Flutter Calculation of Wing with an Elastically Supported Concentrated Weight*.

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Abstract

This paper deals with the effects of mounting spring coefficients and chordwise positions of a concentrated weight on the bending-torsion flutter characteristics of a two-dimensional wing. Calculation was carried out on the relations of the so-called flutter coefficient \( V_F \) and the frequency ratio \( n_F/n_T \) versus the spring ratio \( n_P/n_T \) for various chordwise mounting positions of a concentrated weight. In addition, in some cases the relations of the phase and of the amplitude against the spring ratio were also calculated. Special aspect when the concentrated weight is mounted at the forward quarter chord point is pointed out.

Introduction

Airplane design trends are leading to the placement of heavy concentrated masses on the outer wing panels. Recently several authors\(^1,2\) investigated the effects of such masses mounted rigidly to a three-dimensional wing at various positions on the flutter characteristics and some important results were obtained. But the author has not found the research for the case of elastically supported masses. In our country a study of the qualitative effect of an elastically mounted concentrated mass to a two-dimensional wing upon its bending-torsion flutter characteristics was carried out by the author and T. Matsudaira separately during the war and some interesting results were obtained, but they are unpublished.

The purpose of this report is to present the calculated results by the author of this investigation, reexamining those previously obtained and adding a few calculations to them. As a result of the numerical calculation it is shown that a remarkable reduction of the critical flutter velocity is possible to occur, if a concentrated mass is mounted to the two-dimensional wing at the position of 1/4 chord point by a certain spring which is calculable from design constants. Despite of theoretical imperfection, this seems to be worthy of presenting.

Symbols

\( c \): wing chord, being taken as an unit for all length measurements,
\( \lambda_E = \overrightarrow{FE} \): nondimensional distance of elastic axis from the front neutral point (25% of the chord) measured in chords, positive for positions of elastic axis behind the point \( F \),
\( \lambda_G = \overrightarrow{FG} \): distance of gravitational axis,
\( \lambda_P = \overrightarrow{PF} \): distance of mounting position of concentrated mass \( m_P \) from the point \( F \), positive for positions of mass before the point \( F \),
\( \lambda_{PG} = \overrightarrow{PG} = \lambda_P + \lambda_G \), \( \lambda_{PE} = \overrightarrow{PE} = \lambda_P + \lambda_E \), \( \lambda_{EG} = \overrightarrow{EG} = \lambda_G - \lambda_E \),
\( m' \): mass of wing per unit length,

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\( m_P \): mass of concentrated weight,
\( m_L \): mass of circumscribed air cylinder of a diameter equal to the chord of the wing = \(1/4 \pi \rho \),
\( \rho \): air density,
\( \kappa \): radius of gyration of wing,
\( l_E = \frac{\varepsilon^2 + \lambda_E \zeta}{2} \),
\( l_P = \frac{\varepsilon^2 + \lambda_P \zeta}{2} \),
\( k_B \): stiffness of wing in deflection referred to the elastic axis, corresponding to the unit length,
\( k_T \): torsional stiffness of wing around the point \( E \),
\( k_P \): mounting spring constant of a concentrated weight,
\( \Omega_B = \sqrt{\frac{k_B}{m}} = 2\pi n_B \): uncoupled bending frequency of wing,
\( \Omega_T = \sqrt{\frac{k_T}{mL}} = 2\pi n_T \): uncoupled torsional frequency of wing,
\( \Omega_P = \sqrt{\frac{k_P}{mP}} = 2\pi n_P \): uncoupled natural frequency of a concentrated weight,
\( \nu \): circular frequency of oscillation at the critical speed,
\( \tau = \frac{m_P}{m}, \mu_1 = \frac{m_L}{m} \): mass ratios,
\( \Omega = (\frac{\omega_D}{\omega_T})^2, \Omega_P = (\frac{\omega_P}{\omega_T})^2, Y = (\frac{\omega_P}{\nu})^2 \): frequency ratios,
\( V = \frac{1}{\Omega}(\frac{v_0}{\Omega c}) = \frac{2v_0}{\Omega c} = \frac{1}{\omega} \): velocity number, where \( v \) is the velocity of flight and \( \omega \) is the reduced frequency,
\( V_P = V\sqrt{1/Y} = v_P/\pi c \Omega P \): flutter coefficient.

