On the Dynamics of the Aeroplane and Non-Riemannian Geometry, Part I.*

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Abstract

Modern concepts in the differential geometry of paths are applied to dynamics especially in the theory of the disturbed motion of an aeroplane. It is pointed out that the combined system of an aeroplane and the surrounding fluid affords a space of affine connection, in which the system behaves as it moves along the path defined by usual non-Riemannian geometry. The coefficients of the aerodynamical forces can be regarded as forming a set of geometric objects which take the rôle of the parts of the coefficients of affine connection. The equations of small disturbances obtained categorically by the Jacobi criterion for integrability are compared with the usual ones which have been adopted in ordinary technical treatments.

Might it be probable that the present treatise would be compared with G. Kron’s non-Riemannian dynamics of electrical machinery as its aerodynamical counterpart?

Introduction

We remember that the late Professor L. Prandtl successfully based his theory of lifting surfaces\(^1\) on the conception of acceleration potential, introducing such orthodox procedures as are usual in treating electrodynamical problems as are usual in the treatment of the aerodynamical problems. The present scheme can be regarded as analogous in its objectives in the sense that we will try to introduce a hobby of some electrical engineers into the theory of aeronautical engineering. It consists of conceptions in abstract geometry, which has recently been recognized as affording the most effective universal means for describing the behaviour of complicated electrical machines. We owe this recognition, which is now widely accepted by electrical engineers, mostly to pioneer works by Gabriel Kron in relation to rotating electrical machinery\(^2\).

Since the advent of relativity, both physicists and engineers have been endeavoring to develop the application of non-Euclidean, non-flat spaces aiming at unification of complicated problems in their respective fields of interest.

We can attest to the fact that our former teacher, the late Professor S. Yokota may have been the earliest among the engineers in Japan who have tried to follow this trend. Physics went in such directions rather too fast with a favorable wind apparently going beyond the scope of differential geometry, during the 90 years of the life of the physicist Professor A. Tanakadate. But an entrance to the esoteric realm was pointed out for engineers rather lately, by an electrical engineer, Kron. It was because of the pureness and symmetry of electrotechnical problems, that this inaugural task was undertaken in the field of electrical engineering.

Concerning the character of Kron’s electromechanical problems, we should like to draw the reader’s attention to the fact that exchange takes place in machines between electrical and mechanical energy. This was the main reason that complication should be introduced in the mathematical formulation leading to the introduction of non-Euclidean, curved or twisted spaces, which can be proper objects of modern differential geometry.
Similarly in hydro-or aerodynamical problems, ex-
exchange, taking place between different kinds of
energy, the hydro-aerodynamical and the rigid-
dynamical, can be pointed out to cause exactly
similar effects on the motion of such a rigid
body in a fluid, as a ship or an aeroplane.

The following exposition will be a unifying
treatment of these general dynamical problems,
in which the time variable shall play a no more
specialized rôle than the usual generalized coor-
dinates and many apparently occult mathematical
concepts will be translated into the actual physico-
engineering realities. The author’s desire will at
present be considered satisfied, if we can draw
even a little attention of engineers to the unify-
ning view, whose complete meaning they shall not
be unware of, as were the four eminent pioneers
to whose memory we dedicate the present lecture.

Part I.

1. Aerodynamical Terms in the Equations of
Motion.

Beside the usual gravitational term, the most
conspicuous and characteristic in the equations
of motion of an aeroplane are those which express
the aerodynamical reactions. Apart from the fact
that they have been ever fresh topics of both
experimental and theoretical investigators up to
the present, they deserve our particular attention
in furnishing us adequately and easily geometri-
izable model engineering problems.

1.1 Aerodynamical Forces. If we indicate the
degrees of freedom of our system by Latin
indices

\[ i, j, k, \ldots \]

respectively and the corresponding generalized coordinates by

\[ x_i, x_j, x_k, \ldots \]

the components of the generalized aerodynamical
forces can be expressed by the formulae of the type

\[ P_i = \left( \frac{1}{2} \rho S b^2 \right) C_{ijk} x_k x_j, \]  

(1.1)

where \( \left( \frac{1}{2} \rho S b^2 \right) \) is a quantity which is suitably
chosen so as to make the coefficient \( C_{ijk} \) non-dimen-
sional. The density of the air is always
denoted by \( \rho \). Often \( S \) means the area of the
wing or of the frontal surface, \( b \) is a repre-
sentative linear dimension and \( \epsilon \) is put equal to zero
or one according as \( P_i \) is a force or a moment.
The dotted letters \( x \)'s of course signify the
velocity. Although specialists in aerodynamics
are too much accustomed to the formula (1.1) for
the air force, we should like to give some ex-
planation (and modification for its finale expres-
sion) in what follows.

