THE DISTRIBUTION OF THE HILBERT MODULAR TYPE CUSP FORMS OF WEIGHT TWO AND ITS STATISTICAL MODEL

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ABSTRACT

In this paper we give a distribution of the Hilbert modular type cusp forms over real quadratic number fields, which are closely connected with the Euler number of the Hilbert modular surface, using the trace formula proved by the author. We consider the statistical model for the increasing trend of the dimensions, and analyze their distribution through it.

1. Introduction

The trace formula of Hecke operators acting on the space of cusp forms of weight two belonging to a Hilbert modular group over a totally real algebraic number field has been given by the author in [Ishikawa(1979)]. In this paper, we will give the distribution of cusp forms of weight two of two variables over real quadratic fields $\mathbf{Q}(\sqrt{D})$ where D is a square-free natural number and $1 < D < 10000$. As is well known, a cusp form is an eigenfunction of the Laplace-Beltrami operator belonging to the possible minimal discrete spectrum. Also note that the Euler number of the Hilbert modular surface coincides with the dimension of its space plus 4. The dimension is increasing according as multiplying the discriminant. By using the statistical method, we analyze its distribution. The main term of the dimension formula is its volume part. We estimate its error term as the statistical model, and investigate its secular trend and its periodicity. To get a table, we have to calculate the special values of the Dedekind zeta functions and the ideal class numbers of the CM-fields with computer's aid. In our table, we can see that the possible minimal eigenvalue of discrete type is missing only when $D = 2, 3, 5, 13, 17$.

We used ACOS-6 computer system in Okayama University Computer center for about an hour CPU time.

2. The dimension formula

For a square-free natural number D , let k be a real quadratic field over $\mathbf{Q}$ generated by $\sqrt{D}$. We denote by $\mathfrak{g}, E, E^+$ and $\phi_i (i = 1, 2)$ the ring of integers, the group of units in $\mathfrak{g}$, the subgroup of E consisting of all totally positive elements and two distinct projections of k into its completions $\mathbf{R}$ with respect to the infinite valuations. Put $G = \{g = (g_1, g_2); g_i \in SL_2(\mathbf{R})\}$, $F = \{z = (z_1, z_2); z_i \in \mathbf{C}, \operatorname{Re}(z_i) \neq 0\}$. We consider that G acts on F by the

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Key words : Cusp form; Distribution of dimensions; Hilbert modular; Real quadratic field; Statistical model.

linear fractional transformation. We regard $\Gamma = SL_2(\mathfrak{g})$ as a subgroup of G through the projections ϕ_i .

By a cusp form of weight two belonging to Γ , we understand a function $f(z)$ on F , satisfying the following conditions:

- i) $f(z)$ is holomorphic on each connected component of F
- ii) $f(\gamma z) = j(\gamma, z)^{-1} f(z)$ for $\gamma \in \Gamma$

$$\text{where } j(\gamma, z) = \prod_{i=1,2} (c_i z_i + d_i)^{-2} \quad (\gamma_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix})$$

- iii) $f(z)$ vanishes at every parabolic point of Γ .

The linear space of all $f(z)$ is denoted by $S_2(\Gamma)$. The dimension of $S_2(\Gamma)$ is given by an eigenvalue of the identity Hecke operator.

Theorem 1. ([Ishikawa(1979), Theorem 1])

$$\dim S_2(\Gamma) = t_0 + t_1 + t_2 \quad (1)$$

$$t_0 = \frac{1}{2\pi^4} d_k^{3/2} \zeta_k(2), \quad t_1 = \sum_{n \geq 1} \frac{4a_n}{n}, \quad t_2 = -4 \quad (2)$$

Here t_0, t_1 and t_2 are the contributions from identity, elliptic elements and the Eisenstein series. The notations are as follows. d_k, ζ_k are the discriminant of k , the Dedekind zeta function of k . a_n is the number of inequivalent elliptic elements of order $2n$ in Γ .

3. The method of computations

3.1 By the functional equation of the Dedekind zeta function:

$$\zeta_k(1-s) = \pi^{1-2s} d_k^{s-(1/2)} \zeta_k(s) \Gamma(s/2)^2 / \Gamma((1-s)/2)^2, \quad (3)$$

t_0 is equal to $2\zeta_k(-1)$. The next lemma is due to [Siegel(1969)].

