$$\begin{aligned} x=0, \quad v=w=0, \quad x=l, \quad \partial^2 v / \partial x^2 = \partial^2 w / \partial x^2 = 0 \\ \partial v / \partial x = \partial w / \partial x = 0, \quad \partial^3 v / \partial x^3 = \partial^3 w / \partial x^3 = 0 \\ \varphi = 0, \quad \partial \varphi / \partial x = 0 \end{aligned}$$

この場合紙面の都合上詳細な計算は省略するが、固有値はそれぞれつぎのようになる。

$$\begin{aligned} \kappa_{10} I = \kappa_{10}^{(0)} \left[\left\{ \left(\frac{5}{4} + 2\mu \right) - \left(\frac{1}{2} + \mu \right) \sqrt{\delta \kappa_{10}^{(0)}} \right. \right. \\ \times \frac{\cosh \sqrt{\delta \kappa_{10}^{(0)}} + \cos \sqrt{\delta \kappa_{10}^{(0)}}}{\sinh \sqrt{\delta \kappa_{10}^{(0)}} + \sin \sqrt{\delta \kappa_{10}^{(0)}}} \\ \left. \left. + \left(\frac{1}{3} + \frac{\mu}{2} \right) \sqrt{\delta \kappa_{10}^{(0)}} \right. \right. \\ \left. \left. \times \frac{\sinh \sqrt{\delta \kappa_{10}^{(0)}} - \sin \sqrt{\delta \kappa_{10}^{(0)}}}{\sinh \sqrt{\delta \kappa_{10}^{(0)}} + \sin \sqrt{\delta \kappa_{10}^{(0)}}} \right\} - \cos^2 \beta \right] \end{aligned}$$

したがって

$$\kappa_{10} I = \kappa_{10}^{(0)} [(1.19334 + 1.57088\mu) - \cos^2 \beta]$$

$$\kappa_{20} I = \kappa_{10}^{(0)} [(6.4782 + 8.6471\mu) - \cos^2 \beta]$$

$$\kappa_{30} I = \kappa_{10}^{(0)} [(17.8595 + 24.9521\mu) - \cos^2 \beta]$$

また κ_{10} , κ_{20} , κ_{30} はそれぞれ

$$\begin{aligned} \kappa_{10} = [(1.875104)^4 - 57.74443\sigma^2] \\ + \lambda \{ 1.19334 + 1.57088\mu - \sin^2 \beta + \sigma^2 [-1.61621 \\ - 0.857308\mu - 23.0596 \sin^2 \beta] \} / \{ 1 + \sigma^2 [23.0596 \\ + 23.3263\mu] \} \end{aligned}$$

$$\begin{aligned} \kappa_{20} = [(4.69409)^4 - 17676.2\sigma^2] + \{ \lambda 6.4782 \\ + 8.6471\mu - \sin^2 \beta + \sigma^2 [-939.104 \\ - 1452.27\mu + 164.101 \sin^2 \beta] \} / \{ 1 + \sigma^2 \\ [-164.101 + 2469.56\mu] \} \\ \kappa_{30} = [(7.04790)^4 + 25688.8\sigma^2] \\ + \lambda \{ \sigma^2 [0.908031 + 1.11856\mu \\ - 0.0175080 \sin^2 \beta] \} / \{ 0.5\mu + \sigma^2 (0.0175080 \\ - 3.52124\mu) \} \end{aligned}$$

ここで、計算の便宜上、 $\varepsilon=1$ ととっている。

以上の結果より、断面の重心と振り中心が一致しないことが振動数に及ぼす影響は翼のアスペクト比が小さい場合に大きくなり、また低次振動より高次振動の方が大きいようである。なお最初に記した様に実際の回転翼は幾何学的に傾けた形状をもっておりこのことも振動数に影響を及ぼすが、これについては別の機会にゆずることとする。

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Note on the Flow just behind Three-Dimensional Shock*

By Shigeki MORIOKA**

Abstract

The concise expressions which determine the derivatives of flow quantities just behind a steady three dimensional shock with respect to the space coordinates, are generally derived according to Thomas' procedure, under the assumption of regularity of shock surface. These expressions assume comparatively simple forms when we take the length of the lines of curvature as the parameters. As special cases, they are related to the flow just behind two-dimensional or axially symmetrical shock. The consistency relation is derived for axially symmetrical shock. Then it is shown that the distribution of singular shock directions for axially symmetrical shock that for two-dimensional shock.

1. Introduction

The concise expressions which determine the derivatives of velocity components, density and

pressure just behind a steady two-dimensional shock with respect to space coordinates, have been provided by T. Y. Thomas¹⁾, under the assumption of regularity of shock surface. Those are necessary to determine, for instance, the direction

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of the equi-speed line, the equi-density line, etc., their curvature just behind shock, or the direction of spines in the hodograph plane, or to attempt the expansion of flow quantities from the shock. Further, using those results, he²⁾ has given the consistency relation that connects the curvature of stream line just behind shock and its derivatives along the stream line with the curvature of the shock line and its derivatives along shock. Then he has investigated the distribution of singular shock direction. S. F. Shen³⁾ has discussed this in detail.

