On Torsional Vibrations of a Beam with a Small Amount of Pretwist*

Eric Reissner** and Kyuichiro Washizu***

Abstract

The present paper treats the problem of the pretwisted, narrow cross section beam on the basis of an approximate system of differential equations which is obtained by variational methods from the theory of shallow thin shells.

Application of the equations of the paper to the problem of torsional vibrations shows that a relatively small amount of pretwist may have a considerable influence on the torsional frequencies of the beam.

1. Introduction

In the following we consider the problem of deriving a system of ordinary differential equations for vibrations of pretwisted narrow cross section beams. Introducing a span coordinate x which becomes the independent space coordinate we allow that the width or chord of the beam, as well as the rate of pretwist are given functions of x. We furthermore allow that the thickness of the beam be a given function of x and of a chordwise coordinate y.

Starting with a variational formulation of the problem of thin shallow elastic shells we make approximative assumptions concerning the chordwise modes of deformation of the elements of the shell. We then use the direct methods of the calculus of variations to obtain a system of ordinary differential equations, insofar as the spanwise coordinate is concerned.

We finally apply the equations which are obtained to the problem of torsional vibrations of a beam of constant rectangular cross section and uniform rate of pretwist. For this case we obtain some explicit results which show the appreciable effect of even moderate pretwist in the case of narrower cross section beams.

2. A variational principle for shallow shells

We begin by writing down a variational principle for the linear theory of shallow shells which is the appropriate specialization for this case of a general variational principle for stresses and

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* A report on work initiated in 1954 while the second named author was at MIT. The work was supported in part by the Office of Naval Research of the United States Navy. Received Sept. 28th, 1957.
** Massachusetts Institute of Technology
*** University of Tokyo

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Deformations. Let the middle surface of the shell before deformation be given in a rectangular Cartesian coordinate system in the form

\[ z = z(x, y) \]  

(2.1)

Let the displacement components of the middle surface be \( u, v \) and \( w \), in the directions of the \( x \), \( y \) and \( z \)-axes respectively. Then, under the assumption that any line element of the shell which is perpendicular to the middle surface before the deformation remains perpendicular to the deformed middle surface after the deformation, components of strain \( \varepsilon_x, \varepsilon_y \) and \( \tau_{xy} \) parallel to the middle surface can be expressed in terms of \( u, v \) and \( w \) such that,

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial w}{\partial y} - \frac{\partial^2 w}{\partial y^2} \\
\tau_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x \partial y} \\
&\quad + \frac{\partial z}{\partial y} \frac{\partial w}{\partial x} - 2\zeta \frac{\partial^2 w}{\partial x \partial y}
\end{align*}
\]  

(2.2)

The quantity \( \zeta \) is the distance from a point \( (x, y, z) \) of the middle surface to a point under consideration measured along the normal to the middle surface.

Stress components, corresponding to (2.2) can be expressed as follows in terms of stress resultants and stress couples.

\[
\begin{align*}
\sigma_x &= \frac{N_x}{h} + \frac{M_x}{h^2/6} \cdot \zeta \\
\sigma_y &= \frac{N_y}{h} + \frac{M_y}{h^2/6} \cdot \zeta \\
\tau_{xy} &= \frac{N_{xy}}{h} + \frac{M_{xy}}{h^2/6} \cdot \zeta
\end{align*}
\]  

(2.3)

The quantity \( h = h(x, y) \) is the thickness of the shell.

In the present paper, we consider natural vibration problems. It is well-known that such problems can be reduced to static problems by taking

\[ \rho \omega^2 u, \rho \omega^2 v, \rho \omega^2 w \]  

(2.4)

as body forces in the directions of \( x, y \) and \( z \)-axes respectively, where \( \rho \) is the density of the shell and \( \omega \) is the circular frequency of the natural vibration.

