APPROXIMATIONS TO THE DISTRIBUTION OF A COMBINATION OF THE WILCOXON AND MOOD STATISTICS: A NUMERICAL COMPARISON

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ABSTRACT

On testing hypotheses in two-sample problems, the Lepage-type statistic is often used for testing the location and scale parameters. Various Lepage-type statistics have been proposed and discussed by many authors over the course of many years. One of the most famous and powerful Lepage-type statistics is a combination of the Wilcoxon and Mood statistics, namely T . Deriving the exact critical value of the T statistic is difficult when the sample sizes are increased. In that situation, an approximation to the distribution function of a test statistic is extremely important in statistics. The gamma approximation and the saddlepoint approximations are used to evaluate in upper tail probability for the T statistic under finite sample sizes. The accuracy of various approximations to the exact probability of the T statistic is investigated.

1. Introduction

Testing hypotheses is one of the most important problems in performing nonparametric statistics. On testing hypotheses in two-sample problems, we have to test the location and scale parameters at the same time in many cases. If the scale parameters change, the test statistic for the location parameter is not useful. Similarly, if the location parameters change, the test statistic for the scale parameter is not useful. To resolve the dilemma, Lepage (1971) developed the test statistic that combined the Wilcoxon statistic (Gibbons and Chakraborti, 2003) and the Ansari-Bradley statistic (1960). After the Lepage statistic was proposed, various Lepage-type statistics have been proposed and discussed over the course of many years, e.g. Büning and Thaewald (2000), Neuhäuser (2000), Büning (2002), Murakami (2007). One of the most famous and powerful Lepage-type statistics is a combination of the Wilcoxon and Mood (1954) statistics, namely T , proposed by Pettitt (1976). To calculate the exact critical value of the test statistic is an important problem in testing hypotheses. However, it is difficult to obtain the exact critical value when the sample sizes are increased. Under these circumstances, we must estimate the exact critical value with an approximation method. Hence, considering approximations for evaluating the density or distribution function of the test statistic remains one of the most important topics in statistics. For the approximation presented in this paper, we used a gamma approximation and a saddlepoint approximation proposed by Daniels (1954, 1987) and specifically of the type developed by Lugannani and Rice (1980). The saddlepoint approximations have the advantage in the tails of the distribution. Saddlepoint approximations have been used with great success by many authors, and excellent discussions of their applications to a range

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of distributional problems are provided by Reid (1988), Jensen (1995), Goutis and Casella (1999), Huzurbazar (1999), Kolassa (2006) and Butler (2007). The saddlepoint approximation can be obtained for any statistic that admits a cumulant generating function (CGF). When the CGF is unknown or difficult to derive, Easton and Ronchetti (1986) and Wang (1992) proposed obtaining approximations to the CGF by using expansion.

For conducting a distribution-free test, Giles (2001) and Chen and Giles (2008) treated the saddlepoint approximations to the limiting distribution of the Anderson-Darling statistic (1952, 1954) and found that the saddlepoint approximations were better than the approximation of Sinclair and Spurr (1998). In addition, Murakami (2009) indicated that the saddlepoint approximation was better than the approximation of Sinclair, Spurr and Ahmad (1990) to the limiting distribution of the modified Anderson-Darling statistic (1990).

In nonparametric statistics, researchers are very interested in considering approximations under finite sample sizes. Froda and van Eeden (2000) proposed a uniform saddlepoint expansion to the null distribution of the Wilcoxon-Mann-Whitney test. Additionally, Bean, Froda and van Eeden (2004) compared a saddlepoint approximation of the Wilcoxon-Mann-Whitney test with that of Edgeworth, and determined normal and uniform approximations under finite sample sizes. Recently, Murakami and Kamakura (2009) considered a saddlepoint approximation to the distribution of the Jonckheere-Terpstra test (Gibbons and Chakraborti, 2003) with a continuity correction. In addition, Murakami (2010) investigated the accuracy of a saddlepoint approximation with a continuity correction to the distribution of a test statistic for a competing risks model. Most of the previous works were focused on the statistics for location or scale changes. Once again, we often require testing the location and scale parameters at the same time. In this paper, we focus on a combination of the Wilcoxon and Mood statistics. In Section 2, we introduce the T statistic and list some exact critical values of the T statistic for small sample sizes. In addition, we give the explicit first four cumulants of the T statistic. Note that we might not evaluate the exact critical value of the T statistic when the sample sizes are increased. In that case, by using the cumulants, we suggest the gamma approximation. Since it is also difficult to derive the CGF of the T statistic, we consider the approximation to the CGF of the T statistic. In Section 3, we report on the saddlepoint approximations to the distribution of the T statistic. In Section 4, we compare the accuracy of the gamma approximation, normal-based and nonnormal-based saddlepoint approximations to the distribution of the T statistic under finite sample sizes. Finally, we conclude this paper in Section 5.