In this paper, the general procedure is provided that extends those expressions to the flow just behind three dimensional steady shock. When we take the length of lines of curvature as parameters, we obtain comparatively simple expressions. As special cases, they are related to the flow just behind two-dimensional or axially symmetrical shock⁴⁾.

If the problem is restricted to axially symmetrical shock from the first, the treatment becomes more simple. The consistency relation is derived for axially symmetrical shock. Then it is shown that the distribution of singular shock directions for axially symmetrical shock is the same as that in the two-dimensional case.

These expressions are applicable to the flow of steady, ideal gas just behind a shock by neglecting viscosity and heat conductivity.

2. Fundamental relations

Let ϕ and ψ be two independent parameters, then, a shock surface is expressed by position vector,

$$\mathbf{x} = \mathbf{x}(\phi, \psi) = x_\alpha \mathbf{e}_\alpha,$$

where $\mathbf{e}_\alpha (\alpha = 1, 2, 3)$ are spacial fundamental vectors. Here the summation convention is used and this convention will be used throughout this paper. When we consider $\psi = \text{constant}$ and $\phi = \text{constant}$, we call them ϕ -curve and ψ -curve, respectively, and generally we call them parameter curves.

At a point (ϕ, ψ) on a shock surface, the tangent vectors to ϕ - and ψ -curve are expressed as

$$\frac{\partial \mathbf{x}}{\partial \phi} = \mathbf{x}_{,\phi} = \frac{\partial x_\alpha}{\partial \phi} \mathbf{e}_\alpha, \quad \frac{\partial \mathbf{x}}{\partial \psi} = \mathbf{x}_{,\psi} = \frac{\partial x_\alpha}{\partial \psi} \mathbf{e}_\alpha,$$

The unit vector of surface normal in this point

is expressed as,

$$\boldsymbol{\nu} = \frac{\mathbf{x}_{,\phi} \times \mathbf{x}_{,\psi}}{\sqrt{(\mathbf{x}_{,\phi} \times \mathbf{x}_{,\psi})^2}}.$$

Here, we assume that $\mathbf{x}_{,\phi} \times \mathbf{x}_{,\psi} \neq 0$.

Let, $\mathbf{x}_{,\phi} = \mathbf{E}$, $\mathbf{x}_{,\psi} = \mathbf{F}$, $\mathbf{x}_{,\phi}^2 = G$, and let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be tangent unit vectors to ϕ - and ψ -curve, respectively. Then,

$$\boldsymbol{\lambda} = \lambda_\alpha \mathbf{e}_\alpha = \frac{1}{\sqrt{E}} \mathbf{x}_{,\phi}, \quad \boldsymbol{\mu} = \mu_\alpha \mathbf{e}_\alpha = \frac{1}{\sqrt{G}} \mathbf{x}_{,\psi},$$

$$\boldsymbol{\nu} = \nu_\alpha \mathbf{e}_\alpha = \frac{1}{\sqrt{EG - F^2}} (\mathbf{x}_{,\phi} \times \mathbf{x}_{,\psi}).$$

Among $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, there exist the following relations:

$$\boldsymbol{\lambda} \cdot \boldsymbol{\nu} = \lambda_\alpha \nu_\alpha = 0, \quad \boldsymbol{\mu} \cdot \boldsymbol{\nu} = \mu_\alpha \nu_\alpha = 0,$$

$$\boldsymbol{\lambda} \cdot \boldsymbol{\mu} = \lambda_\alpha \mu_\alpha = \frac{F}{\sqrt{EG}}, \quad \boldsymbol{\lambda} \times \boldsymbol{\mu} = \frac{\sqrt{EG - F^2}}{\sqrt{EG}} \boldsymbol{\nu} = \mathbf{e}_\alpha (\lambda_\beta \mu_\gamma).$$

Let $u_i = u_{i\alpha} \mathbf{e}_\alpha$, p_i and ρ_i be velocity, pressure and density of the flow in front of the shock, and let $u = u_\alpha \mathbf{e}_\alpha$, p , ρ be ones just behind shock. Let the velocity components to surface normal, tangent to ϕ - and ψ -curve distinct by suffixes n , λ and μ , respectively. Then,

$$u_{in} = \mathbf{u}_i \cdot \boldsymbol{\nu} = u_{i\alpha} \nu_\alpha = \frac{\sqrt{EG}}{\sqrt{EG - F^2}} u_{i\alpha} (\lambda_\beta \mu_\gamma), \quad (1)$$

$$u_n = \mathbf{u} \cdot \boldsymbol{\nu} = u_\alpha \nu_\alpha = \frac{\sqrt{EG}}{\sqrt{EG - F^2}} u_\alpha (\lambda_\beta \mu_\gamma), \quad (2)$$

$$u_\lambda = \mathbf{u} \cdot \boldsymbol{\lambda} = u_\alpha \lambda_\alpha, \quad u_\mu = \mathbf{u} \cdot \boldsymbol{\mu} = u_\alpha \mu_\alpha.$$