With the above preliminaries, the general variational principle for the shallow shell with traction free boundaries can be stated as follows [1, 2]:

The governing equations and boundary conditions of the shell problem are the Euler equations of the variational equation \( \delta \Pi = 0 \) where the functional \( \Pi \) is defined by

\[
\Pi = \int \left\{ \left( \frac{\partial u}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial w}{\partial x} \right) N_x + \left( \frac{\partial v}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial w}{\partial y} \right) N_y \\
+ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} \right) N_{xy} \\
- \frac{\partial^2 w}{\partial x^2} M_x - \frac{\partial^2 w}{\partial y^2} M_y - 2 \frac{\partial^2 w}{\partial x \partial y} M_{xy} \\
- \frac{1}{2} \left( \frac{N_x^2}{C_x} + \frac{N_y^2}{C_y} - 2 \nu_D \frac{N_x N_y}{C_x} + \frac{N_{xy}^2}{C_x} \\
+ \frac{M_x^2}{D_x} + \frac{M_y^2}{D_y} - 2 \nu_B \frac{M_x M_y}{D_x} + \frac{M_{xy}^2}{D_{xy}} \right) \right\} dx dy
\]

(2.5)

and the quantities

\( u, v, w, N_x, N_y, N_{xy}, M_x, M_y \) and \( M_{xy} \)

are varied independently.

In equation (2.5),

\( C_x, C_y, C_{xy} \) are extensional rigidities of the shell

\( D_x, D_y, D_{xy} \) are bending rigidities of the shell

\( \nu_D \) is Poisson's ratio in extension

\( \nu_B \) is Poisson's ratio in bending.

\( \eta \) is a parameter which has the value unity for actual shells, but which will be useful to indicate negligibility of the associated inertia contributions.

For isotropic shells the quantities, \( C, D \) and \( \nu \) reduce to

\[
\begin{align*}
C_x &= E h, \quad C_y = E h, \quad C_{xy} = G h \\
D_x &= E h^3/12, \quad D_y = E h^3/12, \quad D_{xy} = G h^3/12 \\
\nu_D &= \nu_B = \nu
\end{align*}
\]

where \( E \) is the Young's modulus, \( \nu \) is the Poisson's ratio and \( E = 2(1+\nu)G \).
3. Formulation of an approximate solution

Since we are interested in solvable approximate equations rather than in unsolvable exact equations, we assume that \( u, v \) and \( w \) are given approximately by,

\[
\begin{align*}
 u &= u_0 + u_1 y, \quad v = v_0 + v_1 y, \quad w = w_0 + w_1 y \\
 u_0, u_1, v_0, v_1, w_0, w_1, f_0 \text{ and } f_1 \text{ are functions of } x \text{ only.} 
\end{align*}
\]

We assume further

\[
z = f_0 + f_1 y
\]

The quantities \( u_0, u_1, v_0, v_1, w_0, w_1, f_0 \) and \( f_1 \) are functions of \( x \) only. By substituting (3.1) and (3.2) into (2.5) and defining

\[
\begin{align*}
 N_z^{(1)} &= \int_C \left( C_1^{(2)} N_z y^2 dy \right) \\
 N_z^{(0)} &= \int_C \left( C_1^{(2)} N_z y^2 dy \right) \\
 N_y^{(1)} &= \int_C \left( C_1^{(2)} M_y y dy \right) \\
 N_y^{(0)} &= \int_C \left( C_1^{(2)} M_y y dy \right) \\
 N_{xy}^{(1)} &= \int_C \left( C_1^{(2)} M_{xy} y dy \right) \\
 N_{xy}^{(0)} &= \int_C \left( C_1^{(2)} M_{xy} y dy \right)
\end{align*}
\]

and

\[
J^{(1)} = \int_C \rho y^2 dy, \quad (i = 0, 1, 2)
\]

where \( C_1(x) \) and \( C_2(x) \) denote the \( y \)-coordinate of the leading and trailing edges of the beam respectively [Fig. 2], the first variation of \( II \) can be transformed into the following form.