**Equation of Motion and Flutter Stability Determinant**

Assuming no external force on the mass \( m_P \), we get the Eqs. of motion

\[
m \ddot{z} + (k_B + k_P)z + m_\zeta \omega_0 \dot{z} + (k_B \lambda_B - k_P \lambda_P)\omega - k_P z_P = L_P, \quad (1)
\]

\[
m \ddot{\theta} + (k_B \lambda_E - k_P \lambda_P)\dot{\theta} + m(\varepsilon^2 + \lambda_E \zeta)\dot{\theta} + (k_B \lambda_E^2 + k_P \lambda_P \zeta)\omega + k_P \lambda_P \dot{z}_P = M_P \quad (2)
\]

where \( L_P \) and \( M_P \) are the aerodynamical lift and moment on the wing referring to the point \( F \), which are expressed by Th. Theodorsen as follows:

\[
L_P = -m_L \left\{ \dot{y} + z + \frac{1}{4} \dot{\alpha} \right\}
- 4m_L \nu C \left\{ \dot{x} + \frac{1}{2} \dot{\alpha} \right\}
\]

\[
M_P = -m_L \left\{ \frac{1}{2} \dot{y} + \frac{3}{32} \dot{z} + \frac{1}{4} \dot{\alpha} \right\}
\]

where \( C \) is a complex function of \( \omega \) and is expressed in terms of Bessel functions of the first and second kind: \( C(\omega) = F(\omega) + jG(\omega) \). At the critical velocity we will obtain a sinusoidal oscillation which is the borderline case between damped and divergent ones, and hence substituting the following expressions

\[
z = A e^{j\omega t}, \quad \omega = B e^{j\omega t}, \quad z_P = C e^{j\omega t} \quad (5)
\]

into the Eqs. of motion and eliminating the complex amplitudes \( A, B, \) and \( C \), we get the flutter stability equation. If there exists any real root of \( \nu \) satisfying this characteristic Eq., a flutter oscillation can occur. In solving this equation we must also evaluate an appropriate value of \( \omega \), the reduced frequency, upon which the aerodynamical terms depend. This will be done if the stability equation is separated into real and imaginary parts and both equations are satisfied by the same real root of \( \nu \) for a certain value of \( \omega \) which is to be sought. These values of \( \nu \) and \( \omega \) give the required frequency and velocity of flutter condition.

The flutter stability equation is written in the following form by using the notations given:

\[
\begin{bmatrix}
\mu_1 k_A \nu + \lambda_0 Y - 1, & \mu_2 k_A - \lambda_0 + \lambda_0 \nu Y,
\mu_1 k_B \nu + \lambda_0 Y, & \mu_1 m_\lambda - \lambda_0 \nu \lambda_0 \nu Y + \lambda_0 \nu \nu Y + \lambda_0 \nu Y + \lambda_0 \nu Y
\end{bmatrix} = 0,
\]

where \( \mu_1, \mu_2 \) are mass ratios, \( \lambda_0 \) is the aerodynamic lift, \( \lambda_0 \) is the aerodynamic moment on the wing, \( \mu_1, \mu_2 \) are mass ratios, \( \lambda_0 \) is the aerodynamic lift, \( \lambda_0 \) is the aerodynamic moment on the wing.

\[
\Omega_P Y \left\{ \begin{array}{c}
\rho_0 Y -
\tau \rho_0 \rho Y
\end{array} \right\} = 0,
\]

or

\[
\rho_0 Y \left\{ \begin{array}{c}
D_0 -
\tau \rho_0 \rho Y
\end{array} \right\} = 0.
\]