According to Thomas' expression, the shock conditions and the differential equations of flow are written as follows:

$$u_\alpha = u_{i\alpha} - \frac{2(\rho_i u_{in}^2 - r p_i) \nu_\alpha}{(r+1) \rho_i u_{in}}, \quad (3)$$

$$p = p_i + \frac{2(\rho_i u_{in}^2 - r p_i)}{r+1}, \quad (4)$$

$$\rho = \rho_i + \frac{2\rho_i(\rho_i u_{in}^2 - r p_i)}{2r p_i + (r-1) \rho_i u_{in}^2}, \quad (5)$$

$$p_{,\alpha} + \rho u_\sigma u_{\sigma,\alpha} = 0, \quad (6)$$

$$\rho_{,\alpha} u_\alpha + \rho u_{\sigma,\sigma} = 0, \quad (7)$$

$$\rho u_{\alpha,\beta} u_\alpha u_\beta - r p u_{\sigma,\sigma} = 0. \quad (8)$$

3. First derivatives of flow quantities

If we partially differentiate the shock conditions (3), (4) and (5) with respect to ϕ or ψ , we obtain

$$\left. \begin{aligned} u_{\alpha,\beta} \lambda_\beta &= \frac{1}{\sqrt{E}} A_{\alpha\beta}, & p_{,\beta} \lambda_\beta &= \frac{1}{\sqrt{E}} B_\beta, \\ \rho_{,\beta} \lambda_\beta &= \frac{1}{\sqrt{E}} C_\beta, & u_{\alpha,\beta} \mu_\beta &= \frac{1}{\sqrt{G}} A_{\alpha\psi}, \\ p_{,\beta} \mu_\beta &= \frac{1}{\sqrt{G}} B_\psi, & \rho_{,\beta} \mu_\beta &= \frac{1}{\sqrt{G}} C_\psi, \end{aligned} \right\} \quad (9)$$

where the quantities indicated in the right hand side; $A_{\alpha\beta}$, B_β , ...etc., are obtained by differentiating the quantities of right hand side of Eqs. (3), (4) and (5) with respect to ϕ or ψ .

We define the quantity ξ_i^α by the matrix form as

$$[\xi_i^\alpha] = \begin{bmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 \\ \xi_2^1 & \xi_2^2 & \xi_2^3 \\ \xi_3^1 & \xi_3^2 & \xi_3^3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \quad (10)$$

Then, its determinant is $|\xi_i^\alpha| = u_\alpha(\lambda_\beta \mu_\gamma) = (\sqrt{EG - F^2}/\sqrt{EG})u_n$.

Let the inverse matrix of $[\xi_i^\alpha]$ be $[\zeta_\alpha^i]$, then we obtain

$$[\zeta_\alpha^i] = \begin{bmatrix} \zeta_1^1 & \zeta_1^2 & \zeta_1^3 \\ \zeta_2^1 & \zeta_2^2 & \zeta_2^3 \\ \zeta_3^1 & \zeta_3^2 & \zeta_3^3 \end{bmatrix} = \frac{\sqrt{EG}}{\sqrt{EG - F^2}} \frac{1}{u_n} \times \begin{bmatrix} (\mu_2 u_3 - \mu_3 u_2) & (\lambda_3 u_2 - \lambda_2 u_3) & (\lambda_2 \mu_3 - \lambda_3 \mu_2) \\ (\mu_3 u_1 - \mu_1 u_3) & (\lambda_1 u_3 - \lambda_3 u_1) & (\lambda_3 \mu_1 - \lambda_1 \mu_3) \\ (\mu_1 u_2 - \mu_2 u_1) & (\lambda_2 u_1 - \lambda_1 u_2) & (\lambda_1 \mu_2 - \lambda_2 \mu_1) \end{bmatrix} \quad (11)$$

From the shock conditions (9) and the differential equations of flow (6), (7) and (8), we obtain

$$u_{\alpha,\beta} \xi_i^\alpha \xi_j^\beta = A_{ij}, \\ \rho_{,\beta} \xi_i^\beta = a_i,$$

where

$$\left. \begin{aligned} A_{11} &= \frac{1}{\sqrt{E}} A_{\alpha\beta} \lambda_\alpha, & A_{12} &= \frac{1}{\sqrt{G}} A_{\alpha\psi} \lambda_\alpha, \\ A_{13} &= -\frac{1}{\sqrt{E}} \frac{B_\beta}{\rho}, & A_{21} &= \frac{1}{\sqrt{E}} A_{\alpha\beta} \mu_\alpha, \\ A_{22} &= \frac{1}{\sqrt{G}} A_{\alpha\psi} \mu_\alpha, & A_{23} &= -\frac{1}{\sqrt{G}} \frac{B_\psi}{\rho}, \\ A_{31} &= \frac{1}{\sqrt{E}} A_{\alpha\beta} u_\alpha, & A_{32} &= \frac{1}{\sqrt{G}} A_{\alpha\psi} u_\alpha, \\ A_{33} &= \frac{rp}{\rho} u_{\sigma,\sigma} = c^2 u_{\sigma,\sigma}. \end{aligned} \right\} \quad (12)$$