Take the simplest case of the steady rectilinear
flight in the plane of symmetry and let the
velocity of flight be denoted by \( V \), then by the
ordinary, dimensional consideration, we get

\[ L = C_L \frac{1}{2} \rho V^2 S, \quad D = C_D \frac{1}{2} \rho V^2 S \]  

(1.2)

for the usual lift and drag force respectively,
where the coefficients of lift and drag, \( C_L \) and
\( C_D \), stand for \( C_{ijk} \) and \( V^2 \) is substituted for the
quadratic factor \( x_k x_j \). The characteristic coef-
ficients \( C_L \) and \( C_D \) are not universal constants.

But they depend on the parameters of the flying
position of the aeroplane such as the angle
of attack, which is the most important and is
usually denoted by \( \alpha \). It is defined by

\[ \tan \alpha = \frac{w}{V}, \]

where \( w \) means the velocity of the down wash
relative to the lifting surface. Thus dependence
is apparent of the coefficients \( C_L \) and \( C_D \) on the
velocity of the air relative to the aeroplane. They
shall not fail to depend also on the angles of the
control surfaces and their velocities, i.e. of the
coordinates and the velocities of the corresponding
degrees of freedom of the dynamical system.

Since the representative non-dimensional coef-
ficients \( C_L \), \( C_D \) can be expanded in Taylor series:

\[ C_L = (C_L)_0 + (C_L)_1 \frac{w}{V} \quad \text{and} \quad C_D = (C_D)_0 + (C_D)_1 \frac{w}{V} \]

where \( (C_L)_1 \), \( (C_D)_1 \) are the linear coefficients
and \( (C_L)_0 \), \( (C_D)_0 \) are the constant terms.

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Therefore, the linearized equations of motion
in the form

\[ \begin{align*}
L &= (C_L)_0 \frac{1}{2} \rho V^2 S + (C_L)_1 \frac{1}{2} \rho V w S \\
D &= (C_D)_0 \frac{1}{2} \rho V^2 S + (C_D)_1 \frac{1}{2} \rho V w S
\end{align*} \]

(1.3.1)
where \((C_{L})_0, (C_{D})_0, (C_{L})'_0, (C_{D})'_0\) are the same as in (1.3), while the third terms consist of such sums of all the contributions of the remaining terms up to an infinite power of \(w/V\) as defined by the coefficient
\[
(C_L'')_0 = \frac{1}{2} (C_{L''})_0 + \frac{1}{3} (C_{L''})_0 \left( \frac{w}{V} \right).
\]
All these coefficients being non-dimensional and \(V, w\) corresponding to \(x_i, x_j, x_k, \ldots\), the formulae (1.3.1) represent just some particular components of (1.1). The same should be true for each of the other degrees of freedom.

1.2 Modification of the dynamical equations.

The above coefficients \(C_{ij;k}\) may further depend on acceleration.

As it has been illustrated in the ordinary theory of a lifting surface in non-uniform motion, the aerodynamical coefficients should by no means depend on derivatives of the coordinate variables higher than the components of velocity and the accelerations owing to the fact that the fundamental equations of hydrodynamics are of the second order, when the fluid is assumed perfect. It is probable that consideration of viscosity sometimes introduces a different effect which may not be very predominant, however, in practical problems.

In any case, the assumption can be justified that the dynamical equations of the second order
\[
\ddot{x} + \Gamma_{ik,j} \dot{x} \dot{x} + \tau^i_{j k} = f^i
\]
(1.4)* solve the originally given ones with respect to the components of acceleration. The coefficients \(\Gamma_{ik,j}, \tau^i_{j k}, f^i\) thereby obtained do not depend on acceleration. We can further make \(\tau^i_{j k}\) and \(f^i\) independent of \(x's\). For, if it happens not to be the case, we can expand these equations in power series of \(x\) and take the first term only, gathering the remaining terms in \(\Gamma_{i,j} \dot{x} \dot{x} \dot{a}^k\) purely formally.

There remains finally only \(\Gamma_{i,j}^{k}\) as the object to be investigated more closely. The corresponding terms contains the usual Coriolis or Christoffel acceleration, which we define by
\[
\{k_{ij} \} = \dddot{x} x^j
\]
where \(\{k_{ij} \}\) is the Christoffel symbol of the second kind in relation to a certain symmetric covariant tensor \(g_{ij}\). In our problem, each component of the tensor represents the mass or the moment of inertia associated with each corresponding degree of freedom of the dynamical system, whose totality we may regard as defining the metric of the underlying space. The covariant and contravariant components \(g_{ij}, g^{ik}\) of the metric tensor and therefore the Christoffel symbols \(\{k_{ij} \} = \dddot{x} x^j\)
\[
[k_{ij} \} = \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)
\]
can be expressed as the function of the coordinates only, for they are determined by the positions of the controlling surfaces and the angle of rotation of the propeller-crankshaft system.