(76)
where
\[ k', = -1 + j2CV = -2(VG+1) \]
\[ + j(2V^F) = k' + jk'' \]
\[ k'' = -\frac{1}{4} + CV + j V\left(\frac{1}{2} + C\right) \]
\[ = \left(V^2F - VG - \frac{1}{4}\right) + j \left(V^2G + VF + \frac{1}{2} k' + \frac{1}{2} k'' V\right) \]
\[ + j \left(\frac{1}{2} k'' - \frac{1}{2} k' V\right), \]
\[ m_\alpha = -\frac{1}{4} \]
\[ m_\beta = -\frac{3}{32} + j \left(\frac{1}{4} \cdot V\right) \]
\[ d_0 = -(G_2 + j 3G_1) \]

Separating Eq. (7) into real and imaginary parts, we obtain
\[ a' = G_2 Y \{d'_2 - \gamma G_2\} - d'_2 = 0, \quad (9) \]
\[ a'' = G_2 Y \{d''_2 - \gamma G_2\} - d''_2 = 0, \quad (10) \]
where G_2 and G_3 present the real and imaginary parts of the second term in parentheses of Eq. (7).

Furthermore we also obtain the following relations for the complex amplitudes, with which the phase relations can be calculated: say A:B:C=(ξ_1+jη_1):(ξ_2+jη_2), then
\[ \xi_1 = \frac{1}{A_1} \left| -\frac{3}{32} \mu_1 + l_p^2 \right| + (G_2 Y^2 + l_p^2) Y, \quad \lambda_p = \frac{\mu_2 V}{Y - 1}, \]
\[ \eta_1 = \frac{1}{A_1} \left| \mu_1 V \left(G_2 Y - 1\right) \right| \]
\[ \xi_2 = \frac{G_2 Y}{A_1} + l_p Y, \quad \lambda_p = \frac{\mu_2 V}{Y - 1} \]
\[ \eta_2 = \frac{G_2 Y}{A_1} \frac{\mu_2}{4} \]
\[ A_1 = \left| G_2 Y \frac{G_2 Y - 1}{\mu_2 + l_p^2} \right| \]

Especially if l_p=0, then A:B:C=(ξ_1+jη_1):1:(G_2 Y/G_2 Y-1)(ξ_1+jη_1), where
\[ \xi_{10} = \frac{-\left(\frac{3}{32} \mu_1 + l_p^2\right) + \left(b(\lambda_p^2 + l_p^2) Y\right)}{\frac{\mu_1}{4} + \lambda_p - \Omega \lambda_p Y}, \]
\[ \eta_{10} = \frac{V}{\mu_2}, \]

and we can understand that when the concentrated weight is mounted elastically at the front neutral point F, the bending motion of wing and the motion of mounting weight become in phase or 180 degrees out of phase.

**Solution of Flutter Stability Determinant**

As above mentioned, the critical flutter velocity ν_p and frequency ν_p are obtained from real roots of Y and V which satisfy both of equation (9) and (10) simultaneously, since Y=(ω_f/ν)^2 and V=2ν/ν. Eqs. (9) and (10) are rewritten as follows:
\[ G_p Y = \frac{1}{1 - \gamma G_2 / \lambda_p^2} = \frac{1}{1 - \gamma G_2 / \lambda_p^2} \quad (11) \]

From this expression we get G_2 / d'_2 = G_2 / d''_2, and this leads to the quadratic equation of Y:
\[ a(V) \cdot Y^2 - b(V) Y + c(V) = 0, \quad (12) \]
where a(V), b(V), c(V) are functions of V and various design constants of wings as follows:
\[ a(V) = \left[ \frac{k + k''}{2} \left(\frac{1}{2} - \lambda_p^2 \right) \lambda_p Y^2 \right] \]
\[ + b(V), \quad c(V) \]
At first from eq. (12) $Y$ is calculated for various values of $V$ and given design constants and next from eq. (11) a corresponding value of $\varepsilon P$ is calculated for a set of $Y$- and $V$-value. And then $V\sqrt{\varepsilon P}(=VF)$ and $\sqrt{\varepsilon P}(=nP/nT)$ are calculated and plotted against the corresponding $\varepsilon P(=nP/nT)$ values.