2. A combination of the Wilcoxon and Mood statistics

In this section, we introduce a modification of the Lepage statistic. Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$ be two random samples of size n and m independent observations, each of which had a continuous distribution described as F_1 and F_2 , respectively. Pettitt (1976) first described the use of the Mood statistic in a Lepage-type test as follows:

$$T = \left(\frac{W - E(W)}{\sqrt{\text{var}(W)}} \right)^2 + \left(\frac{M - E(M)}{\sqrt{\text{var}(M)}} \right)^2,$$

where

$$W = \sum_{i=1}^N iV_i \quad \text{and} \quad M = \sum_{i=1}^N \left(i - \frac{N+1}{2} \right)^2 V_i.$$

Let $V_i = 1$ if the i th smallest of the $N = n + m$ observations is from X and $V_i = 0$ otherwise. $E(\cdot)$ and $\text{var}(\cdot)$ denote the mean and variance under the null hypothesis. The limiting distribution of the T statistic is known to be a chi-square distribution with two degrees of freedom (Pettitt, 1976). Herein, we give the exact first four cumulants of the T statistic as follows:

$$\begin{aligned}\kappa_1 &= 2, \\ \kappa_2 &= \frac{4(A_1 + A_2)}{35nm(N-2)^2(N+1)(N+2)} = \frac{4N^4}{(N-2)^2(N+1)(N+2)} + O(N^{-1}), \\ \kappa_3 &= \frac{16(A_3 + A_4 + A_5 + A_6)}{35035n^2m^2(N-2)^3(N+1)^2(N+2)^2} = \frac{16N^7}{(N-2)^3(N+1)^2(N+2)^2} + O(N^{-1}), \\ \kappa_4 &= \frac{96(A_7 + A_8 + A_9 + A_{10} + A_{11} + A_{12})}{2977975n^3m^3(N-2)^4(N+1)^3(N+2)^3} = \frac{96N^{10}}{(N-2)^4(N+1)^3(N+2)^3} + O(N^{-1}).\end{aligned}$$

See Appendix for A_1 to A_{12} . Additionally, we list the exact critical value of the T statistic in Table 1.

 Table 1: Exact critical value of the T statistic

n	m	$Pr(T \geq t)$	t	n	m	$Pr(T \geq t)$	t
7	7	0.0099	7.25000	10	5	0.0100	7.69484
		0.0245	6.40408			0.0250	6.43484
		0.0490	5.56020			0.0496	5.34045
		0.0997	4.46531			0.0999	4.35955
10	10	0.0100	7.95671	30	5	0.0100	8.89485
		0.0250	6.72156			0.0250	6.89588
		0.0499	5.69229			0.0500	5.48300
		0.0996	4.53541			0.0999	4.27949
13	13	0.0100	8.29581	50	5	0.0100	9.13507
		0.0250	6.89706			0.0250	7.01403
		0.0500	5.76003			0.0500	5.51212
		0.1000	4.55274			0.1000	4.25339

Since researchers find it necessary to know the exact critical values of the T statistic, we offer a program to calculate the exact critical values of the T statistic for small sample sizes. However, for example, we may not evaluate the exact critical value of the T statistic for the case of $n = m = 20$. This is because we have to consider ${}_{40}C_{20} = 137,846,528,820$ combinations to calculate the exact critical value. Therefore it is necessary to approximate the distribution of the T statistic. Then, we first suggest a gamma approximation to the distribution of the T statistic as follows:

$$\Gamma(\alpha, \beta) \quad \text{where} \quad \alpha = \frac{4}{\kappa_2} \quad \text{and} \quad \beta = \frac{\kappa_2}{2}.$$

$\Gamma(\alpha, \beta)$ converges to a chi-square distribution with two degrees of freedom when $N \rightarrow \infty$ (Lepage *et al.*, 1986).