Now we will define a pure system of new aerodynamical forces by introducing the coefficients
\[
A_{kij} = [k_{ij}, i] - g_{ii} \Gamma_{ik,j}^{'}
\]
The intention in adopting such a formal conception is that the fundamental equations (1.4) can be reduced to the form
\[
g_{ij} \dot{x}^i + [k_{ij}, i] \dot{x} \dot{x} \dot{x} + \tau^i_{j k} = f^i + A_{kij} \dot{x} \dot{x} \dot{a}^k
\]
(1.4.1), which are the ordinary dynamical equations of Lagrange modified by the aerodynamical forces
\[
\Pi_i = A_{kij} \dot{x} \dot{x} \dot{a}^k
\]
in the right-hand members. We write
\[
\tau^i_{j k} = g_{ik} \dot{a}^k, \quad f^i = g_{ij} \dot{a}^j
\]
for the covariant components of the respective quantities.

When we can further neglect the terms of higher power in \(w/V\) in the expansion of (1.3), we can assume that the aerodynamical coefficients \(A_{kij}\) in (1.5) or (1.5.1) do not depend on the velocity too. This appears practically sufficient in the dynamics of an aeroplane. The treatment of acceleration is much simplified in our problem.

1.3 Aerodynamical resistance derivatives.

Although the equations (1.4.1) have been reached by a circuitous way, which has been taken purposely in view of the elusiveness of recognition of the aerodynamical reactions, we can arrive at an equivalent equations by adding \(P_t\) of (1.1)
directly to the right-hand member of the ordinary equations of Lagrange. Let the result be
\[ g_{ij}\ddot{x}^j + [kj, i] \dot{x}^j \dot{x}^k + Q_{ij}\dot{x}^j = F_i \]
\[ + \left( \frac{1}{2} \epsilon S\mathbf{b} \right) C_{ijk}\dot{x}^j \dot{x}^k. \]  
(1.6)

These equations show apparently the same structure as (1.4.1), in which \( F_i, Q_{ij} \) and \( \epsilon S\mathbf{b} C_{ijk} \) are substituted for the former \( f_i, r_{ij} \) and \( A_{ijk} \). But the former aerodynamical coefficients \( A_{ijk} \) were independent of acceleration, which is generally not the case here.

The interrelation of the alternative formulations can be more adequately understood by the following consideration which consists essentially in an approximate solution of (1.6) with respect to \( \dot{x}^s \).

By the so-called aerodynamical resistance derivative we mean such quantities derived from \( P_i \) as is expressed by the extensor components.4)
\[ P_i(1)\dot{x}^* = \frac{\partial P_i}{\partial \dot{x}^1}, \quad P_i(2)\dot{x}^* = \frac{\partial P_i}{\partial \dot{x}^2}. \]  
(1.7)

The terms of higher components \( P_i(\dot{x}^h) \) can be considered nonessential on the same practical or fundamental consideration as in \( \ddot{x}^s \). If we further neglect the non-linear terms in \( \dot{x} \), as is usual in the theory of small disturbance retaining only the linear terms in \( \dot{x}^s \) such as
\[ P_i(\ddot{x}^h) \approx \frac{\partial P_i}{\partial \dot{x}^h}, \]
we come to obtain a simpler, more easily handled system of equations than (1.6). In other word the real masses and moments of inertia which have been represented by \( g_{ij} \) should be substituted for apparent masses and moments of inertia
\[ h_{ij} = g_{ij} - P_i(\dot{x}^j), \]  
(1.8)
in order that the dependence of the right-hand members on acceleration, which cause the main difficulty in handling (1.6), can be removed. We are accustomed to proceed to a similar effect in the theory of the rolling of ships. It should be noticed that the tensor \( h_{ij} \) is not always symmetric.