Lemma 1. For a real quadratic field $k = \mathbf{Q}(\sqrt{D})$ (D is square-free), we have

$$\zeta_k(-1) = \begin{cases} \frac{1}{30} \sum_{1 \leq b \leq \sqrt{D}} b \equiv 1 \pmod{2} \sigma((D-b^2)/4) & \text{if } D \equiv 1 \pmod{4} \\ \frac{1}{60} (\sigma(D) + 2 \sum_{1 \leq b \leq \sqrt{D}} \sigma(D-b^2)) & \text{if } D \equiv 2, 3 \pmod{4} \end{cases} \quad (4)$$

where $\sigma(x)$ is the sum of divisors of x .

3.2 Using [Shimizu(1963), No.26], we can rewrite t_1 as follows.

Lemma 2.

$$t_1 = \frac{2}{h_k} \sum_{s, \mathfrak{f}} \frac{h(\mathfrak{o}(s, \mathfrak{f}))}{w(\mathfrak{o}(s, \mathfrak{f}))} (w(\mathfrak{o}(s, \mathfrak{f})) - 1) \quad (5)$$

Here, $s, \mathfrak{f}$ run over integers in k , integral ideals in k such that $4-s^2$ is totally positive and $(s^2-4)\mathfrak{f}^{-2}$ is integral (taken up to $(s \pm \sqrt{s^2-4})/2 \pmod{E}$). $\mathfrak{o} = \mathfrak{o}(s, \mathfrak{f})$, $h(\mathfrak{o})$ and $w(\mathfrak{o})$ are the order in $k(\sqrt{s^2-4})$ with discriminant $(s^2-4)\mathfrak{f}^{-2}$, the class number of $\mathfrak{o}$ and the index of E in the group of units in $\mathfrak{o}$. h_k denotes the class number of k .

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s is only 0 or ± 1 except $D = 5$, when s satisfies the condition above. For $D = 5$, $s = \pm(1 \pm \sqrt{5})/2$ also satisfies it. The possible $\mathbf{f}$ other than $\mathbf{f} = \mathbf{g}$ are $\mathbf{p}, \mathbf{p}^2$ for $s = 0, D \equiv 3 \pmod{4}$, $\mathbf{p}$ for $s = 0, D \equiv 2 \pmod{4}$, and $\mathbf{q}$ for $s = \pm 1, D \equiv 0 \pmod{3}$, where $\mathbf{p}^2 = 2\mathbf{g}, \mathbf{q}^2 = 3\mathbf{g}$. Then

Lemma 3. *When $(s, \mathbf{f})$ satisfies the condition in Lemma 2, we get*

$$w(\mathbf{o}(0, \mathbf{f})) = \begin{cases} 4 & \text{if } D = 2, \mathbf{f} = \mathbf{p} \\ 2 & \text{otherwise} \end{cases}, w(\mathbf{o}(\pm 1, \mathbf{f})) = \begin{cases} 6 & \text{if } D = 3, \mathbf{f} = \mathbf{q} \\ 3 & \text{otherwise} \end{cases} \quad (6)$$

$$w(\mathbf{o}(\pm(1 \pm \sqrt{5})/2, \mathbf{g})) = 5 \quad \text{if } D = 5$$

Put $K = k(\sqrt{s^2 - 4})$. Let $\mathbf{o}_0$ be the principal order of K . Next we will treat $h(\mathbf{o})/w(\mathbf{o})$. Considering the numbers of residue classes for the conductor $\mathbf{f}$ of $\mathbf{o}$ in $\mathbf{o}_0$, we have

Lemma 4. *When $\mathbf{f}$ is the conductor of an order $\mathbf{o}$ in K , then*

$$\frac{h(\mathbf{o})}{w(\mathbf{o})} = \frac{h(\mathbf{o}_0)}{w(\mathbf{o}_0)} N(\mathbf{f}) \prod_{\mathbf{p}|\mathbf{f}} \left(1 - \left(\frac{K}{\mathbf{p}}\right) N(\mathbf{p})^{-1}\right) \quad (7)$$

where $(K/\mathbf{p})$, N stand for the Artin symbol, and for the norm. $\mathbf{p}$ runs over all prime ideals dividing $\mathbf{f}$.

For any case except $s = \pm(1 \pm \sqrt{5})/2$, K is a composite field of k and $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, so $h(\mathbf{o}_0)$ can be expressed by the class numbers of the quadratic fields. The next lemma is given by the consideration of the Dedekind zeta function of K .