It is difficult to derive the cumulant generating function (CGF) of the T statistic; we need to approximate the CGF of the T statistic. Herein, we introduce the approximation

of Easton and Ronchetti (1986) to the CGF of the statistic as follows:

$$K(x) \approx K_1(x) = \sum_{i=1}^4 \frac{\kappa_i x^i}{i!}.$$

Kakizawa and Taniguchi (1994) defined a correction of the $K_1(x)$ as follows:

$$K(x) \approx K_2(x) = \kappa_1 x + \frac{(1 + \kappa_2)x^2}{2!} + \frac{\kappa_3 x^3}{3!} + \frac{\kappa_4 x^4}{4!}.$$

In addition, Easton and Ronchetti (1986) considered another approximation as

$$K(x) \approx K_3(x) = \kappa_1 x + \frac{\kappa_2}{2} x^2 + \log \left(1 + \frac{\kappa_3}{6} x^3 + \frac{3\kappa_4}{72} x^4 + \frac{\kappa_3^2}{72} x^6 \right).$$

Additionally, Wang (1992) proposed a simple modification of $K_1(x)$ as follows:

$$K(x) \approx K_4(x; p) = \kappa_1 x + \frac{\kappa_2}{2!} x^2 + \left(\frac{\kappa_3}{3!} x^3 + \frac{\kappa_4}{4!} x^4 \right) \eta_p(x),$$

where

$$\eta_p(x) = \exp \left\{ -\frac{\kappa_2 p^2 x^2}{2} \right\}.$$

Wang defined p as follows:

$$p = \max \left(\frac{1}{2}, \inf \left\{ q \mid K_4^{(2)}(x; q) \geq 0 \text{ for all } T \right\} \right).$$

By the first four cumulants, we may easily estimate the i th cumulant of the T statistic for $i \geq 5$ as follows:

$$\sum_{i=5}^{\infty} \frac{2^{i-1} 2(i-1)! N^{3(i-1)+1} x^i}{(N-2)^i (N+1)^{i-1} (N+2)^{i-1} i!} + O(N^{-1}). \quad (1)$$

Then we add (1) to $K_1(x)$ and named $K_5(x)$ as follows:

$$\begin{aligned} K(x) &\approx K_1(x) + \sum_{i=5}^{\infty} \frac{2^{i-1} 2(i-1)! N^{3(i-1)+1} x^i}{(N-2)^i (N+1)^{i-1} (N+2)^{i-1} i!} + O(N^{-1}) \\ &\approx K_1(x) + \sum_{i=5}^{\infty} \frac{2^i N^{3i-2} x^i}{(N-2)^i (N+1)^{i-1} (N+2)^{i-1} i} \\ &= K_1(x) - \frac{1}{3(N-2)^4 N^2 (N+1)^3 (N+2)^3} \left[2N^3 x \left\{ 3(N^3 + N^2 - 4N - 4)^3 \right. \right. \\ &\quad \left. \left. + 3N^3 (N^3 + N^2 - 4N - 4)^2 x + 4(N-2)N^6 (N+1)(N+2)x^2 + 6N^9 x^3 \right\} \right. \\ &\quad \left. + 3(N^3 + N^2 - 4N - 4)^4 \log \left(1 - \frac{2N^3 x}{N^3 + N^2 - 4N - 4} \right) \right] \\ &= K_5(x). \end{aligned}$$

We may consider replacing the $K_1(x)$ to $K_2(x)$ in $K_5(x)$ as follows:

$$K(x) \approx K_6(x) = K_2(x) - \frac{1}{3(N-2)^4 N^2 (N+1)^3 (N+2)^3} \left[2N^3 x \left\{ 3(N^3 + N^2 - 4N - 4)^3 \right. \right. \\ \left. \left. + 3N^3 (N^3 + N^2 - 4N - 4)^2 x + 4(N-2)N^6 (N+1)(N+2)x^2 + 6N^9 x^3 \right\} \right. \\ \left. + 3(N^3 + N^2 - 4N - 4)^4 \log \left(1 - \frac{2N^3 x}{N^3 + N^2 - 4N - 4} \right) \right].$$