Thus, the first derivatives of velocity components and density just behind shock can be expressed as

$$u_{\alpha,\beta} = A_{ij} \zeta_\alpha^i \zeta_\beta^j, \quad (13)$$

$$\rho_{,\beta} = a_i \zeta_\beta^i. \quad (14)$$

While the derivatives of pressure can be obtained from Eq. (6) and above results,

$$p_{,\beta} = -\rho A_{i3} \zeta_\beta^i. \quad (15)$$

Among the above expressions, all the coefficients, except A_{33} and a_3 , have the permitted determination, in the sense that these quantities can be expressed by the conditions in front of shock and

the form of the shock surface. While A_{33} and a_3 can be expressed by other coefficients.

4. Computation of coefficients A_{33} and a_3 .

By Weingarten's formula,

$$\frac{\partial \nu}{\partial \phi} = a x_{,\phi} + b x_{,\psi} = \sqrt{E} a \lambda + \sqrt{G} b \mu,$$

$$\frac{\partial \nu}{\partial \psi} a' x_{,\phi} + b' x_{,\psi} = \sqrt{E} a' \lambda + \sqrt{G} b' \mu,$$

where

$$a = \frac{FM - GL}{EG - F^2}, \quad b = \frac{FL - EM}{EG - F^2},$$

$$a' = \frac{FN - GM}{EG - F^2}, \quad b' = \frac{FM - EN}{EG - F^2},$$

and $L = \nu \cdot x_{,\phi\phi}$, $M = \nu \cdot x_{,\phi\psi}$, $N = \nu \cdot x_{,\psi\psi}$.

Thus, from Eqs. (1) and (2), we obtain

$$\frac{\partial u_{1n}}{\partial \phi} = \sqrt{E} a u_\lambda + \sqrt{G} b u_\mu,$$

$$\frac{\partial u_{1n}}{\partial \psi} = \sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu.$$

Here the problem is restricted to the case of steady shock, and it is assumed that u_{1a} , p_1 and ρ_1 are constant.

Differentiating partially the right hand side quantities of Eq. (4) with respect to ϕ or ψ , we obtain

$$B_\beta = \frac{4\rho_1 u_{1n}}{r+1} (\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu), \quad (16)$$

$$B_\psi = \frac{4\rho_1 u_{1n}}{r+1} (\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu). \quad (17)$$

Similarly, from Eq. (3):

$$A_{\alpha\beta} = -\frac{B_\beta \nu_\alpha}{\rho_1 u_{1n}} + \frac{[\rho]}{\rho} \{(\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu) \nu_\alpha - \sqrt{E} a u_{1n} \lambda_\alpha - \sqrt{G} b u_{1n} \mu_\alpha\}, \quad (18)$$

$$A_{\alpha\psi} = -\frac{B_\psi \nu_\alpha}{\rho_1 u_{1n}} + \frac{[\rho]}{\rho} \{(\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu) \nu_\alpha - \sqrt{E} a' u_{1n} \lambda_\alpha - \sqrt{G} b' u_{1n} \mu_\alpha\}, \quad (19)$$

where $[\rho] = \rho - \rho_1$, and from Eq. (5):

$$C_\beta = \frac{-2}{(r+1)u_n} \{(r-1)\rho - (r+1)\rho_1\} (\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu), \quad (20)$$

$$C_\psi = \frac{-2}{(r+1)u_n} \{(r-1)\rho - (r+1)\rho_1\} (\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu). \quad (21)$$

Multiplying λ_α or μ_α to Eq. (18) and summing up with α , we obtain

$$\left. \begin{aligned} A_{\alpha\beta} \lambda_\alpha &= -\frac{[\rho]}{\rho} \frac{1}{\sqrt{E}} (Ea + Fb) u_{1n}, \\ A_{\alpha\beta} \mu_\alpha &= -\frac{[\rho]}{\rho} \frac{1}{\sqrt{G}} (Fa + Gb) u_{1n}, \end{aligned} \right\} \quad (22)$$

Similarly, from Eq. (19):

$$\left. \begin{aligned} A_{\alpha\psi} \lambda_1 &= -\frac{[\rho]}{\rho} \frac{1}{\sqrt{E}} (Ea' + Fb') u_{1n}, \\ A_{\alpha\psi} \mu_1 &= -\frac{[\rho]}{\rho} \frac{1}{\sqrt{G}} (Fa' + Gb') u_{1n}. \end{aligned} \right\} \quad (23)$$

Finally multiplying u_α to Eqs. (18) and (17) and summing up with α we obtain

$$\left. \begin{aligned} A_{\alpha\beta} u_\alpha &= -\left(\frac{4}{r+1} + \frac{[\rho]^2}{\rho_1 \rho}\right) (\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu) u_n, \\ A_{\alpha\psi} u_\alpha &= -\left(\frac{4}{r+1} + \frac{[\rho]^2}{\rho_1 \rho}\right) (\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu) u_n. \end{aligned} \right\} \quad (24)$$

