SENSITIVITY ANALYSIS IN LATENT CLASS ANALYSIS

Tsukio Morita* and Yutaka Tanaka†

ABSTRACT

The present paper proposes a procedure for evaluating the stability of Green's solution of latent class analysis from the viewpoint of sensitivity analysis. For this purpose, the first order differential coefficients are derived for the quantities contained in the solution with respect to a perturbation parameter. A numerical example is given for illustration.

1. Introduction

More than fifty years have passed since Lazarsfeld (1950) proposed the latent class analysis or, in short, LCA. The purpose of LCA is to explore the structure of association among manifest categorical variables by introducing a latent class. There are several typical methods for obtaining estimates of parameters in LCA such as Green's method (Green, 1951), Gibson's method (Gibson, 1955) and the maximum likelihood method by McHugh (1956). However, none of these methods guarantees a stable solution. Okamoto and Isogai (1978), Morita (1980) and others pointed out by numerical studies that LCA frequently results in improper solutions when a small sampling error is introduced. In such situations, even if we fortunately obtained a proper solution, the reliability of the solution is doubtful. On the other hand, the sensitivity analysis approach has been studied in various multivariate methods including regression analysis, principal component analysis (Critchley, 1985 and Tanaka, 1988), factor analysis (Tanaka and Odaka, 1989), and the method of quantification (Tanaka and Tarumi, 1986). There are two aspects in sensitivity analysis. One is to detect influential observations and the other is to evaluate the stability of the results. No paper on LCA has discussed the stability from the viewpoint of sensitivity analysis. In this paper, we choose Green's method and propose a procedure for evaluating the influence on estimates produced by a small perturbation on the weights of observations. In Section 2 we describe Green's method somewhat in detail. This gives a preparation for deriving the first order differential coefficients of several matrices which appear in the process of obtaining the estimates. In Section 3, by giving a particular perturbation on the observations, we show the sensitivity of the estimates. A numerical example is given in Section 4 on the basis of artificial data and in Section 5 we will describe the concluding remarks.

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Key words: Green's method; Latent class analysis; Sensitivity analysis
2. Latent class analysis and Green’s method

Suppose that we have response patterns of \( N \) individuals to \( n \) items as in Table 1, where

\[
x_{i\alpha} = \begin{cases} 
1; & \text{if individual } \alpha \text{ responds positively to item } i, \\
0; & \text{otherwise}.
\end{cases}
\]

Suppose that there are \( n \) items and \( m \) latent classes and that for each item, an individual belongs to one and only one of \( m \) classes. We define the probabilities \( p_{ij} \) and \( p_{ijk} \) which are called the manifest probabilities as follows:

- \( p_{i} = \) the probability that an individual responds positively to item \( i \),
- \( p_{ij} = \) the probability that an individual responds positively to both items \( i \) and \( j \),
- \( p_{ijk} = \) the probability that an individual responds positively to items \( i, j \) and \( k \), simultaneously.

Let us denote by \( w_t \) the probability that a randomly selected individual belongs to class \( t \) and by \( \pi_{it} \) the probability that an individual belonging to class \( t \) responds positively to item \( i \). It is assumed in LCA that the responses up to three items are conditionally independent with each other given that the individual belongs to a latent class. Then we obtain the following equations which express the relations between the manifest probabilities and the model parameters:

\[
\begin{align*}
\pi_{i} & = \sum_{t=1}^{m} w_{t} \pi_{it} \quad (i = 1, \ldots, n), \\
p_{ij} & = \sum_{t=1}^{m} w_{t} \pi_{it} \pi_{jt} \quad (i, j = 1, \ldots, n), \\
p_{ijk} & = \sum_{t=1}^{m} w_{t} \pi_{it} \pi_{jt} \pi_{kt} \quad (i, j, k = 1, \ldots, n)
\end{align*}
\]

with \( \sum_{t=1}^{m} w_{t} = 1 \). The parameters \( w_{t} \) and \( \pi_{it} \) for \( t = 1, \ldots, m \); \( i = 1, \ldots, n \) (also, see Table 2) are estimated on the basis of the observations in Table 1.