Lemma 5. *For $K = \mathbf{Q}(\sqrt{d_1}, \sqrt{-d_2})$ ($d_1 \geq 2, d_2 \geq 1$), the ideal class number of K is given by*

$$h(\mathbf{o}_0) = \delta_2(1/2) h(d_1) h(-d_2) h(-d_1 d_2) \quad (8)$$

where $h(d)$ denotes the class number of $\mathbf{Q}(\sqrt{d})$. $\delta_2 = 2$ or 1 , according to $(d_1, d_2) = (2, 1), (2, 2)$ or not.

On the other hand, by the direct calculation, we get

$$h(\mathbf{o}(\pm(1 \pm \sqrt{5})/2, \mathbf{g})) = 1 \quad (9)$$

Summing up above things, we obtain

Theorem 2.

$$t_1 = a(D)h(-D) + b(D)h(-3D) + c(D) \quad (10)$$

$a(D)$, $b(D)$ and $c(D)$ are given in the following tables.

D	$D \equiv 1 \pmod{4}$	$D \equiv 2 \pmod{4}$ $D \neq 2$	$D \equiv 3 \pmod{8}$ $D \neq 3$	$D \equiv 7 \pmod{8}$	$D = 2$	$D = 3$
$2a(D)$	1	3	10	4	5	3

D	$D \equiv 1, 2 \pmod{3}$	$D \equiv 3 \pmod{9}$ $D \neq 3$	$D \equiv 6 \pmod{9}$	$D = 3$	D	$D = 5$	$D \neq 5$
$6b(D)$	4	20	12	13	$5c(D)$	8	0

4. The statistical model

We can calculate the dimension $S_2(\Gamma)$ with the aid of Theorem 2. In this section, we regard the dimension as a function of discriminant x of a real quadratic field, and consider the relation $(x, \dim S_2(x))$ statistically.

We have computed $\dim S_2(x)$ for x below 10000 using the computer, and put up the correlation diagram of $(x, \dim S_2(x))$ in Figure 1.

From the arguments in §§2–3, its dimension consists of the volume part and the elliptic part, which are referred to the special value of the Dedekind zeta function and the sum of class numbers of the imaginary quadratic fields, respectively. We estimate the volume part as

$$\begin{aligned} \frac{1}{2\pi^4} x^{3/2} \zeta_k(2) &= \frac{1}{2\pi^4} x^{3/2} \zeta_Q(2) \Pi_p(1 - (x/p)p^{-2})^{-1} \\ &\sim \frac{1}{2\pi^4} x^{3/2} \zeta_Q(2) \Pi_p(1 + p^{-2})^{-1} = \frac{1}{180} x^{3/2} \quad (x \rightarrow \infty) \end{aligned} \quad (11)$$

Fig. 1

Fig. 2

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where ζ_Q denotes the Riemann zeta function. Comparing above with the estimates of the class numbers of the imaginary quadratic fields, in the asymptotic expansion, the main term of $\dim S_2(x)$ becomes to $(1/180)x^{3/2}$. We define an error term by

$$\varepsilon(x) = \dim S_2(x) - (1/180)x^{3/2} \quad (12)$$

Now we deal with the error term $\varepsilon(x)$. We give the correlation diagram of $(\log(x), \log \varepsilon(x))$ in Figure 2. Moreover we get the interesting results that the coefficient of correlation between $(\log(x), \log \varepsilon(x))$ is 0.8801 ($1 < x < 10000$). To get its secular trend, we treat the data with the moving average method. In figure 3, we show a graph of the error variance with the data processed by the moving average method of step below 20. It is decreasing rapidly on $[5, 8]$, and slowly on $[9, 16]$. As in Figure 4, we present the correlation diagram of the data with the moving average method of step 10, whose error variance is lower than 0.1.

Fig. 3

Fig. 4

Following from Fig. 4, its secular trend increases almost straight in this interval. Therefore we consider the following statistical model for this error term.

$$\varepsilon(x) = \alpha x^\beta \quad (13)$$

By means of the regression analysis, we estimate (α, β) for a sample space of $(\log(x), \log \varepsilon(x))$ ($1 < x < 10000$). The exponent β is more important than α in our case. The

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estimates of (α, β) are given in the following table.

interval	α	β	error variance
1 ~ 9999	0.01670	1.295	0.452
1 ~ 999	0.04180	1.152	0.105
1001 ~ 1999	0.01683	1.284	0.280
2001 ~ 2999	0.00974	1.356	0.383
3001 ~ 3999	0.00762	1.385	0.439
4001 ~ 4999	0.02122	1.264	0.493
5001 ~ 5999	0.01162	1.336	0.514
6001 ~ 6999	0.01408	1.316	0.539
7001 ~ 7999	0.04205	1.195	0.581
8001 ~ 8999	0.00395	1.461	0.554
9001 ~ 9999	0.00654	1.404	0.582

The table above says that β increases slightly in this interval. β is approximately equal to 1.295, say β_0 .