3. Saddlepoint approximation

In this section, we consider a saddlepoint approximation (Daniels; 1954, 1987) to the distribution of the T statistic. To obtain the saddlepoint approximation, we evaluate the first four derivatives of the CGF of the T statistic. To determine the saddlepoint approximation to $\Pr(T \geq v)$, we solve the saddlepoint equation, $K^{(1)}(x) = v$, and use the unique solution ($x = \hat{x}$) to calculate

$$\hat{w} = \sqrt{2(\hat{x}v - K(\hat{x}))} \text{sgn}(\hat{x}) \quad \text{and} \quad \hat{u} = \hat{x} \sqrt{K^{(2)}(\hat{x})},$$

where $\text{sgn}(\hat{x}) = \pm 1, 0$ if $\hat{x}$ is positive, negative or zero and $K^{(j)}(\cdot)$ denotes the j th derivative of $K(\cdot)$. The saddlepoint approximations to the cumulative distribution function of the T statistic by using the normal-based Lugannani and Rice (L-R) formula for the tail probability is given by

$$\Pr_1(T \geq v) = 1 - \Phi(\hat{w}) + \phi(\hat{w}) \left\{ \frac{1}{\hat{u}} - \frac{1}{\hat{w}} \right\}$$

and

$$\Pr_2(T \geq v) = 1 - \Phi(\hat{w}) + \phi(\hat{w})S_1,$$

where $\phi(\cdot)$ is the standard Normal density function and $\Phi(\cdot)$ is the corresponding cumulative distribution function and

$$S_1 = \frac{1}{\hat{u}} - \frac{1}{\hat{w}} + \frac{1}{\hat{u}} \left(\frac{\zeta_4}{8} - \frac{5\zeta_3^2}{24} \right) - \frac{\zeta_3}{2\hat{u}^2} - \frac{1}{\hat{u}^3} + \frac{1}{\hat{w}^3} \quad \text{and} \quad \zeta_r = \frac{K^{(r)}(\hat{x})}{\{K^{(2)}(\hat{x})\}^{r/2}}.$$

Wood *et al.* (1993) generalized the L-R formula to a nonnormal-based saddlepoint approximation. Suppose that $G(w)$ is the CGF of the base distribution. For each ξ , let w_ξ be the unique solution to the saddlepoint equation

$$G^{(1)}(w_\xi) = \xi,$$

where $G^{(j)}(\cdot)$ denotes the j th derivative of $G(\cdot)$. Then we find $\hat{\xi}$ to satisfy

$$G(w_\xi) - \xi w_\xi = K^{(1)}(\hat{x}) - v\hat{x},$$

where $\hat{x}$ is obtained from $K^{(1)}(\hat{x}) = v$.

Suppose now that Γ and γ are the cumulative distribution function and density of the distribution whose CGF is $G(w)$. Then we have the generalized L-R formula as follows:

$$\Pr_3(T \geq v) = 1 - \Gamma(\hat{\xi}) + \gamma(\hat{\xi}) \left\{ \frac{1}{\hat{u}_{\hat{\xi}}} - \frac{1}{w_{\hat{\xi}}} \right\}$$

and

$$\Pr_4(T \geq v) = 1 - \Gamma(\hat{\xi}) + \frac{\gamma(\hat{\xi})}{\delta_{\hat{\xi}}} \left\{ \frac{1}{\hat{u}_{\hat{\xi}}} - \frac{1}{w_{\hat{\xi}}} + S_2 \right\},$$

where

$$\hat{u}_{\hat{\xi}} = \hat{u} \sqrt{G^{(2)}(w_{\hat{\xi}})}, \quad \tau_r = \frac{G^{(r)}(\hat{\xi})}{\{G^{(2)}(\hat{\xi})\}^{r/2}}, \quad \delta_{\hat{\xi}} = 1 + \frac{1}{8}\tau_4 - \frac{5}{24}\tau_3^2,$$