The result can be written
\[ \ddot{x} + (h^{-1})^{ij}[kj, h] \dot{x}^j \dot{x}^k + r_{ij}\dot{x}^j = f^i + (P^i)_a \]
where
\[ (h^{-1})^{ij} h_{kj} = \delta_f, \quad r_{ij} = (h^{-1})^{ij} Q_{kj}, \quad f^i = (h^{-1})^{ij} F_j \]
and \( (P^i)_a \) is the first term in the expansion
\[ P^i = (h^{-1})^{ij} \{ P_j(\dot{x}) \dot{x}^j \} \]  
with respect to \( \dot{x}^s \). Comparing them with (1.4.1), whose approximation they are, we obtain
\[ f^i = (h^{-1})^{ij} F_j, \quad r_{ij} = (h^{-1})^{ij} Q_{kj}, \]
\[ g_{ij} h_{kj} \dot{x}^j \dot{x}^k = (P^i)_a - (h^{-1})^{ij} P_j(\dot{x}^j), \]  
whence
\[ f_i = g_{ij} h_{kj} F_k, \quad r_{ij} = g_{il}(h^{-1})^{lk} Q_{kj}, \]  
(1.9)

and
\[ A_{ijk} = \left( \frac{1}{2} \epsilon S\mathbf{b} \right) C_{ijk}[g_{ij}(h^{-1})^{kn}] + [\delta^h - g_{dh}(h^{-1})^{kn}] \]  
(1.10)
where \( C_{ijk} \) denotes the usual aerodynamical coefficients in absence of acceleration. These approximate expressions for the quantities defined in (1.2) in terms of the actual aerodynamical coefficients and the virtual mass appear to be sufficient for practical purposes in so far as the problems of aeroplanes are concerned, in spite of the possibility of further lessening arbitrary assumptions, which will, if performed faithfully, introduce extensors of higher orders.

In the formulation of (1.4.1), apparently statical aerodynamical forces \( H_i \) have appeared on the stage instead of the apparent mass or inertia. By "apparently statical" we mean such interventions of the mass or inertia as is shown in (1.10), but by no means, the absence of aerodynamical effects.

2. Geometrical interpretations.5)

We have been given ground for adopting the equations (1.4) or (1.4.1) in which stands the second member. These equations will be reduced to a simpler from
\[ \ddot{x} + \Gamma_{ij}^{\dot{x}} \dot{x}^i \dot{x}^j = 0, \]  
when the external force \( f^i \) and the dissipation terms \( r_{ij}\dot{x}^j \) are absent. The last equations can be regarded as the equations of paths or geodesics in a multidimensional manifold (space of affine connection), where \( \Gamma_{ij}^{\dot{x}} \) stands for the coefficient of affine connection. Wherefore, our problem will be reduced to mathematical problems in affine geometry of paths in this simplest case.

In the case of the existence of the external force and the dissipation, the trajectory of the dynamical system will deviate from the above
normal path, being controlled by the general equations (1.4.1). But the system can be regarded as operating in the same space of affine connection and the geometry of affine connection will here also be the most efficient implement for representing its dynamical behaviour.

In order to make our arguments smooth and simple in the following, we shall herewith assume that $f_{ij}$ and $\zeta_{ij}$ are independent of the velocity, which condition we have pointed out is practically true.

2.1 The film-space. In view of the fact that fundamental equations of a general dynamical system of $n$ degrees of freedom are different from the equations of paths of the underlying $n$-dimensional manifold, we shall adopt the film-space of $n+1$ dimensions, whose coordinate variables $x^\kappa$, $\kappa=0,1,\ldots,n$ are so elected that

$$x^0=t, \quad x^i=x^i, \quad i=1,\ldots,n.$$  

It has been proved that the equations of motion can be put in the form

$$\frac{d^2x^\kappa}{dt^2} + \lambda_{\mu\kappa} \frac{dx^\mu}{dt} \frac{dx^\kappa}{dt} = 0,$$  

which can be regarded as the equations of paths in an $(n+1)$-dimensional space of affine connection, where we shall take

$$\lambda_{ij} = \Gamma_{ij} - \Lambda_{ij},$$  

$$\lambda_{ij} = \lambda_{j}^{i} = \gamma_{j}^{i}, \quad \gamma_{j}^{i} = g^{ij} \gamma_{ij},$$  

$$\lambda_{ij} = f_{ij} = g_{ij}.$$  

Since no restriction is imposed for the case $\alpha=0$, we will put

$$\lambda_{ij} = \lambda_{ij} = 0$$  

arbitrarily.

The behaviour of our dynamical system of $n$ degrees of freedom is now described by the worldline in an $(n+1)$-dimensional film-space, whose $n$-dimensional projection on the space of $n$ degrees of freedom is observed as the usual trajectory. These formalisms have been deliberately adopted rather in view of the well-recognized effect of geometrization (in recent time) even in relation to practical engineering problems than because of any special taste of the writer. It should be all the more emphasized in consideration of the fact that engineering problems can be the most adequate objects of the modern geometry whose birth is essentially connected with utmost physical reality.

