう推進機関の空気力学が発生した。筆者たちはその豊富な実験資料の裏づけにより、精力的に広範囲な対象を取り扱い、特に新しい手法は盛られていないが、この新しい分野に押し進もうとする意志に溢れた、異色のある力作といえる。ただドイツ人一流の、力が余って読者の理解を考えない強引な書き振りが、それは過剰な記号の導入などに見られ、もう少し実験例のみによる直観的な説得力が望ましいのではないかと考える。この本の必要は、実際の設計に際して最も緊急に感ぜられるであろうから、もっとも著者たちは既にそのことに気がついていて、序文にも、この本は設計図表若しくは必要な「処方」を与えるものではないと断ってある通り、この方面の問題の処理に困っている読者には期待したほどの面白さが無いかもしれない。

第1章 序 論

第2章 推進法の1次元取扱い 熱力学的基礎, ジェット推進, プロペラ推進の基本概念を与える。

第3章 2次元非等方流 流体力学的基础概念と、渦層による自由境界面の決定法を述べているが、もう少し親切に書かないと流体力学に縁遠い読者にはわかり難いと思われる。

第4章 空気取入口 2次元、円形その他の形の空

気取入口に働く空気力、圧力分布を求める方法と実験例を述べ、ジェット発動機を対象としている興味ある章。空気圧縮性も考慮してある。

第5章 有限長さの整形 前章が空気取入口の頭部のみを問題としていたのに対し、ジェット発動機ナセルのような場合の空気力と圧力分布を論じている。なお円筒のまわりの3次元流は渦環法によって解く。

第6章 覆いつきプロペラ 船舶で Kort 環といっている、短い円筒の中に入れたプロペラの性能を論じたもので、航空機に使用し得るか否か疑問である。

第7章 ラムジェット発動機 空気力学的及び熱力学的に取扱ったものであるが、V-1に使われたアルグース・シュミットのパルスジェットにちょっと触れてあるだけで、理論計算が主である。

第8章 ターボジェット発動機 概論のみ。

第9章 ジェット発動機の装備法 本章は従来見られなかったもので、空気取入法による各種損失、各種ナセル取付法の利害など、ピストン発動機の場合のプロペラと胴体の干渉に相当する問題を、かなり詳細に論じている。

第10章 ジェットの問題 ジェット発動機の噴流の性質、翼、胴体との干渉を主として実験的に述べている。

第11章 自然界における空気力学的推進法 鳥や昆虫の羽ばたき推進法の簡単な議論。

第12章 冷却法 覆いつき冷却器の各種装備法による冷却能力と空気抵抗を論じたもの。

附 録 点、円板および環状吹き出し、ダブレット環、渦層、渦筒、渦環など、流れを数学的に表現するために必要な要素の式および数値表を掲げる。

その他各章ごとにおびたしい文献表を添えてある。（佐貫亦男）