Expression of Initial Stresses Based on Green's Functions*

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Synopsis
During the cooling or heating of a body whose boundary is free from any external force, when will the induced thermal stress become maximum, and what is the maximum? This problem seems interesting. In order to answer the question generally when the "Eigenspannungs-quellen" \( e_{ij} \) are given, the authors tried to express the initial stress by the incompatibility and the dislocation of the Eigenspannungs-quellen using Green's functions, for the distribution of the initial stress depends essentially on them.

Riemann's curvature tensor constructed by the fundamental tensor, \( \frac{1}{2}(e_{ij} - \delta_{ij}) \), represents the incompatibility of \( e_{ij} \), and the topology of the body relates to the dislocation induced by \( e_{ij} \). An effect of the geometrical shape of the body on the initial stress is represented by Green's functions, which are independent of the Eigenspannungs-quellen.

1. Introduction
A body is assumed to be free from any external force but under the Eigenspannungs-quellen, \( e_{ij} \), which are prescribed by the other conditions. For instance, when the body is heated, the Eigenspannungs-quellen \( e_{ij} \) are equal to \( \alpha T \delta_{ij} \), where \( \alpha \), \( T \), and \( \delta_{ij} \) mean the thermal linear expansion coefficient, the temperature of the body, and Kronecker's delta respectively.

Hook's law for the strain tensor \( \varepsilon_{ij} \) and the stress tensor \( \sigma_{ij} \) is

\[
\varepsilon_{ij} = \frac{1}{2\mu} (e_{ij} - \delta_{ij}) + \frac{1}{2\nu} \sigma_{ij} + e_{ij}, \quad (1)
\]

or

\[
\sigma_{ij} = 2\mu (e_{ij} - \delta_{ij}) + \lambda (e - \varepsilon)^2, \quad (1')
\]

where

\[
\begin{align*}
\sigma & = \sigma_{ij} \delta^{ij} / 2\varepsilon^{ij} \delta^{ij}, \\
\varepsilon & = \varepsilon_{ij} \delta^{ij}, \\
e & = e_{ij} \delta^{ij}
\end{align*}
\]

\( \mu \) and \( \nu \) denote the shear modulus and Poisson's ratio respectively, and there are the relations

\[
\begin{align*}
3k & = 3\lambda + 2\mu \quad \text{for 3 dimension} \\
3k & = 2\lambda + 2\mu \quad \text{for 2 dimension} \\
\lambda & = 2\mu \nu / (1 - 2\nu) \quad \text{for plane strain} \\
\lambda & = 2\mu \nu / (1 - \nu) \quad \text{for plane stress.}
\end{align*}
\]

The equations of compatibility are

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (3)
\]

The equations of equilibrium are

\[
\sigma_{ij,j} = 0 \quad \text{in } D, \quad (4)
\]

where \( D \) is the domain of the body.

The boundary conditions are

\[
\sigma_{ij} n_j = 0 \quad \text{on } S, \quad (5)
\]

where \( S \) is the boundary of \( D \) and \( n_j \) are components of a normal vector on the surface \( S \).

The system of (1)-(5) is equivalent to the usual elastic problem under the mass force \( (-2\mu e_{ij} - \lambda \delta_{ij})_j \) and the surface force \( (2\mu e_{ij} + \lambda \delta_{ij}) n_j \). Let \( \sigma'_{ij} \) denote the solution of the above equivalent elastic problem, we can write the initial stress as:

\[
\sigma_{ij} = -2\mu e_{ij} - \lambda \delta_{ij} + \sigma'_{ij}, \quad (6)
\]

Especially, when \( e_{ij} = \alpha T \delta_{ij} \), the equation (4) may be put in an equation

\[
\mu u_{i,j,j} + (\lambda + \mu) u_{j,j,i} - 3k (\alpha T)_i = 0. \quad (7)
\]

Therefore, we can find the particular solution of (7) by introducing a thermal potential \( \phi \)

\[
u_i = \phi, \quad (8)
\]
where $\phi$ must satisfy Poisson's equation
\begin{equation}
(\lambda+2\mu)\phi_{ij,ij}=3k\alpha T. \tag{9}
\end{equation}

Though these methods reduced to the equivalent elastic problems have been useful for analyses of several practical problems, but they have an inconvenience such that the equivalent surface or mass forces and the right side of (9) may consist of the compatible parts which have no significance in the Eigenspannungs-quellen, for instance, when $T$ is constant, Eq. (9) has a solution, nevertheless, no thermal stress occurs. Such a circumstance makes it inconvenient to estimate the order of magnitude of initial stress. The purpose of this research is to express the initial stress with the essential parts of the Eigenspannungs-quellen which are incompatible, by the help of Green's functions.