\[
\begin{align*}
 \frac{\delta II}{\delta f_{0,1}} &= - \int_0^1 \left[ \left\{ N_z^{(1)} \right\}' + \alpha \gamma (J^{(0)} u_0 + J^{(1)} u_1) \right] \delta u_0 \\
 &\quad + \left\{ N_z^{(0)} \right\}' - N_{xy}^{(0)} + \alpha \gamma (J^{(0)} u_0 + J^{(1)} u_1) \delta u_1 \\
 &\quad + \left\{ N_x^{(1)} \right\}' + \omega \gamma (J^{(1)} v_0 + J^{(0)} v_1) \right] \delta v_0 \\
 &\quad + \left\{ N_x^{(0)} \right\}' - N_y^{(0)} + \omega \gamma (J^{(1)} v_0 + J^{(0)} v_1) \delta v_1 \\
 &\quad + \left\{ M_z^{(0)} \right\}' - \omega \gamma (J^{(0)} w_0 + J^{(1)} w_1) \right] \delta w_0 \\
 &\quad + \left\{ M_z^{(1)} \right\}' - 2 \omega \gamma (J^{(1)} w_0 + J^{(0)} w_1) \right] \delta w_1 \\
 &\quad + \left\{ M_y^{(1)} \right\}' + \left\{ f_N^{(0)} N_z^{(0)} \right\}' - \left\{ f_N^{(0)} N_z^{(1)} \right\}' + \left\{ f_N^{(1)} N_z^{(0)} \right\}' - \left\{ f_N^{(1)} N_z^{(1)} \right\}' + \omega \gamma (J^{(0)} w_0 + J^{(1)} w_1) \right] \delta w_0 \\
 &\quad + \left\{ M_{xy}^{(0)} \right\}' + \left\{ f_N^{(0)} N_{xy}^{(0)} \right\}' + \left\{ f_N^{(1)} N_{xy}^{(1)} \right\}' + \left\{ f_N^{(1)} N_{xy}^{(0)} \right\}' + f_N^{(0)} N_{xy}^{(1)} \right] \delta w_0 \\
 &\quad + \left\{ M_{xy}^{(1)} \right\}' + \left\{ f_N^{(1)} N_{xy}^{(1)} \right\}' + \left\{ f_N^{(0)} N_{xy}^{(0)} \right\}' + f_N^{(1)} N_{xy}^{(0)} \right] \delta w_1.
\end{align*}
\]

Fig. 2

\[
\begin{align*}
 \text{(1) Stress-strain relations} \\
 (N_z/C_2) - \nu_{y} (N_y/C_2) &= u_0 + f_0' y^2 \\
 &\quad + (u_0' + f_0'' y^2 + f_0''' y^3) y + f_0'' y^2 \\
 &\quad - \nu_{y} (N_z/C_2) + (N_y/C_2) = v_0 + f_1' y^2 \\
 &\quad + (v_0' + f_1'' y^2 + f_1''' y^3) y + f_1'' y^2 \\
 &\quad - \nu_{y} (N_z/C_2) + (N_y/C_2) = w_0 + f_2' y^2 \\
 &\quad + (w_0' + f_2'' y^2 + f_2''' y^3) y + f_2'' y^2 \\
 &\quad - \nu_{y} (N_z/C_2) + (N_y/C_2) = 0
\end{align*}
\]

\[
\begin{align*}
 \text{(2) Equations of motion} \\
 (M_z/Z_2) - \nu_{y} (M_y/Z_2) &= \rho y^2 (M_z/Z_2) - (M_y/Z_2) = -2 \omega y \\
 &\quad + (f_N^{(0)} N_z^{(0)} + f_N^{(1)} N_z^{(1)})' + (f_N^{(0)} N_z^{(1)} + f_N^{(1)} N_z^{(0)})' + \omega \gamma (J^{(0)} w_0 + J^{(1)} w_1) \right] \delta w_0 \\
 &\quad + (f_N^{(1)} N_y^{(1)} + f_N^{(0)} N_y^{(0)})' + \omega \gamma (J^{(1)} v_0 + J^{(0)} v_1) \right] \delta v_1
\end{align*}
\]