To show the procedure for Green’s method, let us now define the following:

\[
D_{w} = \text{diag}(w_{1}, \ldots, w_{m}), \quad D_{k} = \text{diag}(\pi_{k1}, \ldots, \pi_{km}),
\]

--- 348 ---
Sensitivity Analysis in Latent Class Analysis

$L = \begin{pmatrix} 
1 & 1 & \ldots & 1 \\
\pi_{11} & \pi_{12} & \ldots & \pi_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{n1} & \pi_{n2} & \ldots & \pi_{nm} 
\end{pmatrix}, \quad Q = \begin{pmatrix} 
1 & p_1 & \ldots & p_n \\
p_1 & p_{11} & \ldots & p_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
p_n & p_{n1} & \ldots & p_{nn} 
\end{pmatrix}, \quad R_k = \begin{pmatrix} 
p_k & p_{1k} & \ldots & p_{nk} \\
p_{1k} & p_{11k} & \ldots & p_{1nk} \\
\vdots & \vdots & \ddots & \vdots \\
p_{nk} & p_{n1k} & \ldots & p_{nnk} 
\end{pmatrix}$

Table 2: LCA Model (with $n$ items and $m$ classes)

<table>
<thead>
<tr>
<th>Class</th>
<th>Item</th>
<th>Item</th>
<th>Item</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$w_1$</td>
<td>$\pi_{11}$</td>
<td>$\pi_{12}$</td>
<td>$\pi_{1i}$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$t$</td>
<td>$w_t$</td>
<td>$\pi_{1t}$</td>
<td>$\pi_{2t}$</td>
<td>$\pi_{it}$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$m$</td>
<td>$w_m$</td>
<td>$\pi_{1m}$</td>
<td>$\pi_{2m}$</td>
<td>$\pi_{im}$</td>
</tr>
</tbody>
</table>

where $D_w$ and $D_k$ are matrices of order $m \times m$, $L$ is of order $(n+1) \times m$ and $Q$ and $R_k$ are of order $(n+1) \times (n+1)$. Then equation (1) can be expressed as

$$Q = LD_w L', \quad R_k = LD_w D_k L'.$$

Define $R = \sum_{k=1}^{n} R_k$. Then

$$R = LD_w \left( \sum_{k=1}^{n} D_k \right) L' = LD_w D_L L',$$

where $D_L = \text{diag}(\pi_1, \ldots, \pi_m)$, $\pi_t = \sum_{k=1}^{n} \pi_{kt}$ ($t = 1, \ldots, m$). We assume that $w_t > 0$ ($t = 1, \ldots, m$), $\text{rank}(L) = m$ and that $\pi_t (t = 1, \ldots, m)$ are distinct. Put $A = LD_w^{1/2}$, then

$$Q = (LD_w^{1/2})(LD_w^{1/2})' = AA', \quad R = (LD_w^{1/2})D_L (LD_w^{1/2})' = AD_L A'.$$

Let $B$ be an $(n+1) \times m$ matrix satisfying $Q = AA' = BB'$ and $\text{rank}(B) = m$. Then there exists $F \in O(m)$ such that $A = BF$. Since

$$(B'B)^{-1}B'RB(B'B)^{-1} = (B'B)^{-1}B'AD_L A'B(B'B)^{-1}$$

$$= (B'B)^{-1}B'(BFD_L F'B')(B'B)^{-1} = FD_L F',$$

$D_L$ and $F$ are the eigenvalue matrix and the eigenvector matrix of $T = (B'B)^{-1}B'RB(B'B)^{-1}$, respectively. Since $A = LD_w^{1/2}$,

$$w_t = a_{0t}^2 \quad (t = 1, \ldots, m),$$

$$\pi_{it} = a_{it}/a_{0t} \quad (t = 1, \ldots, m; i = 1, \ldots, n),$$

where $a_{0t}$ and $a_{it}$ are $(1, t)$ and $(i + 1, t)$ elements of $A$, respectively.
The estimation procedure of Green’s method can be described in the following steps:

**Step 1.** Estimate the non-duplicated elements of $\hat{Q}$ and $\hat{R}$ using Equation (3) with $u^*_\alpha = 1, \alpha = 1, \ldots, N$, and substitute appropriate initial values to the duplicated elements.