and

$$S_2 = \frac{1}{8} * \left(\frac{\zeta_4}{\hat{u}_{\hat{\xi}}} - \frac{\tau_4}{w_{\hat{\xi}}} \right) - \frac{5}{24} * \left(\frac{\zeta_3^2}{\hat{u}_{\hat{\xi}}} - \frac{\tau_3^2}{w_{\hat{\xi}}} \right) - \frac{1}{2G^{(2)}(\hat{\xi})^{\frac{1}{2}}} * \left(\frac{\zeta_3}{\hat{u}_{\hat{\xi}}} - \frac{\tau_3}{w_{\hat{\xi}}} \right) \\ - \frac{1}{G^{(2)}(\hat{\xi})} * \left(\frac{1}{\hat{u}_{\hat{\xi}}^3} - \frac{1}{w_{\hat{\xi}}^3} \right).$$

4. Numerical comparison

In this section, we report on the accuracy of various approximations. We use the gamma, normal-based saddlepoint and nonnormal-based saddlepoint approximations. In the tables, the critical values were estimated by a simulation study which was 10,000,000 times repetition. We list the upper tail probability of the proposed gamma approximation, namely A_G , and the limiting distribution, namely A_C for each critical value of the T statistic. In addition, SA_{NL} and SA_{NH} denote the normal-based saddlepoint approximations corresponding to $\Pr_1(\cdot)$ and $\Pr_2(\cdot)$, respectively. For a nonnormal-based saddlepoint approximation, specifically, we choose the chi-square distribution with one and two degrees of freedom in this paper because the limiting distribution is the chi-square distribution. The CGF of the chi-square distribution is

$$G(w_{\hat{\xi}}) = -\frac{\nu}{2} \log(1 - 2w_{\hat{\xi}}),$$

where ν denotes the degrees of freedom. In the table of numerical results, $SA_{CL\nu}$ and $SA_{CH\nu}$ denote the chi-square-based saddlepoint approximations for $\Pr_3(\cdot)$ and $\Pr_4(\cdot)$, respectively.

Using an extra term approximation doesn't necessarily guarantee an improvement for any particular fixed finite sample sizes. Sometimes, it is possible for the corrections from the extra terms to worsen the approximation. Then, in this paper, we compared the numerical results of the saddlepoint approximations based on $K_1(x)$, $K_2(x)$, $K_5(x)$ and $K_6(x)$ for equal and unequal sample sizes. Since the saddlepoint approximation based on $K_3(x)$ and $K_4(x; p)$ appeared to give a poorer approximation in the examples considered than that based on $K_1(x)$, we will not include this approach in the numerical examples presented in this section.

The results of Table 2 show that the saddlepoint approximation SA_{NH} based on $K_6(x)$ is more accurate than the other saddlepoint approximations in upper tail probability. In addition, the A_G approximation is also efficient to approximate the upper tail probability. The A_G and SA_{NH} based on $K_5(x)$ approximations are more satisfactory than the A_C approximation for equal sample sizes.

Table 2: Tail probability for $n = m = 30$

		4.583	5.897	7.189	8.844
	Simulation	0.100	0.050	0.025	0.010
	A_C	0.102	0.052	0.027	0.012
	A_G	0.098	0.049	0.025	0.010
$K_1(x)$	SA_{NL}	0.115	0.056	0.026	0.009
	SA_{NH}	0.117	0.057	0.026	0.009
	SA_{CL1}	0.109	0.052	0.024	0.008
	SA_{CH1}	0.120	0.055	0.025	0.008
	SA_{CL2}	0.113	0.054	0.025	0.008
	SA_{CH2}	0.102	0.050	0.023	0.008
$K_2(x)$	SA_{NL}	0.129	0.064	0.031	0.011
	SA_{NH}	0.130	0.065	0.031	0.011
	SA_{CL1}	0.122	0.060	0.028	0.010
	SA_{CH1}	0.137	0.065	0.030	0.011
	SA_{CL2}	0.127	0.063	0.030	0.010
	SA_{CH2}	0.122	0.061	0.029	0.010
$K_5(x)$	SA_{NL}	0.092	0.047	0.024	0.010
	SA_{NH}	0.087	0.045	0.023	0.010
	SA_{CL1}	0.090	0.045	0.023	0.010
	SA_{CH1}	0.090	0.047	0.025	0.011
	SA_{CL2}	0.092	0.047	0.024	0.010
	SA_{CH2}	0.088	0.045	0.023	0.010
$K_6(x)$	SA_{NL}	0.106	0.054	0.028	0.012
	SA_{NH}	0.101	0.051	0.026	0.011
	SA_{CL1}	0.103	0.052	0.026	0.011
	SA_{CH1}	0.093	0.052	0.028	0.012
	SA_{CL2}	0.106	0.053	0.027	0.012
	SA_{CH2}	0.103	0.052	0.027	0.011