It should be noted that $\lambda_{PE}$ is always found in a form of the product with $\Omega$ in $a(V)$ and $b(V)$ and then the aspect which is obtainable by varying the value of $\lambda$ in the case of $\Omega=0$ is the same in the case of $\Omega\neq0$ and $\lambda_{PE}=0$, that is, when the position of a concentrated weight coincides with the elastic axis and the position of it is changed.

Some Examples of Calculation

(a) Effect of a concentrated mass  In Fig. 2 the effect of the concentrated mass and mounting spring are shown. Various design constants are given as follows: $\alpha G=0.13$ (C. G. axis 38%), $\alpha E=0.05$ (elastic axis 30%), $\alpha P=0.45$ (mounting position 20% ahead of L.E.), $\lambda=(nB/nT)(nB/nT)^2=0.26$, $\kappa=0.38$, $\gamma=mP/m=0.35$. Curves in this Fig. are concerned with the flutter of type A which will be explained later, and show that a larger wing mass (i.e. $\alpha_1$ small) has a larger effect upon the relation between the critical velocity and the spring ratio $nP/nT$.

(b) Effect of the mounting position  In Fig. 3 and 4 the effects of the mounting position are shown. In these calculations the design constants are given as follows: $\lambda_G=0.15$ (C. G. axis 38%), $\lambda_E=0.05$ (elastic axis 30%), $\lambda_P=0.45$ (mounting position 20% ahead of L.E.), $\Omega=(nP/nT)^2=0.26$, $\kappa=0.38$, $\gamma=mP/m=0.35$. Curves in this Fig. are concerned with the flutter of type A which will be explained later, and show that a larger wing mass (i.e. $\mu_1$ small) has a larger effect upon the relation between the critical velocity and the spring ratio $nP/nT$.

Fig. 2 Flutter coefficient $VF$ vs. spring ratio $nP/nT$ for various values of mass ratio $\mu_1$.
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Fig. 3 Calculated results in the case of \( \Omega = 0 \). (Effect of mounting positions.)

(a) Plot of \( Y \) VS. \( V \).

(b) Flutter coefficient \( VF \) and frequency ratio \( nF/nT \) against spring ratio \( nP/nT \).

(c) Variation of phases against spring ratio \( nP/nT \).

(d) Relation between amplitudes against spring ratio \( nP/nT \).

With use of this approximation we can obtain the approximate expression for \( a(V) \), \( b(V) \), \( c(V) \) for small \( V \) as follows:

\[
a(V) = a'V, \quad b(V) = b'V, \quad c(V) = c'V,
\]

where

\[
a' = -\mu (a_1 + a_2 \lambda P), \quad b' = \mu^2 \lambda P (a_1 + a_2 \lambda P) (a_3 + a_4 \lambda P), \quad c' = -\mu^2 \{ (1 + 8 \lambda P) / 2 \}^2 - \mu \lambda^2 (a_2 + a_3 \lambda P) (a_3 + a_4 \lambda P) + \lambda^2 (a_3 + a_4 \lambda P), \quad a_3 = 1 + 2 \lambda P, \quad a_4 = 1 + 2 \lambda P.
\]