(In the numerical example in Section 4, we selected $\hat{p}_{ij} = \hat{p}_{ij}^{3/4}$ and $\hat{p}_{ij} = \hat{p}_{ij}^{3/4}$ as the initial values (see Morita, 1980).)

**Step 2.** Solve the eigenequation of $\hat{Q}$ and compute $\hat{B} := \hat{V} \hat{A}^{1/2}$, where $\hat{A}$ and $\hat{V}$ are the matrices of the largest $m$ eigenvalues and the associated eigenvectors, respectively.

**Step 3.** Solve the eigenequation of $\hat{T}$, and obtain the eigenvalues $\hat{D}_r$ and the eigenvectors $\hat{F}$. Then compute $\hat{A} := \hat{B}\hat{F}$.

**Step 4.** Based on the elements of $\hat{A}$, compute $\hat{w}_t$ and $\hat{\pi}_{it}$ for $t = 1, \ldots, m; i = 1, \ldots, n$.

**Step 5.** Compute the duplicated probabilities $\hat{p}_{ii}$ and $\hat{p}_{ij}$ using $\hat{w}_t$ and $\hat{\pi}_{it}$ as follows;

$$\hat{p}_{ii} = \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it}^2 \quad (i = 1, \ldots, n),$$

$$\hat{p}_{ij} = \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it} \hat{\pi}_{jt} \quad (i, j = 1, \ldots, n).$$

**Step 6.** Compare the calculated $\hat{p}_{ii}$ and $\hat{p}_{ij}$ with the corresponding elements of $\hat{Q}$ and $\hat{R}$. If the discrepancies are smaller than pre-assigned constants, then stop the iteration. Otherwise, substitute the new values into $\hat{Q}$ and $\hat{R}$ and go back to step 2.

3. Perturbation and sensitivity

In sensitivity analysis in multivariate methods such as regression analysis, principal component analysis, and factor analysis, the effects on estimates due to a small change in case-weight assigned to each individual are frequently studied. However, in LCA we need a lot of individuals to obtain stable solutions because of instability even if the number of items is small. Therefore, we consider the case that multiple observations are disturbed.

Let $u^*_\alpha$ be a weight for an individual $\alpha$. Then the estimates of the manifest probabilities can be expressed as

$$\hat{p}_i = \frac{1}{N} \sum_{\alpha=1}^{N} u^*_\alpha x_{i\alpha},$$

$$\hat{p}_{ij} = \frac{1}{N} \sum_{\alpha=1}^{N} u^*_\alpha x_{i\alpha} x_{j\alpha} \quad (i \neq j),$$

$$\hat{p}_{ijk} = \frac{1}{N} \sum_{\alpha=1}^{N} u^*_\alpha x_{i\alpha} x_{j\alpha} x_{k\alpha} \quad (i \neq j \neq k \neq i).$$

Define

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nN} \end{pmatrix}, \quad U^* = \text{diag}(u^*_1, \ldots, u^*_N)',$$

$$1_n = (1, \ldots, 1)', \quad 1_N = (1, \ldots, 1)'.$$
Then, Equation (3) can be rewritten in matrix form as follows:

\[
\begin{align*}
\hat{p} &= (\hat{p}_i) = \frac{1}{N} X U^* 1_N, \\
\hat{P} &= (\hat{p}_{ij}) = \frac{1}{N} X U^* X', \\
\hat{P}_k &= (\hat{p}_{ijk}) = \frac{1}{N} X U^* \Lambda_k X' (k = 1, \ldots, n),
\end{align*}
\]

where \( \Lambda_k = \text{diag}(x_{k1}, \ldots, x_{kN}) \).