Next we study the cyclical variation of $\varepsilon(x)$. Put

$$\eta(x) = \log \varepsilon(x) - \beta_0 \log(x) - \log \alpha_0 \quad (\alpha_0 = 0.01670) \quad (14)$$

We carry out the auto-correlation analysis and the spectral analysis for the data $\{\eta(x); 1 < x < 10000\}$. We show its correlogram, its power spectrum in Figure 5, 6. There exists $\eta(x)$ only when x is a discriminant of real quadratic field; the characteristic function has a period of 4. The data restricted to $x \equiv 1 \pmod{4}$ account for three quarter of all the data. It follows from (11) that, as to the volume part, the estimate of t_0 for $x \equiv 1 \pmod{8}$ and that for $x \equiv 5 \pmod{8}$ are in the ratio 5 : 3. Therefore we expect there is a period of 8, which appears in Fig. 6.

Fig. 5

Fig. 6

To examine its structure minutely, we give the correlogram and the power spectrum for the data $\{\eta(x); x \equiv 1 \pmod{8}\}$ in Figure 7, 8. For the case of $x \equiv 5 \pmod{8}$, $x \equiv 8 \pmod{16}$ or

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	0	1	2	3	4	5	6	7	8	9
1	**	16-	38	52	96-	106	200-	158-	**	236-
2	0-	84	220-	352	556	828	916	**	1740	1604
3	0	90	196	392	632	706	**	1380	1392	1828
5	0-	16	36	64	**	152	**	224	204	296
6	2	100-	216	**	684	796	1108	1500	1540	1964
7	2	90	**	406	544	**	1110	1152	1624	2098
9	**	16-	40	58	112-	92-	188	172-	268-	**
10	4-	88	228	444	556	820	1216-	1180	**	2124
11	6	100	270	390	616	968	1032	**	1810	1866
13	0-	16-	34	72-	76	**	140-	198	190	344
14	8	108	268	384-	**	932	1024	1404	1912	1880-
15	8	104	220	**	692	768	1068	1524	1492	1868
17	0-	**	44	52-	106	108	164-	154	288	196
18	**	108	224-	388	632	732	1052	1496	1432-	**
19	10	104	260	484	594	912	1230	1290	**	2318
21	2	**	36	78	84-	140-	**	260	184-	346
22	12	96-	244	452	552	**	1128	**	1556	2080-
23	14	112	258	380	**	894	968	1332	1834	1664
26	16-	**	292-	424	652	1000	1016-	**	1916	1900
27	**	126	226	412	672	768	1108	1474	1450	**
29	4-	22	36-	76	88	**	124-	**	208-	312-
30	20	128-	244	444	696	784-	**	1564-	1528	1892
31	18	122	284	506	622	**	1322	1324	1724	**
33	2	20	44-	**	116-	100-	186	168-	**	234
34	20	128	**	504	620	932	1328-	1276	1796	2240
35	24	**	304	412	660	948	992	**	1920	1876
37	4-	20-	42	84-	76	146	**	216	**	360-
38	24	124	292	**	660	912-	1032	**	1744	1732
39	28	146	274	480	750	**	**	1650	1578	2156
41	4-	22	52-	60	**	120-	192-	168	**	212-
42	28	144	**	**	724-	780	1076	1532	1540-	1948
43	30	124	**	**	614	908	1198	1250	1592	2092
45	**	24-	**	80	92-	144	144	264	**	**
46	32	136	308	520-	644	948	1364	1380-	**	2400
47	30	**	316	430	640	990	1022	**	**	1794
49	**	24-	54	64-	112-	**	224	156	314	260-
51	40	166	290	**	784	860	1228	1722	1672	2112
53	8-	**	44	72-	86	164	132-	246	204-	316-
54	**	164	292	516	792	884-	1228	1632-	1660	**
55	40	148	300	532	644	964	1336	1300	**	2244
57	6	24-	48-	64	128-	108-	**	188-	260-	224
58	40-	140	292	528	612-	**	1260	1308	1724	2188
59	42	168	356	486	**	1072	1166	1484	2034	1972
61	8-	28	**	**	84-	168	148-	236-	216	**
62	44	**	320	428-	676	1048	1084	1380	1956	1776-
63	**	170	298	**	778	826	1156	1604	1582	**
65	8-	28	56-	68-	124	116-	188	**	336-	220-
66	52	184	308	540	832	884	**	1700	1644	2204
67	50	150	320	542	638	**	1332	1340	**	2278
69	10	**	44-	**	100	164-	150	284-	196	364
70	52	164-	**	552-	668	940	1312	1364	1844	2372-
71	54	**	370	492	788	1142	1148	1616	2044	1994
73	8-	28-	56	76-	112	118	228-	152-	**	252
74	60-	180	392-	500	756	1116	1180	**	2068	2040
77	12	30	52-	80	**	172-	136-	252	224-	320-
78	60	192	292	**	820	**	1156	1644-	1544	2028
79	70	186	**	618	702	1004	1448	1488	1920	2436
81	**	28-	60-	74	144-	116	226	204	296-	**
82	64-	172	348	580	660	988	1308	1300	**	2340
83	62	196	370	470	724	1080	1110	**	2038	1822
85	12-	32-	52	100	92-	**	156-	240-	212	400-
86	68	212	400	540	**	1156-	**	1636	2148	1980-
87	72	200	328	**	830	866	1244	1594	1586	2092
89	12-	**	**	72-	134	140	208	190	356	244
90	**	216	336-	548	**	928	1308	1724	1684	**
91	80	206	388	640	766	1040	1450	1468	**	2590
93	14	36-	48-	98	100-	152-	**	288	204	346
94	84	204	**	660-	760	**	1500	1456-	1924	2532
95	80	212	424	508	**	1156	1192	1604	2032	2028
97	12-	32-	**	72-	124	130	236-	164-	316	260-
98	**	**	412-	488	788	1084	1128-	1508	2060	1944
99	**	242	364	612	926	1026	1344	1852	1744	**