2. Green's Functions

Consider two systems of deformation states in a domain $D$, one of which is the considered system (1)-(5) and the other has no Eigenspannungs-quellen but a singularity at a point $\bar{x}$. Let displacements, strains, and stresses in the latter system be denoted by a star *. Since the elastic strain energy is homogeneous with respect to the elastic strain, and the stress, we have the following identity
\begin{equation}
\sigma_{ij}^{*} \epsilon_{ij}^{*} = \sigma_{ij}^{*} (\epsilon_{ij}^{*} - \epsilon_{ij}). \tag{10}
\end{equation}

$\epsilon_{ij}^{*}$ are induced from $u_{ij}^{*}$ like (3) and relate to $\sigma_{ij}^{*}$ by Hooke's law. Let $G$ denote a spheroid having a center at the point $\bar{x}$ and $\Sigma$, its boundary.

By integrating (10) over the whole domain $D-G$, we have
\begin{equation}
\int_{D-G} \sigma_{ij}^{*} u_{ij}^{*} n_{j} ds - \int_{D-G} \sigma_{ij}^{*} u_{ij} n_{j} ds = - \int_{D-G} \sigma_{ij}^{*} e_{ij} ds. \tag{11}
\end{equation}

We use the following elementary solutions\(^{3),4}\) which are the displacements induced by a dislocation in $\alpha$ direction on the surface normal to $\beta$ direction which is located at the origin of the coordinate
\begin{equation}
\Gamma^{\alpha}(x) = k_{1} (\beta_{\alpha}^{\alpha} x_{\alpha}^{2} + \beta_{\alpha}^{\beta} x_{\alpha} x_{\beta} - x_{\alpha} \delta_{\alpha}^{\beta})/r^{3} + 3x_{\alpha} x_{\beta} x_{\gamma} / r^{5}, \tag{12}
\end{equation}
where $r = (x_{\alpha} x_{\beta})^{1/2}$ and $k_{1} = 1 - 2\nu$.

For the two dimensional case we have
\begin{equation}
\Gamma_{\alpha\beta}(x) = \frac{1}{2(\lambda + \mu)} (\delta_{\alpha}^{\alpha} x_{\beta}^{2} + \delta_{\alpha}^{\beta} x_{\alpha} x_{\beta} - x_{\alpha} \delta_{\alpha}^{\beta})/r^{2} + \frac{1}{\mu} x_{\alpha} x_{\beta} / r^{4}. \tag{13}
\end{equation}

We take $\Gamma^{\alpha}$ as the displacements of the * system:
\begin{equation}
u_{*}(x) = \Gamma^{\alpha}(x - \bar{x}). \tag{14}
\end{equation}

When the radius of $G$ tends to zero, Eq. (11) becomes
\begin{equation}
8\pi(1 - \nu) \sigma_{\alpha\beta}(\bar{x}) + \frac{8\pi\mu}{\lambda + 2\mu} - I(14 - 10\nu) \sigma_{\alpha\beta} + (2 + 10\nu) \sigma_{\beta\alpha} + \int_{S} \sigma_{ij} u_{ij} n_{j} ds = \int_{D} \sigma_{ij} e_{ij} ds, \tag{15}
\end{equation}
and for the two dimensional case we have
\begin{equation}
\frac{\pi}{\mu} \lambda + 2\mu \sigma_{\alpha\beta}(\bar{x}) + \frac{\pi}{2} (2\sigma_{\alpha\beta} + \sigma_{\beta\alpha}) + \int_{S} \sigma_{ij} u_{ij} n_{j} ds = \int_{D} \sigma_{ij} e_{ij} ds. \tag{16}
\end{equation}

It is seen that $\Gamma^{\alpha}(x - \bar{x})$ is the principal part of Green's function for obtaining $\sigma_{\alpha\beta}(\bar{x})$. If a regular displacement is added to $\Gamma^{\alpha}(x - \bar{x})$ in order to satisfy the homogeneous condition $\sigma_{\alpha\beta} n_{j} = 0$, we shall have Green's functions which are components of a tensor.