The results of Table 3 reveal that the SA_{CH2} based on $K_1(x)$ and SA_{NH} based on $K_6(x)$ are more accurate than the other saddlepoint approximations. In addition, both of the saddlepoint approximations are more accurate and satisfactory than the A_C approximation. However, the A_G approximation was more suitable than the saddlepoint approximations for upper tail probability for unequal sample sizes. Therefore, the SA_{NH} based on $K_6(x)$ and A_G were efficient for approximating the distribution of the T statistic.

5. Conclusion

In this paper, we consider a gamma approximation and various saddlepoint approximations to the distribution of the Lepage-type test which is a combination of the Wilcoxon and Mood statistic, named T . The first four cumulants of the T statistic are derived. By the numerical results, the precision of the gamma approximation and SA_{NH} based on $K_6(x)$ are shown to be superior to that of other normal-based and nonnormal-based saddlepoint approximations to the distribution of the Lepage-type test. In any case, the A_G approximation is more accurate than the A_C approximation. In addition, the approximation A_G was more efficient than the SA_{NH} based $K_6(x)$ approximation for equal and unequal sample

Table 3: Tail probability for $n = 40, m = 20$

		4.571	5.877	7.161	8.829
	Simulation	0.100	0.050	0.025	0.010
	A_C	0.102	0.053	0.028	0.012
	A_G	0.098	0.050	0.025	0.011
$K_1(x)$	SA_{NL}	0.116	0.057	0.027	0.009
	SA_{NH}	0.117	0.058	0.027	0.009
	SA_{CL1}	0.110	0.053	0.024	0.008
	SA_{CH1}	0.121	0.056	0.026	0.009
	SA_{CL2}	0.114	0.055	0.026	0.009
	SA_{CH2}	0.103	0.051	0.024	0.008
$K_2(x)$	SA_{NL}	0.130	0.066	0.031	0.011
	SA_{NH}	0.130	0.066	0.032	0.011
	SA_{CL1}	0.122	0.061	0.029	0.010
	SA_{CH1}	0.139	0.067	0.031	0.011
	SA_{CL2}	0.127	0.064	0.030	0.011
	SA_{CH2}	0.122	0.061	0.029	0.010
$K_5(x)$	SA_{NL}	0.093	0.048	0.025	0.011
	SA_{NH}	0.088	0.046	0.024	0.010
	SA_{CL1}	0.091	0.046	0.024	0.010
	SA_{CH1}	0.092	0.048	0.025	0.011
	SA_{CL2}	0.093	0.047	0.025	0.011
	SA_{CH2}	0.089	0.046	0.024	0.010
$K_6(x)$	SA_{NL}	0.107	0.055	0.028	0.012
	SA_{NH}	0.102	0.052	0.027	0.012
	SA_{CL1}	0.104	0.053	0.027	0.012
	SA_{CH1}	0.095	0.053	0.028	0.012
	SA_{CL2}	0.107	0.053	0.028	0.012
	SA_{CH2}	0.104	0.053	0.027	0.012

sizes, especially unequal sample sizes. Therefore, the A_G statistic was suitable to estimate the critical value of the T statistic.