Now when the discriminant of Eq. (12) is examined with these approximate expressions for \( a(V) \), \( b(V) \) and \( c(V) \), \( b^2 - 4ac' \) becomes zero equally, whatever the value of \( \lambda P \) as well as \( V \) may be, and hence it cannot be concluded that Eq. (12) has equal roots only at \( V = 0 \) for \( \lambda P \approx 0 \). But taking account of higher order terms of \( V \) for these coefficients, it appears that the discriminant of Eq. (12) becomes zero as \( V \to 0 \).
When \( \lambda_P \approx 0 \) generally. In fact numerical calculation reveals that it holds for \( \lambda_P = 0 \) and equal roots \( Y_0 \) and corresponding \( \Omega_P \)-value \( \Omega_{P0} \) are determined as follows:

\[
\begin{align*}
Y_0 &= \frac{V}{2a^2} = \frac{2lx^2 - k_l - \frac{1}{16} \mu_1}{2lx^2 + 2\Omega_k (l_k - \frac{1}{2})} \\
\Omega_{P0} &= \left\{ \left( l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) \right)^2 + \left( \frac{1}{Y_0} \right) \left( l_k - l_k^2 \right) \right\} \\
&\quad \times \left\{ l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) Y_0 + \left( l_k - l_k^2 - \frac{1}{4} \right) \right\} \\
&\quad \times \left\{ \frac{1}{4} - \frac{3}{32} \mu_1 \right\} \\
&\quad \times \left( \frac{3}{32} \mu_1 - \frac{1}{4} \right)
\end{align*}
\]

Of course the flutter coefficient \( V_F = V\sqrt{1/Y} \) becomes zero for these values.

\[
\begin{align*}
\Omega_0 &= \frac{2l_k^2 - k_l - \frac{1}{16} \mu_1}{2l_k^2 + 2\Omega_k (l_k - \frac{1}{2})} \\
\Omega_{P0} &= \left\{ \left( l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) \right)^2 + \left( \frac{1}{Y_0} \right) \left( l_k - l_k^2 \right) \right\} \\
&\quad \times \left\{ l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) Y_0 + \left( l_k - l_k^2 - \frac{1}{4} \right) \right\} \\
&\quad \times \left\{ \frac{1}{4} - \frac{3}{32} \mu_1 \right\} \\
&\quad \times \left( \frac{3}{32} \mu_1 - \frac{1}{4} \right)
\end{align*}
\]

When \( \lambda_P \approx 0 \) generally. In fact numerical calculation reveals that it holds for \( \lambda_P = 0 \) and equal roots \( Y_0 \) and corresponding \( \Omega_P \)-value \( \Omega_{P0} \) are determined as follows:

\[
\begin{align*}
Y_0 &= \frac{V}{2a^2} = \frac{2l_k^2 - k_l - \frac{1}{16} \mu_1}{2l_k^2 + 2\Omega_k (l_k - \frac{1}{2})} \\
\Omega_{P0} &= \left\{ \left( l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) \right)^2 + \left( \frac{1}{Y_0} \right) \left( l_k - l_k^2 \right) \right\} \\
&\quad \times \left\{ l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) Y_0 + \left( l_k - l_k^2 - \frac{1}{4} \right) \right\} \\
&\quad \times \left\{ \frac{1}{4} - \frac{3}{32} \mu_1 \right\} \\
&\quad \times \left( \frac{3}{32} \mu_1 - \frac{1}{4} \right)
\end{align*}
\]

Of course the flutter coefficient \( V_F = V\sqrt{1/Y} \) becomes zero for these values.

\[
\begin{align*}
\Omega_0 &= \frac{2l_k^2 - k_l - \frac{1}{16} \mu_1}{2l_k^2 + 2\Omega_k (l_k - \frac{1}{2})} \\
\Omega_{P0} &= \left\{ \left( l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) \right)^2 + \left( \frac{1}{Y_0} \right) \left( l_k - l_k^2 \right) \right\} \\
&\quad \times \left\{ l_k^2 + \Omega_k \left( l_k - \frac{1}{2} \right) Y_0 + \left( l_k - l_k^2 - \frac{1}{4} \right) \right\} \\
&\quad \times \left\{ \frac{1}{4} - \frac{3}{32} \mu_1 \right\} \\
&\quad \times \left( \frac{3}{32} \mu_1 - \frac{1}{4} \right)
\end{align*}
\]

Of course the flutter coefficient \( V_F = V\sqrt{1/Y} \) becomes zero for these values.