\[
\begin{align*}
 \text{(3) Boundary conditions at the free edges} \\
 N_z^{(0)} = 0, \quad N_z^{(1)} = 0, \quad N_y^{(0)} = 0, \quad N_y^{(1)} = 0 \\
 [M_z^{(0)}]' + f_N^{(0)} N_z^{(0)} + f_N^{(1)} N_z^{(1)} + f_N^{(0)} N_z^{(1)} + f_N^{(1)} N_z^{(0)} = 0 \\
 M_z^{(0)} = 0 \\
 M_z^{(1)} = -2 M_{xy}^{(0)} + f_N^{(1)} N_z^{(0)} + f_N^{(0)} N_z^{(1)} + f_N^{(1)} N_y^{(1)} + f_N^{(0)} N_y^{(0)} = 0 \\
 f_N^{(1)} N_{xy}^{(1)} = 0, \quad M_z^{(1)} = 0
\end{align*}
\]

It may be added that the expression (3.4) suggests the following two types of boundary conditions for shell edges which are not traction free.

(a) For a clamped-edge:

\[
\begin{align*}
 u_0 &= 0, \quad u_1 = 0, \quad v_0 = 0, \quad v_1 = 0 \\
 w_0 &= 0, \quad w_1 = 0, \quad w_1' = 0, \quad w_1'' = 0 \\
 &\quad \text{subject to}
\end{align*}
\]
(b) For a simply supported edge with no restraint in the \( xy \) plane:

\[
\begin{align*}
\{ u, v, \psi, \theta, w, M_{x}, M_{y}, M_{xy}, N_{x}, N_{y}, N_{xy}, M_{x}, M_{y}, M_{xy}, N_{x}, N_{y}, N_{xy}, M_{x}, M_{y}, M_{xy} \}
\end{align*}
\]

To convert the system (3.15) to (3.12), we obtain from (3.5) the following system of six equations for \( N_{x} \) to \( N_{xy} \) and from (3.6) the following system of three equations for \( M_{x}, M_{xy}, \) and use is made of the defining equations (3.3).

\[
\begin{align*}
N_{x}^{(0)} &= (u_{0}' + f_{1}w_{1}') \int_{0}^{l} \frac{C_{x}}{C_{y}} \left( \frac{C_{x}}{C_{y}} - \nu_{D} \right) y^{2} dy \\
&+ (v_{0}' + f_{1}w_{1}') \int_{0}^{l} \frac{C_{y}}{C_{x}} \left( \frac{C_{x}}{C_{y}} - \nu_{D} \right) y^{2} dy \\
&+ f_{1}w_{1}' \int_{0}^{l} \frac{C_{y}}{C_{x}} \left( \frac{C_{x}}{C_{y}} - \nu_{D} \right) y^{2} dy, \\
N_{y}^{(0)} &= (u_{0}' + f_{1}w_{1}') \int_{0}^{l} \frac{C_{x}}{C_{y}} \left( \frac{C_{x}}{C_{y}} - \nu_{D} \right) y^{2} dy \\
&+ (v_{0}' + f_{1}w_{1}') \int_{0}^{l} \frac{C_{y}}{C_{x}} \left( \frac{C_{x}}{C_{y}} - \nu_{D} \right) y^{2} dy \\
&+ f_{1}w_{1}' \int_{0}^{l} \frac{C_{y}}{C_{x}} \left( \frac{C_{x}}{C_{y}} - \nu_{D} \right) y^{2} dy, \\
M_{xy}^{(0)} &= -2w_{1}' \int_{0}^{l} \frac{D_{y}}{D_{x}} \left( \frac{D_{x}}{D_{y}} - \nu_{D} \right) y^{2} dy \\
&+ f_{1}w_{1}' \int_{0}^{l} \frac{D_{y}}{D_{x}} \left( \frac{D_{x}}{D_{y}} - \nu_{D} \right) y^{2} dy \\
M_{xy}^{(0)} &= -2w_{1}' \int_{0}^{l} \frac{D_{y}}{D_{x}} \left( \frac{D_{x}}{D_{y}} - \nu_{D} \right) y^{2} dy,
\end{align*}
\]

where \( \int \) dy is used instead of \( \int \) dy for the sake of brevity.