Appendix First four cumulants of the T statistic

First cumulant of the T statistic:

$$\kappa_1 = 2.$$

Second cumulant of the T statistic:

$$\kappa_2 = \frac{4(A_1 + A_2)}{35nm(N-2)^2(N+1)(N+2)},$$

where

$$\begin{aligned} A_1 &= nm(35N^4 - 127N^3 - 626N^2 + 1588N + 5184), \\ A_2 &= -4N(N+1)(2N^3 - 29N^2 + 37N + 251). \end{aligned}$$

Third cumulant of the T statistic:

$$\kappa_3 = \frac{16(A_3 + A_4 + A_5 + A_6)}{35035n^2m^2(N-2)^3(N+1)^2(N+2)^2},$$

where

$$\begin{aligned} A_3 &= n^3(N+m)(-35035N^7 + 498212N^6 + 2563853N^5 - 17007490N^4 \\ &\quad - 81801784N^3 + 181994024N^2 + 1340455728N + 1740082176), \\ A_4 &= n^2N(35035N^8 - 529672N^7 - 3470117N^6 + 17886274N^5 + 114223564N^4 \\ &\quad - 154233296N^3 - 1803614736N^2 - 3209149192N - 1331007552), \\ A_5 &= 4nN^2(7865N^7 + 226566N^6 - 219696N^5 - 8105445N^4 - 6940182N^3 \\ &\quad + 115789752N^2 + 367266754N + 332751888), \\ A_6 &= -6N(7523N^8 + 58206N^7 - 348346N^6 - 2060124N^5 + 3905103N^4 \\ &\quad + 35872734N^3 + 60357736N^2 + 19827864N - 10223336). \end{aligned}$$

Fourth cumulant of the T statistic:

$$\kappa_4 = \frac{96(A_7 + A_8 + A_9 + A_{10} + A_{11} + A_{12})}{2977975n^3m^3(N-2)^4(N+1)^3(N+2)^3},$$

where

$$\begin{aligned} A_7 &= n^5(2N+m)(2977975N^{10} - 119447185N^9 - 416453777N^8 \\ &\quad + 9662189097N^7 + 36504126558N^6 - 348955046364N^5 \\ &\quad - 2043389380984N^4 + 3044326120816N^3 + 48316473341600N^2 \\ &\quad + 133264055195712N + 122585067270144), \\ A_8 &= n^3N^2(2977975N^{11} - 149008145N^{10} - 1058925665N^9 + 12716457417N^8 \\ &\quad + 91920338798N^7 - 415906988748N^6 - 4510464677176N^5 \\ &\quad - 2171634664576N^4 + 89662808498144N^3 + 383917313837760N^2 \\ &\quad + 651986874923008N + 410763198283776), \\ A_9 &= -n^4N(8933925N^{11} - 373122035N^{10} - 1570597275N^9 + 30513701451N^8 \\ &\quad + 137220485794N^7 - 1080341110284N^6 - 7363705791048N^5 \\ &\quad + 6524997969752N^4 + 165622587603072N^3 + 525118794908160N^2 \\ &\quad + 632456105636864N + 205381599141888), \\ A_{10} &= -8nN^2(4512684N^{11} + 26145374N^{10} - 469694095N^9 - 2464865903N^8 \\ &\quad + 19034565943N^7 + 144838776267N^6 - 85995851346N^5 \\ &\quad - 3470780397296N^4 - 12770870514146N^3 - 18874143095350N^2 \\ &\quad - 7710625009828N + 4397374185240), \\ A_{11} &= 120N(158220N^{12} + 70674N^{11} - 15606667N^{10} - 16323560N^9 \\ &\quad + 702834509N^8 + 2370214388N^7 - 10772853205N^6 - 84809116296N^5 \\ &\quad - 199114230869N^4 - 155586112366N^3 + 51248243068N^2 \\ &\quad + 55123084552N - 24966727664), \end{aligned}$$

$$\begin{aligned}
A_{12} = & 8n^2N(1847560N^{12} + 44667177N^{11} - 164746396N^{10} - 3933207360N^9 \\
& + 1719630496N^8 + 173226771955N^7 + 470836325354N^6 \\
& - 2670141798630N^5 - 19136609062424N^4 - 45858483492450N^3 \\
& - 44546842988086N^2 - 7710625009828N + 4397374185240).
\end{aligned}$$

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