In order to evaluate the influence of a subset of individuals we consider the following perturbation. Let \( S = \{i_1, \ldots, i_l\} \) be a subset of the individuals in the set \( \{1, \ldots, \alpha, \ldots, N\} \) and let the weight for the individual \( \alpha \) be \( u^*_\alpha = N u_\alpha / \sum_{\beta=1}^{N} u_\beta \), \( (\alpha = 1, \ldots, N) \). Then we give the perturbation

\[
u_{0}(\varepsilon) = (1, \ldots, 1, \ldots, 1)' \rightarrow u(\varepsilon) = (u_1(\varepsilon), \ldots, u_\alpha(\varepsilon), \ldots, u_N(\varepsilon))',
\]

where

\[
u_{\alpha}(\varepsilon) = \begin{cases} 
1 - \varepsilon & \alpha \in S, \\
1 & \alpha \notin S.
\end{cases}
\]

The perturbation on individuals results in the perturbed \( \hat{Q}(\varepsilon) \) and \( \hat{R}(\varepsilon) \). The differential coefficient of \( U^* \) at \( \varepsilon = 0 \) is \( U^{*}(1) = (1/N)I_N - E_{i_1,\ldots,i_l} \), where \( E_{i_1,\ldots,i_l} = \text{diag}(1, \ldots, 0, 1, \ldots, 0) \) is an \( N \times N \) diagonal matrix with 1's in columns \( i_1, \ldots, i_l \) and \( I_N \) an identity matrix. The differential coefficients of \( \hat{Q} \) and \( \hat{R} \) at \( \varepsilon = 0 \), which we denote by \( \hat{Q}^{(1)} \) and \( \hat{R}^{(1)} \), respectively, are obtained from Equation (4) by replacing \( U^* \) with \( U_{*}^{(1)} \). Note that \( \hat{Q}_{11}^{(1)} = 0 \). We note that \( \hat{p}_{ii}^{(1)} \) and \( \hat{p}_{ij}^{(1)} \), the differential coefficients of \( \hat{p}_{ii} \) and \( \hat{p}_{ij} \) at \( \varepsilon = 0 \), respectively, cannot directly be obtained by perturbation in the observations as in the case of \( p_{ii} \) and \( p_{ij} \). We will discuss this issue at the end of this section.

Suppose that the matrix \( \mathbf{T} \) changes to \( \mathbf{T}(\varepsilon) \) due to a small perturbation \( \varepsilon \) and that \( \hat{\mathbf{T}}(\varepsilon) \) is expressed as a convergent power series in the parameter \( \varepsilon \):

\[
\hat{T}(\varepsilon) = \hat{T} + \varepsilon \hat{T}^{(1)} + O(\varepsilon^2).
\]

Then, as shown in Rellich (1969) the eigenvalues \( \hat{\lambda}(\varepsilon) \)'s and eigenvectors \( \hat{f}(\varepsilon) \)'s of \( \hat{T}(\varepsilon) \) can be expanded in convergent power series as

\[
\hat{\lambda}_s(\varepsilon) = \lambda_s + \varepsilon \hat{\lambda}_s^{(1)} + O(\varepsilon^2),
\]

\[
\hat{f}_s(\varepsilon) = f_s + \varepsilon \hat{f}_s^{(1)} + O(\varepsilon^2), \quad s = 1, \ldots, m,
\]

where \( \hat{T} \hat{f}_s = \lambda_s \hat{f}_s \), \( s = 1, \ldots, m \).

Assume that the eigenvalues \( \hat{\lambda}_s (s = 1, \ldots, m) \) of \( \hat{T} \) are simple. Then, as given in Sibson (1979) the coefficients of the first terms of the eigenvalues and the associated eigenvectors can be expressed as

\[
\hat{\lambda}_s^{(1)} = \hat{f}_s^{(1)} \hat{T}^{(1)} \hat{f}_s,
\]

\[
\hat{f}_s^{(1)} = \sum_{r \neq s} (\hat{\lambda}_s - \hat{\lambda}_r)^{-1} (\hat{f}_r^{(1)} \hat{T}^{(1)} \hat{f}_s) \hat{f}_r, \quad s = 1, \ldots, m.
\]
Since $\mathcal{T}(\varepsilon) = (\mathcal{B}'(\varepsilon)\mathcal{B}(\varepsilon))^{-1}\mathcal{B}'(\varepsilon)\mathcal{R}(\varepsilon)\mathcal{B}(\varepsilon)(\mathcal{B}'(\varepsilon)\mathcal{B}(\varepsilon))^{-1}$ (see Section 2), the differential coefficient of $\mathcal{T}(\varepsilon)$ at $\varepsilon = 0$ is given as