Using the relations (16) to (24), the coefficients of the right hand side of Eq. (12) can be determined, except A_{33} and a_3 , as follows:

$$\left. \begin{aligned} A_{11} &= -\frac{[\rho]}{\rho_1} \frac{Ea + Fb}{E} u_n, \\ A_{12} &= -\frac{[\rho]}{\rho_1} \frac{Ea' + Fb'}{\sqrt{EG}} u_n, \\ A_{13} &= -\frac{4}{r+1} \frac{\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu}{\sqrt{E}} u_n, \\ A_{21} &= -\frac{[\rho]}{\rho_1} \frac{Fa + Gb}{\sqrt{EG}} u_n, \\ A_{22} &= -\frac{[\rho]}{\rho_1} \frac{Fa' + Gb'}{G} u_n, \\ A_{23} &= -\frac{4}{r+1} \frac{\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu}{\sqrt{G}} u_n, \\ A_{31} &= -\left(\frac{4}{r+1} + \frac{[\rho]^2}{\rho_1 \rho}\right) \times \frac{\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu}{\sqrt{E}} u_n, \\ A_{32} &= -\left(\frac{4}{r+1} + \frac{[\rho]^2}{\rho_1 \rho}\right) \times \frac{\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu}{\sqrt{G}} u_n. \end{aligned} \right\} \quad (25)$$

Now we notice that $\zeta_\sigma^1, \zeta_\sigma^2$ and ζ_σ^3 have expressed the components of vector $(\sqrt{EG}/\sqrt{EG-F^2}) (1/u_n) \cdot (\mu \times u)$, $-(\sqrt{EG}/\sqrt{EG-F^2}) (1/u_n) \cdot (\lambda \times u)$, $(\sqrt{EG}/\sqrt{EG-F^2}) (1/u_n) \cdot (\lambda \times \mu)$, hence

$$\left. \begin{aligned} \zeta_\sigma^1 \zeta_\sigma^1 &= \frac{EG}{EG-F^2} \frac{u^2 - u_\mu^2}{u_n^2}, \\ \zeta_\sigma^2 \zeta_\sigma^2 &= \frac{EG}{EG-F^2} \frac{u^2 - u_\lambda^2}{u_n^2}, \quad \zeta_\sigma^3 \zeta_\sigma^3 = \frac{1}{u_n^2}, \\ \zeta_\sigma^1 \zeta_\sigma^2 &= \frac{EG}{EG-F^2} \frac{1}{u_n^2} \left(u_\lambda u_\mu - \frac{F}{\sqrt{EG}} u^2 \right), \\ \zeta_\sigma^1 \zeta_\sigma^3 &= \frac{EG}{EG-F^2} \frac{1}{u_n^2} \left(\frac{F}{\sqrt{EG}} u_\mu - u_\lambda \right), \end{aligned} \right\}$$

$$\zeta_\sigma^2 \zeta_\sigma^3 = \frac{EG}{EG-F^2} \frac{1}{u_n^2} \left(\frac{F}{\sqrt{EG}} u_\lambda - u_\mu \right),$$

Thus, inserting these relations into $A_{33} = c^2 u_{\alpha, \sigma} = c^2 A_{\alpha j} \zeta_\sigma^j$, we obtain

$$\left. \begin{aligned} A_{33} &= \frac{-u_n}{\left(\frac{u_n^2}{c^2} - 1\right)} \frac{EG}{EG-F^2} \left\{ \left[\frac{[\rho]}{\rho_1} \right. \right. \\ &\quad \times \frac{Ea + Fb}{E} (u^2 - u_\mu^2) \\ &\quad + \frac{[\rho]}{\rho_1} \frac{Fa + Gb}{\sqrt{EG}} \left(u_\lambda u_\mu - \frac{F}{\sqrt{EG}} u^2 \right) \\ &\quad + \left(\frac{8}{r+1} + \frac{[\rho]^2}{\rho_1 \rho} \right) \left(\frac{\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu}{\sqrt{E}} \right) \\ &\quad \times \left(\frac{F}{\sqrt{EG}} u_\mu - u_\lambda \right) \left. \right\} \\ &\quad + \left\{ \frac{[\rho]}{\rho_1} \frac{Ea' + Fb'}{E} (u^2 - u_\lambda^2) \right. \\ &\quad + \frac{[\rho]}{\rho_1} \frac{Fa' + Gb'}{\sqrt{EG}} \left(u_\lambda u_\mu - \frac{F}{\sqrt{EG}} u^2 \right) \\ &\quad + \left(\frac{8}{r+1} + \frac{[\rho]^2}{\rho_1 \rho} \right) \left(\frac{\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu}{\sqrt{G}} \right) \\ &\quad \times \left(\frac{F}{\sqrt{EG}} u_\lambda - u_\mu \right) \left. \right\}. \end{aligned} \right\} \quad (26)$$