3. Incompatibility and Dislocation

In order to express that the incompatibility and the dislocation of the Eigenspannungs-quellen only are sources of the initial stresses in a free body, we can rewrite the right sides of (15) and (16). First, consider a two dimensional body and use the following Airy's stress function\(^{4}\)
\begin{equation}
F_{\alpha\beta}(x) = - \delta_{\alpha\beta} \log r + \frac{1}{2} x_{\alpha} x_{\beta} / r^{2}. \tag{17}
\end{equation}

The displacements which are calculated from (17) are equal to (13). Then, Green's function expressed with Airy's function becomes
\begin{equation}
F(x) = F^{\alpha}(x - \bar{x}) + G^{\alpha}(x), \tag{18}
\end{equation}
where $G^{\alpha}(x)$ is the additive regular function in order to satisfy the condition of stress-free boundary, thus, we have:
\begin{equation}
\begin{aligned}
\epsilon_{ij}^{*} &= F_{,kj} \delta_{ij} - F_{ij}, \tag{19} \\
\epsilon_{ij}^{*} n_{j} &= 0 \quad \text{on } S, \tag{20} \\
x_{k} \epsilon_{ij}^{*} n_{j} - x_{i} \epsilon_{kj}^{*} n_{j} &= 0 \quad \text{on } S. \tag{21}
\end{aligned}
\end{equation}

The last two conditions are equivalent to
\begin{equation}
\frac{dF_{ij}}{ds} = 0, \quad \frac{dF_{ij}}{ds} = 0 \quad \text{on } S. \tag{22}
\end{equation}
\[
\frac{d}{ds}(xF_{i,n} + yF_{i,y} - F) = 0 \quad \text{on } S, \quad (23)
\]

where \( ds \) means a line element along the boundary \( S \). The right side of (16) can be transformed to

\[
\int_D \sigma^{ij} e_{ij} d\nu = \int_D F(e_{nk} - e_{ij,ij}) d\nu + \frac{n}{2} (2\sigma^{xy} + \sigma^{xy}) + \int_S F(e_{nk} - e_{ij,ij}) n_j ds + \int_S (F_{ij} e_{ij} - F_{ij}) n_j ds. \quad (24)
\]

Furthermore, the last two integrals in (24) are transformed to

\[
\int_S (xF_{i,n} + yF_{i,y} - F) d\omega - \int_S F_{ij} d\xi + \int_S F_{ij} d\eta,
\]

where

\[
\begin{align*}
&d\xi =\left( e_{nx,y} + y(e_{nx,n} - e_{nx,n}) \right) dx \\
&+\left( e_{ny,n} - e_{ny,n} \right) dy \\
&d\eta =\left( e_{nx,y} + x(e_{nx,n} - e_{nx,n}) \right) dx \\
&+\left( e_{ny,n} + x(e_{nx,n} - e_{nx,n}) \right) dy \\
&d\omega =\left( -e_{nx,n} - e_{nx,n} \right) dx +\left( e_{nx,n} - e_{nx,n} \right) dy
\end{align*}
\]

\[(25)
\]

\( d\xi \) and \( d\eta \) are coordinates of the origin of the \( x' \) coordinate system which is fixed at the point \( x + dx \) with respect to the \( x \) coordinate system; \( d\omega \) is a rotation of the \( x' \) system relative to the \( x \) system caused by such a releasing. Taking into account that \( F_{ij} \) and \( xF_{i,n} - F \) are constants on the boundary \( S \), we have

\[
\frac{\pi}{\mu} \frac{\lambda + 2\mu}{\lambda + \mu} \sigma^{ij}(\vec{r}) = \int_D (e_{nk} - e_{ij,ij}) F d\nu + \sum (xF_{i,n} + yF_{i,y} - F) \int_S d\omega - F_{ij} \int_S d\xi + F_{ij} \int_S d\eta, \quad (27)
\]

where the summation \( \Sigma \) is taken for every boundary. \( e_{nk} - e_{ij,ij} \) is called the incompatibility and equivalent to the Riemann's curvature with the fundamental tensor \( \frac{1}{2}(e_{ij} - \delta_{ij}) \) neglected in the second order term. \( \int_S d\xi, \int_S d\eta \) and \( \int_S d\omega \) are called the dislocation and relate to the topological properties of the body.