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	10	11	12	13	14	15	16	17	18	19
1	356	306	564-	336-	654	476	688-	**	1032-	596-
2	2204	2656	2448	3100	3976-	3500	**	5348	4500	5364
3	2380	2282	2736	3614	3128	**	4752	4296	5228	5964
5	264	468-	308	**	428	628	488	916	**	976
6	2512	2436	**	3808-	3344	4212	5020	4532-	5540	6808-
7	1884	**	3000	2786	3540	4468	3958	4576	5824	4946
9	436-	272-	504	412	592-	478	856-	500-	**	676
10	2080-	2564	**	3064	3548	4412	3964	**	6072-	5052
11	2308	2916	2760	3476	4348	3810	**	5496	4926	**
13	224-	420	360-	496-	**	740	428-	782	**	852-
14	**	3088-	2716	**	4244	3852-	4676	5576-	5028	5868
15	2532	2400	**	3552	3388	3924	4964	**	**	6632
17	**	320-	424-	374	656-	396	**	588	804	**
18	2384-	2216	2820	3640	3244-	3868	4896-	4124	**	6172
19	2150	2660	3520	3010	3748	**	4294	**	6152	5568
21	272-	432	340	632-	**	**	572-	772-	618	1108-
22	1948	2596	3284	2860-	**	4368-	3884	4604	5788	**
23	2260	2738	2566	**	4062	3590	4188	5294	4598	5592
26	**	3000	2884-	3364	4260	3852	4852	5876	4896	**
27	2420	**	2884	3698	3198	4056	4658	4192	**	6376
29	**	512-	316-	586	476-	648	**	956	564	1034
30	2612	2364-	2900	3756	3468	**	4896	4648-	5332	6420-
31	2230	2828	3402	**	**	4738	4324	5228	6406	5278
33	424-	252	**	380	548-	448	836	496-	856	688-
34	2108	**	3536	3244	3884	4936	4216	**	6292	5688
35	**	3036	2676	3388	4164	3804	4652	5412	4992	**
37	244-	426	340-	492	414	768	432-	**	612	848-
38	2276	2928-	2556	3316	3868	3604	**	5548	4784	5524
39	2698	2436	3172	4012	3658	**	5460	4644	5428	6860
41	410	340	492-	**	720	428	794	600-	860	674
42	2416-	2340	**	3608	3332	4020	5060-	4280	5372	6132
43	2076	**	3300	2960	3692	4502	3888	4916	5976	5128
45	292	424-	336	648	**	696	540	784-	**	1080
46	2188	2700	3532	3260	4092	4796-	4320	**	6236	5532
47	2276	2888	2740	3248	3998	3620	**	5506	4670	5868
49	380-	318	588-	356	**	520-	740-	548	**	596-
51	2766	2494	**	4044	3622	4508	5340	4844	5592	6854
53	**	484-	300	576	464-	600-	504	896-	500-	**
54	2844	2540	3252	3916-	3504	4372	5556	4904-	**	7068
55	2096	2700	3504	3172	3828	4836	4144	**	6140	5396
57	436	284-	522	388	580	**	816-	476	910	692
58	**	2620	3300-	3028	**	4572	3936-	4996	5748	5028
59	2528	3240	2882	**	4374	3950	4844	6042	**	6192
61	256-	**	396-	552-	422	828	492	902	652-	960
62	**	2884	2760	3460	3988	3756	4436	5236	**	**
63	2542	2410	2924	3584	3348	4264	5094	4292	**	6392
65	420	324-	476	388	736-	428-	**	628-	816-	620
66	2828-	2672	3180	4132	3640-	**	**	4832	5836	6816
67	2092	2640	3312	3074	**	4506	4146	4836	5750	5268
69	304-	452	**	**	396	766	556-	816-	632	1176
70	2208	**	3468	3044-	**	4872-	4324	4940	6268	5304-
71	**	3326	3052	3508	4570	3938	4712	5996	5454	**
73	368-	320	560	348-	626	**	700	**	1036-	568-
74	2452	3180	**	3732	4592	4016	**	5872	5136	6268
77	266	512	308-	**	452	656	516	916-	536-	966
78	**	2460	**	3744-	3528	4044	4976	4468	5460	6604
79	2396	**	3634	3404	4236	5070	4356	5384	6454	5562
81	468	284-	552	452-	644-	480	**	516-	**	732
82	2216-	2708	3428	3008	3660	4764	**	**	5968-	5276
83	**	**	2838	3440	4234	3886	**	5506	4792	5924
85	252	484	388-	536-	**	808-	452-	884	684	960-
86	2612	3268	2936	**	4736	4144-	4900	6180	5308	6180
87	2742	2522	**	3948	3350	**	5208	4634	5484	6358
89	**	340-	516-	390	756-	452	858	640-	888-	**
90	2736-	2524	3324	4128	3488-	4484	**	4584	**	6940
91	2314	3028	3718	3344	4132	5116	4632	**	6632	5632
93	308-	420-	342	644	392-	**	544-	792	570	1088-
94	2408	3020	3624	3460	**	4960-	**	5580	6696	5628-
95	2468	3132	2860	**	4524	3916	4908	5724	5112	6092
97	364-	**	596-	356	634	520-	688-	550	1036	588-
98	**	3128	2728	3412	4324	3932	4756	5604	5012-	**
99	2904	2796	3484	4062	3894	4556	5582	4856	**	7282