Equations (3.16) and (3.17), together with (3.7) to (3.12) are a system of fifteen ordinary differential equations for fifteen variables.

A simple system of six equations for the six displacement functions \( u_{0}, v_{0}, w_{0} \) is obtained by introducing (3.16) and (3.17) in (3.7) to (3.12).

### 4. Torsional vibrations of a pretwisted beam

In the following sections, only torsional vibrations of a pretwisted rectangular beam with constant chord \( c \) and constant rectangular cross section, shall be dealt with [Fig. 3]. For this case we may assume that \( u, v, \) and \( w \) can be expressed as

\[
\begin{align*}
u = u_{0} + v_{1}y, \quad w = w_{0}y
\end{align*}
\]

and that the initial deflection of the shell is given by

\[
\begin{align*}
z = y f(x)
\end{align*}
\]

For the sake of simplicity, the material of the beam is assumed to be isotropic. Then, the same procedure as in the preceding section provides the following stress-strain relations, equilibrium equations and boundary conditions.

(1) Stress-strain relations

\[
\begin{align*}
N_{x} &= \frac{Eh}{1 - \nu^{2}} (\epsilon_{x} + \nu \epsilon_{y}) \\
N_{y} &= \frac{Eh}{1 - \nu^{2}} (\nu \epsilon_{x} + \epsilon_{y}) \\
M_{x} &= \frac{Eh}{12(1 - \nu^{2})} (\epsilon_{x}, \epsilon_{y}) \\
M_{xy} &= \frac{Gh}{12(1 - \nu^{2})} (\epsilon_{x}, \epsilon_{y})
\end{align*}
\]

where

\[
\begin{align*}
\epsilon_{x} &= u_{0}' + f_{1}w_{1}'y, \\
\epsilon_{y} &= v_{1} + f_{1}w_{1}, \\
\kappa_{x} &= -w_{1}'y, \\
\kappa_{y} &= 0, \\
\gamma_{xy} &= (v_{1}' + f_{1}w_{1}'y) + f_{1}w_{1}'y
\end{align*}
\]

(2) Equations of motion

\[
\begin{align*}
\left[ N_{x}^{(0)} \right]' + \rho \omega^{2} J_{x}^{(0)} u_{0} = 0 \\
\left[ M_{x}^{(0)} \right]' - \frac{2}{4} [ M_{x}^{(0)} ]' - \frac{f}{4} [ N_{x}^{(0)} ]' + \omega \omega \omega J_{x}^{(0)} u_{0} = 0
\end{align*}
\]

(3) Boundary conditions

For a clamped edge:

\[
\begin{align*}
u_{0} = 0, \quad v_{0} = 0, \quad w_{0}' = 0
\end{align*}
\]

For a free edge:

\[
\begin{align*}
N_{x}^{(0)} = 0, \quad M_{x}^{(0)} = 0
\end{align*}
\]
For a simply supported edge with no restraint in \( y \) plane:
\[
N_x^{(0)} = 0, \quad N_x^{(1)} = 0, \quad w_1 = 0, \quad M_x^{(0)} = 0 \quad (4.10)
\]

We shall assume further that the longitudinal inertial terms containing \( \eta \) in (4.5) and (4.6) can be neglected. Then, by restricting our subsequent discussions to cases in which at least one of the two edges, \( x=0 \) or \( x=l \), is prescribed to be simply supported or kept free, the equations of motion can be simplified into:
\[
N_x^{(0)} = 0 \quad (4.11)
\]
\[
[N_{xy}^{(0)}]'-N_y^{(0)} = 0 \quad (4.12)
\]
\[
[M_x^{(0)}]'-2[M_{xy}^{(0)}]+[f'N_x^{(0)}]' + \omega^2 J_{xy} w_1 = 0 \quad (4.13)
\]