In the above argument, we have assumed the continuity of \( e_{ij} \). When \( e_{ij} \) are discontinuous but finite along a closed curve \( c \), the right side of (27) has the additive terms;

\[
-\int_S F_{ij,ij} d\omega + \int_S F_{ij} d\eta, \quad (28)
\]

where \([ \ ]\) means the difference between the left and the right side value of, for instance, \( d\omega \).

For example, when \( e_{ij} = \delta_{ij} \alpha T \) and \( T = \Delta T = \text{constant} \) in the inner part bounded by \( C \) and \( T = 0 \) in the other part, we have

\[
\frac{\pi}{\mu} \frac{\lambda + 2\mu}{\lambda + \mu} \sigma^{ij}(\vec{r}) = \alpha \Delta T \int_S F_{ij} n_i ds. \quad (29)
\]

For a singly connected body, it is easy to investigate the instant when the thermal stress becomes maximum during cooling or heating of the body and the upper bound of the absolute maximum value of the thermal stresses.

Since the arbitrary constants of \( F \) and \( F_{ij} \) on the boundary can be taken as zero, (27) can be written

\[
\frac{\pi}{\mu} \frac{\lambda + 2\mu}{\lambda + \mu} \sigma^{ij}(\vec{r}) = \alpha \int_D F \rho \rho T d\nu, \quad (30)
\]

where \( \rho = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \). When the flow of heat is stationary, the right side of (3) vanishes, therefore, no stress occurs. The extremum of \( \sigma^{ij} \) with respect to time \( t \) arises when

\[
\int_F \frac{\partial}{\partial t} (\rho T) d\nu = 0. \quad (31)
\]

This is convenient because of the timely independency of \( F \). An upper bound of the absolute maximum of \( \sigma^{ij} \) is obtained directly from the following inequality

\[
\frac{\pi}{\mu} \frac{\lambda + 2\mu}{\lambda + \mu} |\sigma^{ij}| = a |\sigma T|_{\text{max}}, \quad (32)
\]

The upper bound \( a |\Delta T|_{\text{max}} \int \left| F d\nu \right|_{\text{max}} \) is valid throughout the time process and the space domain. It must be noted that \( F \) is unique for a given body.

4. Three Dimensional Expression

An extension of (27) to three dimensional problems can be made easily by using the following components of a tensor instead of \( F^{ij} \) (17):

\[
F_{ijkl}(\vec{r}) = \mu \rho \eta (k_l - \delta_{kl}) (2(1 - 2\nu)(\delta_{kl} - \delta_{ki} + \delta_{il} + \delta_{il})) / \rho + (4\nu - 1) \sigma^{ij} \delta_{ij} \delta_{kl} / \rho^2 - 2(1 - 2\nu) (\delta_{ij} \delta_{kl} - \delta_{i} \delta_{k} \delta_{l}) / \rho^2 + 6 \sigma^{ij} \delta_{ij} \delta_{kl} \delta_{ij} \delta_{kl} / \rho^2, \quad (33)
\]
with a convention that

\[ \varepsilon_{i,j,k} = \varepsilon_{j,i,k} = 0 \]

when any two of the indices are equal;

\[ \varepsilon_{i,j,k} = +1 \]

when \( i, j, k \) is an even permutation of the numbers 1, 2, 3;

\[ \varepsilon_{i,j,k} = -1 \]

when \( i, j, k \) is an odd permutation of the numbers 1, 2, 3.