2.2 Small disturbances. The chief merit of adopting the film-space has been shown to be apparent in deducing the law of the disturbed motion of a dynamical system that it is subjected to the equations of the Jacobi criterion for integrability of the deviation:

$$\frac{d^2W_k}{dt^2} + \zeta_{kl} \frac{dW_k}{dt} \frac{dW_l}{dt} W^v = 0$$  

where $\zeta$ signifies the $(n+1)$-dimensional covariant differentiation associated with the coefficient of affine connection $\zeta_{\mu\nu}$. $\zeta_{kl}$ stands for the $(n+1)$-dimensional Riemann-Christoffel curvature tensor of this film-space and $W_k$ denotes the deviation from the normal path of the system, of which the components of the dotted quantity $\dot{x}^\kappa$ are known as the velocity $\dot{x}^0$ and $\dot{x}^i=1$.

The equations (2.4) naturally break up (can be separated) into two different sets

$$\frac{d^2W_0}{dt^2} + \zeta_{0\mu} \frac{dW_0}{dt} \frac{dW_\mu}{dt} W^v = 0$$  

and

$$\frac{d^2W_i}{dt^2} + \zeta_{ij} \frac{dW_j}{dt} \frac{dW_i}{dt} W^v = 0.$$  

It is easy to see that the first equation (2.4.1) holds identically. The second set (2.4.2) can be brought to the form

$$\delta f = \delta^2 W_i \frac{dW_i}{dt^2} + \delta^2 W_j \frac{dW_j}{dt} + \delta^2 \frac{dW_i}{dt} + \delta^2 \frac{dW_j}{dt} W_k + R_{ijk} \delta^2 W_k W^i + R_{ijk} \delta W_k W^i.$$  

where $\delta$ stands for the $n$-dimensional covariant differentiation for the real degrees of freedom associated with the affine connection $\Gamma_{ik}$ and $R_{ijk}$ for the corresponding $n$-dimensional Riemann-Christoffel curvature tensor. We have, therefore,

$$\frac{\delta W_i}{dt} = \frac{dW_i}{dt} + \delta^2 \Gamma_{ik} \frac{dW_k}{dt}$$  

and

$$R_{ijk} = 2(\theta_{ij} \Gamma_{k}^{x} + \Gamma_{x}^{i} \Gamma_{k}^{x} + \Gamma_{x}^{i} \Gamma_{j}^{x})$$  

It happens that each component of $R_{ijk}$ as defined in (2.6) coincides with the corresponding $(n+1)$-dimensional one, because of (2.3.1). The mixed tensor of the third order in (2.5) is defined by
\[ r_{ij} = \frac{1}{2} \nabla_k r_{kj} \]

where
\[ \nabla_k = \frac{\partial}{\partial x^k} + \Gamma^k_{ij} \delta^j_i. \]

The vector quantity in the left-hand member \( df^i \) is the small deviation of the external force from the normal condition.

The above equations of small disturbances (2.5) are applicable not only to the motion of an airplane, but also to all other dynamical systems working under similar geometrical conditions. They are essentially the same with the equations of hunting used in the famed investigation of G. Kron on the performance of rotating electrical machinery. They are characterized by the intervention of the terms associated with
\[ r_{ijk}, \quad R_{lkji} \]
which we will call the "Coefficients of Hunting" in accordance with Kron. Especially \( r_{ijk} \) is called the "resistance tensor of the third order" in his electrotechnical nomenclature.

In the usual dynamics of an airplane, we can assume \( Q_{ij} = 0 \) whence also the resistance tensor of the third order will not appear in the ordinary theory of small disturbances. It is quite probable that this term will assume a more important rôle in the near future, when we come to question the coupled condition of the mechanical and aerodynamical parts of an airplane.

The Riemann-Christoffel curvature tensor \( R_{lkji} \) consists of the three kinds of terms

1. The inertial curvature:
\[ R_{lkji}^i = 2(\partial_l \chi_{kji} + \chi_{kli} \partial_j), \]
2. The aerodynamical curvature:
\[ -A_{lkji}^i = -2(\partial_l \chi_{kji} + \chi_{kli} \partial_j), \]
3. The interaction:
\[ -2(\partial_l \chi_{kji} \chi_{kji} + \partial_l \chi_{kji} \chi_{kji}), \]

(To be continued)

References
4) H. V. Craig, Vector and Tensor Analysis, 1st Ed. 5th. Impression, Mc Graw-Hill 1934, Parts B & C.