By the use of (4.3) and (4.4), the above three equations are written down in terms of \( u_0 \), \( v_1 \) and \( w_1 \) as follows.
\[
v_1 + f w_1 = -\left( \frac{1}{\mu} \right) \left[ \theta u_0 + \left( \frac{c_1}{12l} \right) \partial^2 \theta \cdot \partial w_1 \right] \quad (4.14)
\]
\[
\left[ \theta^2 - 24(1+\nu) \left( \frac{l}{c} \right)^2 \right] \left[ \theta u_0 + \left( \frac{c_1}{12l} \right) \partial^2 \theta \cdot \partial w_1 \right] = 0 \quad (4.15)
\]
\[
\theta w_1 - \theta \left[ 24(1-\nu) \left( \frac{l}{c} \right)^2 \right] \partial w_1 = 0 \quad (4.16)
\]

where \( x = \xi \), \( \theta = d \partial / d \xi \) and
\[
\lambda = \frac{12(1-\nu)\rho h}{E h^2} \quad (4.17)
\]

Let us restrict our problem further and assume that the beam is twisted linearly by writing
\[
f(x) = \beta x = \beta \xi \]

where \( \beta \) is a constant. Then (4.14), (4.15) and (4.16) are reduced to
\[
v_1 + \beta \xi w_1 = -\left( \frac{1}{\mu} \right) \theta \left( u_0 + \frac{c_1}{12l} \beta w_1 \right) \quad (4.18)
\]
\[
[\theta^2 - 24(1+\nu) \left( \frac{l}{c} \right)^2 \theta \left( u_0 + \frac{c_1}{12l} \beta w_1 \right)] = 0 \quad (4.19)
\]
\[
\theta w_1 - 2 \mu \theta^2 w_1 - \lambda w_1 = 0 \quad (4.20)
\]

where
\[
\mu = 12(1-\nu) \left( \frac{l}{c} \right)^2 + 2 \left( \frac{c_1 \beta}{h} \right)^2 \quad (4.21)
\]

5. Torsional vibration of a pretwisted beam simply supported at the two edges \( x=0 \) and \( x=l \)

For the present case the boundary conditions for the simply supported edges can be written pown as,
\[
at \xi = 0 \quad \text{and} \quad \xi = 1 \quad (5.1)
\]
\[
\theta(v_1 + \beta \xi w_1) = 0, \quad w_1 = 0, \quad \theta w_1 = 0 \quad (5.2)
\]

The natural frequencies of vibration are
\[
\omega_n^2 = \frac{E h^2}{12(1-\nu)\rho h} \left( \frac{\pi}{l} \right)^2 \left( \frac{\pi}{l} \right)^2 \quad (n=1, 2, 3, \ldots) \quad (5.3)
\]

In equation (5.3), the first term in bracket represents the effect of cross sectional warping restraint (or differential bending), the second term represents the effect of the usual torsional stiffness and the third term represents the effect of pretwist.

It may be seen that the pretwist has considerable effect on the frequencies of the beam as soon as \( c_1 \beta / h \) is of order of magnitude unity.

6. Torsional vibration of a pretwisted cantilever beam

Next, let us try to solve (4.18), (4.19) and (4.20) for a pretwisted cantilever beam. The corresponding boundary conditions are:
\[
at \xi = 0 \quad \text{and} \quad \xi = 1 \quad (6.1)
\]
\[
w_0 = 0, \quad \theta w_1 = 0 \quad (6.2)
\]
\[
\theta w_1 = 0, \quad \theta w_1 = 0 \quad (6.3)
\]
\[
\theta w_1 = 0, \quad \theta w_1 = 0 \quad (6.4)
\]

Now, by putting
\[
w_1 = e^{\xi} \quad (6.5)
\]

and substituting (6.5) into (4.20), we obtain an algebraic equation which determines the values of \( \alpha \), namely,
\[
\alpha^2 - 2\mu \alpha^2 - \lambda = 0 \quad (6.6)
\]

The four roots of (6.6) are \( p_1, -p_1, iq \) and \( -iq \), where
\[
p^2 = \mu + \sqrt{\mu^2 + \lambda}, \quad q^2 = -\mu + \sqrt{\mu^2 + \lambda} \quad (6.7)
\]

and \( \lambda \) is given by (4.17) and \( \mu \) is given by (4.21). The general solution for \( w_1 \) is
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where \( A', B', C' \) and \( D' \) are arbitrary constants.