The components of the stress \( \sigma^{ij} \) in (15) are expressed by the \( F^{ik}_{ik}(x-\bar{x}) \) as follows:

\[ \sigma^{ij} = \varepsilon^{ijk} \varepsilon^{klm} F^{ik}_{ik} \]

(34)

with

\[ F^{ik}_{ik}(x) = F^{ik}_{ik}(x-\bar{x}) + G^{ik}_{ik}(x) \]

where \( G^{ik}_{ik} \) are the additive terms such that the stress components (34) satisfy the homogeneous boundary condition

\[ \sigma^{ij} n_j = 0 \]

on \( S \).

(36)

Then the last term of the left side of (15) vanishes and its right side becomes

[\int_D \sigma^{ij} e_{ij} dv = - \int_S F^{ik}_{ik} R^{ik} dv + \frac{8\pi \mu}{15} \{(14 - 10\nu)\sigma^p(\bar{x}) + (2 + 10\nu)\sigma^q(\bar{x})\} \]

(37)

with

\[ R^{ij} = \varepsilon^{ijk} \varepsilon^{klm} e_{ik}, \]

(38)

\( R^{ij} \) mean the components of a curvature tensor in the Riemannian space having the fundamental tensor \( \frac{1}{2}(e_{ij} - \delta_{ij}) \). Now we can define

\[ d\sigma^p = e^{ik} F_{i,k} dx^k \]

\[ d\sigma^q = (e_{ik} + e_{ik}) e^{ik} F_{i,k} dx^k \]

\[ d\varepsilon^p = (e_{ik} + e_{ik}) e^{ik} e_{ik}, \]

(39)

Consider the following line integrals taken along a closed curve on the surface \( S \) of the body which may be put in the surface integrals by Stokes' theorem:

\[ \oint \sigma^p d\sigma^p = \int e^{ik} F_{i,k} n_\sigma dS \]

\[ \oint d\sigma^q = \int (d\sigma^p + \varepsilon_{ik} e^{ik} p_k F_{i,k}) dS \]

\[ = \int e^{ik} e_{ik} e^{ik} F_{i,k} n_\sigma dS \]

\[ = \int e_{ik} \sigma^{ik} e^{ik} n_\sigma dS \]

(40)

Because of vanishment of (36), we can define \( \sigma^p, \sigma^q \) on the surface of the body. Now we consider the following line integral taken along the closed curve which is bounded by the whole surface \( S \) of the body

\[ I = \oint [\sigma^p d\varepsilon^p + \varepsilon^q d\sigma^q] \]

(41)

This can be transformed to the surface integral taken over the whole surface \( S \) of the body by Stokes' theorem:

\[ I = \varepsilon^{ijk} \varepsilon^{klm} \int_S (F^{ik}_{ik} e_{ij} n_\sigma - F^{ik}_{ik} e_{ij} n_\sigma) ds \]

\[ + \int_S (e_{ik} e^{ik} e_{ik} p_k F_{i,k}) n_\sigma ds \]

(42)

Considering (15), (37), and (42), we have

\[ 8\pi(1 - \nu)\sigma^p(\bar{x}) = \int_S F^{ik}_{ik} R^{ik} dv \]

\[ = \int_S (e_{ik} e^{ik} e_{ik} p_k F_{i,k} + \varepsilon_{ik} e^{ik} \varepsilon_{ik} n_\sigma) ds \]

(43)

The integral path of (41) is taken twice; once in regular direction and once in reverse direction along the canonical sections\(^9\) of the surface \( S \) of the body. Along the canonical sections, the differences of values of \( \sigma^p \) and \( \sigma^q \) have the same values on both sides of the sections. Thus, the integral (41) may be put in the form:

\[ I = \oint [\sigma^p] d\sigma^q + [\sigma^q] d\sigma^p \]

(44)

where \([\sigma^p]\) means the differences of values of \( \sigma^p \) between the both sides of the canonical sections. This rests on the fact that an arbitrary orientable surface consisted of 2p canonical sections is topologically equivalent\(^9\) to a 4p-sided polygon with pairs of sides identified according to a definite rule.

Consequently, it is easy to see that the necessary and sufficient conditions of vanishment of initial stresses are the vanishments of all \( R^{ij} \) and (44).

References