Fig. 4 Calculated results in the case of \( \Omega = 0.1 \). (Effect of mounting position.)

\[
\begin{align*}
J_0(\omega) &= 1 - \left( \frac{\omega}{2} \right)^2, \\
Y_0(\omega) &= \frac{2}{\pi} \log \frac{\omega}{2} + r_1, \\
J_1(\omega) &= -\frac{d}{d\omega} J_0(\omega) = \frac{\omega}{2}, \\
Y_1(\omega) &= -\frac{d}{d\omega} Y_0(\omega) = -\frac{1}{\pi \omega},
\end{align*}
\]

\((r_1 = \text{Euler's const.})\)

Using these expressions, the limiting values of \( F(\omega), G(\omega) \) as \( \omega \to 0 \) are

\[
F(\omega) = \frac{J_1(J_1 + Y_0) + Y_1(Y_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \to 1,
\]

\[
G(\omega) = \frac{Y_1(Y_1 + J_0) + Y_1(J_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \to \omega \log \frac{1}{\omega},
\]

and therefore we can obtain the approximate expression for \( k' \) and \( k'' \) as follows:

\[
k' = -(2V^2 + 1) \approx 2 \log V, \quad k'' = 2VF = 2V
\]

By using these approximate expressions it yields that Eq. (12) has a finite root and an infinite root as \( V \to \infty \) and the former is calculable rather exactly, as shown in Figs. (a).
Together with the curves $Y$ versus $V$ thus obtained, pieces of curve $Y_1$ and $Y_2$ are written in. The curve $Y_1$ satisfies the Eq. $\phi'' = 0$ and $Y_2$ the Eq. $\phi'' - r G(V) = 0$ (with $r = 0.1$), and the intersection point $Y_A$ of $Y$ and $Y_1$-curve corresponds to $\phi_P = 0$ which means that no concentrated weight is mounted, while the point $Y_B$ of $Y$ and $Y_2$ corresponds to $\phi_P = \infty$ which implies the situation of fixed weight. A positive value of $\phi_P$ is calculated either for a point on the positive part of $Y$-curve below $Y_A$ or for a point on the upper part of $Y_B$, but there exists no positive value of $\phi_P$ for a point on the part between $Y_A$ and $Y_B$.

As shown in phase calculations, the vibration mode, in which the phase difference between wing translationary and concentrated weight motions is about 180 degrees, belongs to the part below $Y_A$ and the mode, in which they are about in phase, to that above $Y_B$. We shall denote the former as the flutter of type A and the latter as the one of type B.

The value of $V_P$ and $n_P/n_T$ for comparatively small value of $n_P/n_T$ in the case of type B cannot be rather precise, because of uncertainty of $F(\omega)$ and $G(\omega)$ for small $\omega$, i.e. for large value of $V$ ($V=10 \sim 50$), but for $V \rightarrow \infty$ use may be made of the above mentioned approximation in order to find exactly the outset of type B as $n_P/n_T$ is increased. These positions are represented in Figs. (b) by dotted lines. In Fig. 5 the vibratory characteristics in still air are shown for the purpose of comparison with those in flight.

**Conclusion**

The effects of elastically mounted concentrated weight upon the two-dimensional wing flutter have been calculated. Two types of flutter (A and B type) are possible to occur. Generally in the case of chordwise forward supporting, the type A, in which wing translationary and concentrated weight motions have about 180° phase difference, is liable to occur for softer spring and the type B, in which both motions are about in phase, for harder spring. But in the case of rearward supporting, the type B is apt to occur over all spring stiffnesses. It is interesting that chordwise position of support has a great effect on the A type flutter. Especially a remarkable reduction of critical velocity is always possible to occur, if the mass is mounted to wing near the front neutral point (25 % of the chord) by the spring stiffness which is calculable from Eq. (14).

But it should be noted that this calculation is based on the assumption of the two-dimensional wing with no damping, and then further investigation of such effects will be necessary for the actual wing. Effect of wing density also was shown.

**References**