While a_i can be expressed as follows:

$$\left. \begin{aligned} a_1 &= \frac{-2}{(r+1)u_n} \{ (r-1)\rho \\ &\quad - (r+1)\rho_1 \} \frac{\sqrt{E} a u_\lambda + \sqrt{G} b u_\mu}{\sqrt{E}}, \\ a_2 &= \frac{-2}{(r+1)u_n} \{ (r-1)\rho \\ &\quad - (r+1)\rho_1 \} \frac{\sqrt{E} a' u_\lambda + \sqrt{G} b' u_\mu}{\sqrt{G}}, \\ a_3 &= -\frac{\rho}{c^2} A_{33}. \end{aligned} \right\} \quad (27)$$

Now if we choose two Cartesian coordinates x_1 and x_2 as parameters ϕ and ψ , and if we assume that shock surface can be expressed by $x_3 = x_s(x_1, x_2)$, and let be $\partial x_s / \partial x_1 = p$, $\partial x_s / \partial x_2 = q$, $\partial^2 x_s / \partial x_1^2 = r$, $\partial^2 x_s / \partial x_1 \partial x_2 = s$, $\partial^2 x_s / \partial x_2^2 = t$, then, we obtain

$$\left. \begin{aligned} E &= 1 + p^2, \quad F = pq, \quad G = 1 + q^2, \\ L &= \frac{r}{\sqrt{1+p^2+q^2}}, \quad M = \frac{s}{\sqrt{1+p^2+q^2}}, \\ N &= \frac{t}{\sqrt{1+p^2+q^2}}, \end{aligned} \right\} \quad (28)$$

While, if we choose the length of lines of curvature as parameters, we obtain $E=G=1$, and at a point except umbilic point (i. e. $L:M:N \neq$

$E:F:G$, $F=M=0$, $L=\kappa_\lambda$ and $N=\kappa_\mu$, where κ_λ and κ_μ are the normal curvature for ϕ - and ψ -curve at the point considered. Hence,

$$a=-\kappa_\lambda, \quad b=a'=0, \quad b'=-\kappa_\mu.$$

Thus, in this case, we get the comparatively simple form for coefficients A_{ij} and a_i as

$$\left. \begin{aligned} A_{11} &= \frac{[\rho]}{\rho_1} u_n \kappa_\lambda, & A_{12} &= 0, \\ A_{13} &= \frac{4}{r+1} u_n u_\lambda \kappa_\lambda, & A_{21} &= 0, \\ A_{22} &= \frac{[\rho]}{\rho_1} u_n \kappa_\mu, & A_{23} &= \frac{4}{r+1} u_n u_\mu \kappa_\mu, \\ A_{31} &= \left(\frac{4}{r+1} + \frac{[\rho]^2}{\rho_1 \rho} \right) u_n u_\lambda \kappa_\lambda, \\ A_{32} &= \left(\frac{4}{r+1} + \frac{[\rho]^2}{\rho_1 \rho} \right) u_n u_\mu \kappa_\mu, \\ A_{33} &= \frac{n_n}{u_n^2 - 1} \left[\left\{ \frac{[\rho]}{\rho_1} (u^2 - u_\lambda^2) \right. \right. \\ &\quad \left. \left. - \left(\frac{8}{r+1} + \frac{[\rho]^2}{\rho_1 \rho} \right) u_\lambda^2 \right\} \kappa_\lambda \right. \\ &\quad \left. + \left\{ \frac{[\rho]}{\rho_1} (u^2 - u_\mu^2) \right. \right. \\ &\quad \left. \left. - \left(\frac{8}{r+1} + \frac{[\rho]^2}{\rho_1 \rho} \right) u_\mu^2 \right\} \kappa_\mu \right], \end{aligned} \right\} \quad (29)$$

$$\left. \begin{aligned} a_1 &= \frac{2u_\lambda \kappa_\lambda}{(r+1)u_n} \{ (r-1)\rho - (r+1)\rho_1 \}, \\ a_2 &= \frac{2u_\mu \kappa_\mu}{(r+1)u_n} \{ (r-1)\rho - (r+1)\rho_1 \}, \\ a_3 &= -\frac{\rho}{c^2} A_{33}. \end{aligned} \right\} \quad (30)$$

5. Curvature of stream line just behind shock

If we indicate the length along a stream line by η , the curvature of stream line K is defined as,

$$K^2 = \frac{\partial^2 \mathbf{x}}{\partial \eta^2} \cdot \frac{\partial^2 \mathbf{x}}{\partial \eta^2}, \quad (31)$$

namely, K is the magnitude of vector $\partial^2 \mathbf{x} / \partial \eta^2$.