The use of the conditions (6.2), the expression (6.8) is transformed into,

\[
w_1 = A'( \cosh p \xi - \cos q \xi ) + B'( q \sinh p \xi - p \sin q \xi )
\]

where \( A \) and \( B \) are newly introduced arbitrary constants.

The conditions (6.4), combined with (6.9), then give:

\[
\begin{align*}
pq(q \sinh p - p \sin q)A & + pq(q^2 \cosh p + p^2 \cos q)B = 0 \\
(p^2 \cosh p + q^2 \cos q)A & + pq(p \sinh p + q \sin q)B = 0
\end{align*}
\]

Consequently, the frequencies are given by the roots of the following determinantal equation,

\[
q \sinh p - p \sin q = 0
\]

where \( q = 0 \), \( q = 1 \), ..., 10 and tabulated below.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>5.516</td>
<td>4.607</td>
<td>3.696</td>
<td>2.812</td>
<td>10.96</td>
<td>13.11</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>22.04</td>
<td>23.46</td>
<td>27.19</td>
<td>38.33</td>
<td>77.69</td>
<td>43.36</td>
</tr>
</tbody>
</table>

The effect of pretwist on the torsional frequencies of the beam is seen by means of the equation (3.21), which indicates that pretwist increases the torsional rigidity given by Saint-Venant torsion theory by a factor of \( 1 + \left[ (1/30)(1-\nu) \right] (c^2/\beta h)^2 \).

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>15.38</td>
<td>17.46</td>
<td>19.64</td>
<td>21.83</td>
<td>23.61</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>42.59</td>
<td>56.06</td>
<td>62.30</td>
<td>68.60</td>
<td>74.96</td>
</tr>
</tbody>
</table>

Bibliography


Appendix

It may be of interest to observe that the differential equations and boundary conditions governing the torsional vibration of a pretwisted cantilever beam as treated in the section 6 of the present paper, have the same structure as those governing the buckling of an uniformly compressed rectangular plate simply supported along two opposite sides perpendicular to the direction of compression and having one of the other two sides built in and another side kept free. We refer Timoshenko's book "Theory of Elastic-Stability" McGraw-Hill, 1936, page 341.

Following Timoshenko's notations, the lateral deflection of the plate is given by

\[
w = f(y) \sin \frac{m \pi x}{a} \quad (A-1)
\]

where \( f(y) \) is determined by the differential equation,

\[
\partial f - 2\left( m \pi b/a \right)^2 \partial f
\]

\[
. + (m \pi b/a)^2 ([m b/a] - k) f = 0 \quad (A-2)
\]

under the boundary conditions:

at \( \eta = 0 \): \( f = 0 \), \( \partial f / \partial \eta = 0 \) \quad (A-3)

at \( \eta = 1 \): \( \partial^2 f - 2(1 - \nu)(m \pi b/a)^2 \partial^2 f = 0 \)

\[
\partial^2 f - \nu(m \pi b/a)^2 f = 0 \quad (A-4)
\]

where \( y = b \eta \) and \( \partial^2 \equiv \partial / \partial \eta \). \( a \) and \( b \) are the length and width of the rectangular plate respectively. \( k \) is defined by \( N_x = k \pi^2 D/b^4 \) and \( D = Eh^3/12(1-\nu^2) \), where \( N_x \) is the buckling compressive force per unit length of the loaded side.

It is easily observed that by comparing (4.20), (6.2) and (6.4) with (A-2), (A-3) and (A-4) and putting \( \nu \) in the latter equal to zero, the following corresponding relations hold between them.

<table>
<thead>
<tr>
<th>Vibration</th>
<th>Buckling</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>( (m \pi b/a)^2 )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>( -(m \pi b/a)^2 ([m b/a]^2 - k) )</td>
</tr>
</tbody>
</table>