The distribution of the Hilbert modular type cusp forms

$x \equiv 12 \pmod{16}$, they are nearly the same as for the case of $x \equiv 1 \pmod{8}$. So we omit them. The period of 24 which appears in Fig. 8 are derived by the elliptic part $b(x)h(-3x)$. Fig. 8 shows that there are not any other periodic factors in $\eta(x)$. We think that the fact above is reflected in the random appearance of elliptic points.

We conclude from these studies that the statistical model is useful to analyze the number theoretical data quantitatively.

Fig. 7

Fig. 8

5. Table of the dimensions of $S_2(\Gamma)$

For a square-free number D ($1 < D < 2000$), we tabulate the dimensions of $S_2(\Gamma)$ over real quadratic fields $\mathbb{Q}(\sqrt{D})$. The number D is given by

$$D = i + 100j \quad (i = \text{row number}, j = \text{column number}) \quad (15)$$

When the mark ‘-’ appears after a figure, $\mathbb{Q}(\sqrt{D})$ has a unit of negative norm. In this case, the dimension of the space of cusp forms on upper half space of $\mathbb{C}^2$ is equal to the quarter of the dimension in the table. The mark ‘**’ means D is not square-free. For the case of $D = 2, 3, 5, 13, 17$, $\dim S_2(\Gamma)$ vanishes exactly.

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(Received April 1988; Revised May 1988.)