Now, taking account of $\partial \mathbf{x} / \partial \eta = (u_\alpha / u) \mathbf{e}_\alpha$ and $u_\alpha u_\alpha = u^2$,

$$K^2 = \frac{1}{u^2} \left\{ \frac{\partial u_\alpha}{\partial \eta} \frac{\partial u_\alpha}{\partial \eta} - \left(\frac{\partial u}{\partial \eta} \right)^2 \right\}.$$

Further, since $(\partial u_\alpha / \partial \eta)(\partial u_\alpha / \partial \eta) = (1/u^2)(u_{\alpha,\beta} u_{\alpha,\beta})$, $\partial u / \partial \eta = (1/u^2) u_{\alpha,\beta} u_\alpha u_\beta$, and $u_{\alpha,\beta} u_\beta = A_{13} \zeta_\alpha$, $u_{\alpha,\beta} u_\alpha u_\beta = A_{33}$,

$$K^2 = \frac{1}{u^4} \left\{ (A_{13} \zeta_\alpha) (A_{13} \zeta_\alpha) - \frac{1}{u^2} A_{33}^2 \right\}. \quad (32)$$

In particular, if we choose the length of lines of curvature as parameters,

$$K^2 = \frac{1}{u^4 u_n^2} \left\{ \left(A_{13} u - A_{33} \frac{u_\lambda}{u} \right)^2 + \left(A_{23} u - A_{33} \frac{u_\mu}{u} \right)^2 - (A_{13} u_\mu - A_{23} u_\lambda)^2 \right\}, \quad (33)$$

where A_{13} , A_{23} and A_{33} are obtained by Eqs. (29).

In the case of two dimensional shock, it is assumed that κ_μ and u_μ (or κ_λ and u_λ) vanish identically. Thus, we have

$$K^2 = \frac{1}{u^4 u_n^2} \left(A_{13} u - A_{33} \frac{u_\lambda}{u} \right)^2, \quad (34)$$

where $A_{13} = \{4/(\gamma+1)\} u_n u_\lambda \kappa_\lambda$, $A_{33} = \{u_n / (u_n^2 / c^2 - 1)\} \cdot \{([\rho]/\rho_1) u^2 - \{8/(\gamma+1) + ([\rho]^2 / \rho_1 \rho) u_\lambda^2\} \kappa_\lambda$,

This is the result derived by Thomas.

In the case of axially symmetrical shock, the lines of curvature are intersections of the shock surface with the meridian plane and perpendicular to the axis. We choose them ϕ - and ψ -curve, respectively. Then $u_\mu = 0$ and hence $A_{23} = 0$. Thus

$$K^2 = \frac{1}{u^4 u_n^2} \left(A_{13} u - A_{33} \frac{u_\lambda}{u} \right)^2, \quad (35)$$

where $A_{13} = \{4/(\gamma+1)\} u_n u_\lambda \kappa_\lambda$, $A_{33} = \{u_n / (u_n^2 / c^2 - 1)\} \cdot \{([\rho]/\rho_1) u^2 - \{8/(\gamma+1) + ([\rho]^2 / \rho_1 \rho) u_\lambda^2\} \kappa_\lambda + ([\rho]/\rho_1) (u^2 - u_\lambda^2) \kappa_\mu\}$. Let d and α be radii of ϕ -curve (circle) and the shock angle at the point considered, respectively. Then $\kappa_\mu = (1/d) \cos \alpha$.

When we define the curvature as positive or negative according as the curve is concave upward or downward, respectively, the expression for K must take the negative root.

6. Singular shock directions for axially symmetrical shock

If the problem is restricted to axially symmetrical shock from the first, the treatment is very simple. In this case, the differential equations of flow can be written as follows:

$$\left. \begin{aligned} p_{,\alpha} + \rho u_\alpha u_{\alpha,\sigma} &= 0, \\ \rho_{,\alpha} u_\alpha + \rho u_{\sigma,\sigma} + \rho \frac{\bar{u}}{x} &= 0, \\ \rho u_{\alpha,\beta} u_\alpha u_\beta - \gamma p \left(u_{\sigma,\sigma} + \frac{\bar{u}}{x} \right) &= 0, \end{aligned} \right\} \quad (36)$$

where $\bar{x}$ is the distance from the symmetrical axis to the point considered, and $\bar{u}$ is velocity component perpendicular to this axis at this point and in these expressions the suffixes take only 1

and 2. The shock conditions are the same as in the two-dimensional case.

Using rotated coordinates²⁾ and taking into account only the terms which involve the highest derivative of velocity components, we have the same expression for n -th derivative of curvature of streamline just behind shock as in the two dimensional case, namely

$$\frac{d^n K}{d\eta^n} = \frac{u_{2,11\dots 1}}{v} \quad (37)$$

By taking account of

$$\left(\frac{1}{x}\right)_{,\beta\dots\gamma} = \frac{\delta_\beta\dots\delta_\gamma}{(-\sin\alpha)^n} \frac{d^n K_\mu}{dS^n}, \quad (38)$$

just behind shock, where $\delta_1 = \sin\alpha$, $\delta_2 = \cos\alpha$, α is deflection angle and s is arc length along the shock surface in a meridian plane, finally, $u_{2,11\dots 1}$ can be determined by solving the following systems which are obtained from the differential equations of flow and the shock conditions:

$$\left. \begin{aligned} & \left(\frac{u^2}{c^2} - 1 \right) u_{1,1\beta\dots\gamma} - u_{2,2\beta\dots\gamma} \\ & = \bar{u} \frac{\delta_\beta\dots\delta_\gamma}{(-\sin\alpha)^n} \frac{d^n K_\mu}{dS^n}, \\ & u_{\alpha,1\beta\dots\gamma} \lambda_\alpha \lambda_\beta\dots\lambda_\gamma = -\frac{B}{\rho u} \frac{d^n K_\lambda}{dS^n}, \\ & u_{2,\alpha\beta\dots\gamma} \lambda_\alpha \lambda_\beta\dots\lambda_\gamma = A_2 \frac{d^n K_\lambda}{dS^n}, \end{aligned} \right\} \quad (39)$$

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The Theory of Hydrodynamic Stability: C. C. Lin, Cambridge, Cambridge 1955, 155 頁, 15 × 23 cm, 22 s. d.

‘流体力学的安定理論’について初の成書が Cambridge Monographs の一つとして現われた。著者 C. C. Lin は MIT の応用数学の教授で、安定理論については戦時中から今まで数多くの指導的な仕事をし来た人である。

本書で取扱われている安定理論は、微小擾乱に対する所謂線型安定理論である。これは元来、層流から乱流への遷移理論の一部分をなすものであるが、今日では具体的な結果を与える只一つの理論として、遷移理論の中でも独立した一つの体系を形造っている。前世紀の末に始まる長い歴史をもつこの理論を、著者はかなり要領よくこの小冊子にまとめている。

第1章 緒論：微小擾乱法による安定理論の定式化と二三の典型的な流れに対する結果の概観。第2章 Couette 運動の安定性：Taylor の計算の紹介と実験との比較。第3章 二次元 Poiseuille 流の安定性：安定の充分条件についての古典的結果と、Heisenberg に始まる漸近解による固有値問題の解法とを述べる。第4章 流体力学的安定の一般理論：不安定の動因を遠心力などの外力によるもの（外力型）と粘性を媒介とするもの（粘性型）とに分類し、各々について物理的機構を考察する。第5章 平板に沿う境界層：実用的に重要な圧縮性境界層の安定について、Lees, Lin, Dunn 等の計算を紹介する。第6章 他のほとんど平行な流れ：境界層の安定に対する主流の乱れ等のパラ

where B and A_2 are same expressions as obtained by Thomas²⁾ for two dimensional case.

Thus, we prove the following consistency relation,

$$\frac{d^n K}{d\eta^n} = G_{1n}(M, \alpha) \frac{d^n K_\lambda}{dS^n} + G_{2n}(M, \alpha) \frac{d^n K_\mu}{dS^n} + H_n, \quad (40)$$

where H_n is the polynomial of κ_λ , κ_μ and their derivatives less than n -th order with respect to s . Further G_{1n} , G_{2n} and coefficients of H_n are rational functions of M^2 the coefficients of which are functions of α , and both G_{1n} and G_{2n} have the zeros only in the subsonic region.

$d^n \kappa_\mu / dS^n$ can be determined by κ_λ , κ_μ and their derivatives less than n -th order and $G_{1n}(\alpha, M)$ consist with two-dimensional one. Hence the singular direction where the power series approach based on Eq. (40) will fail, is given by the angle α at which any of G_{1n} 's vanish at given M , and its distribution is the same as two-dimensional one. But it has been assumed that κ_μ is finite.

References

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メターの影響を調べ、助走流・混合流など境界層に類似の流についての計算結果を述べる。第7章 天体及び地球物理学的に興味ある安定問題の例：大気循環、対流など外力型不安定の事例を簡単に述べる。第8章 平行流の安定の数学的議論：粘性型不安定に伴う数学的問題、殊に高 Reynolds 数に対する漸近解が多価性をもつことによる困難の解決を、Lin, Wasaw, Morawetz の研究を中心として述べる。この章は第3章と共に本書中の圧巻である。巻末の文献表は豊富で安定理論についてのほとんど全業績を尽している。

不安定の二つの型のうち、乱流への遷移に主として関係するのは粘性型不安定であり、本書の記述の重点もまたここに置かれている。ただ、漸近解法についての詳しい議論が第8章まで後回しになっているが、これは第3章と一緒にしてもっと丁寧に書いてくれた方が有難かったと思う。他にも欲をいえば、叙述のキメがかなり荒く、論文から理屈抜きで結果の式を引用してあることの多い第5章以後など、読むのに相当骨が折れる。これは本書の意図が、安定理論の応用例の紹介よりも、理論そのものの構造の解説にあると思われるだけに残念である。また、初版の故でもあるうが誤植が割合に多く、殊に式の中のミスはいささか読者泣かせである。

とはいえ、内容的にも多岐に亘り、また厄介な数式的議論の欠かせない流体力学的安定理論を、ここまでまとめ上げた著者の腕は非凡である。この方面の最初の専門書として関心ある諸氏の一読をお勧めしたい。

(巽 友正)