$$\hat{\mathcal{T}}^{(1)} = -(\mathcal{B}' \mathcal{B})^{-1}C_1\mathcal{T} + (\mathcal{B}' \mathcal{B})^{-1}C_2(\mathcal{B}' \mathcal{B})^{-1} - \mathcal{T}C_1(\mathcal{B}' \mathcal{B})^{-1},$$

where $C_1 = \mathcal{B}'(1) \mathcal{B} + \mathcal{B}'(1) \mathcal{B}$ and $C_2 = \mathcal{B}(1) \mathcal{R} \mathcal{B} + \mathcal{B}(1) \mathcal{R} \mathcal{B} + \mathcal{B}(1) \mathcal{R} \mathcal{B}(1)$. $\mathcal{B}(1)$ included in $\hat{\mathcal{T}}^{(1)}$ is obtained from $\mathcal{B}(\varepsilon) = \mathcal{V}(\varepsilon) \mathcal{A}(\varepsilon)$ by the spectrum decomposition of $\mathcal{Q}(\varepsilon)$, where $\mathcal{A}(\varepsilon) = \text{diag}(\lambda_1(\varepsilon), \ldots, \lambda_m(\varepsilon))$ and $\mathcal{V}(\varepsilon) = (v_1(\varepsilon), \ldots, v_m(\varepsilon))$ are the matrices of $m$ eigenvalues and the associated eigenvectors of $\mathcal{Q}(\varepsilon)$, respectively (see Step 2 in Section 2). Then the differential coefficient of $\mathcal{B}(\varepsilon)$ at $\varepsilon = 0$ can be expressed as

$$\mathcal{B}(1) = \mathcal{V}(1) \mathcal{A}(1)^2 + (1/2) \mathcal{V} \hat{\mathcal{A}} - 1/2 \mathcal{A}(1).$$

If the eigenvalues $\hat{\lambda}_s (s = 1, \ldots, m)$ of $\hat{\mathcal{Q}}$ are simple, then $\hat{\mathcal{A}}(1) = \text{diag}(\hat{\lambda}_1(1), \ldots, \hat{\lambda}_m(1))$ and $\hat{\mathcal{V}}(1) = (v_1(1), \ldots, v_m(1))$ are given as

$$\hat{\lambda}_s(1) = v_s \mathcal{Q}(1) v_s,$$

$$\hat{v}_s(1) = \sum_{r \neq s} (\hat{v}_s - \hat{v}_r)^{-1}(\hat{v}_r \mathcal{Q}(1) \hat{v}_s) \hat{v}_r, \quad s = 1, \ldots, m,$$

where $\hat{v}_s (s = 1, \ldots, m)$ and $\hat{v}_r (r = 1, \ldots, n + 1)$ are the eigenvalues and the associated eigenvectors of $\hat{\mathcal{Q}}$, respectively.

Since $\mathcal{A}(\varepsilon) = \mathcal{B}(\varepsilon) \mathcal{F}(\varepsilon)$ (see Section 2), we can obtain at $\varepsilon = 0$.

$$\hat{\mathcal{A}}(1) = \hat{\mathcal{B}}(1) \mathcal{F} + \mathcal{F}(1),$$

where $\mathcal{F}$ is the eigenvector matrix of $\hat{\mathcal{T}}$ and $\hat{\mathcal{B}}(1)$ is the differential coefficient of the eigenvector matrix of $\mathcal{T}(\varepsilon)$ at $\varepsilon = 0$. From (2) we can obtain the sensitivity of $\hat{w}_t$ and $\hat{\pi}_{it}$ as

$$\hat{w}_t(1) = 2 \hat{a}_{0t} \hat{a}_{0t}^2,$$

$$\hat{\pi}_{it}(1) = (\hat{a}_{0t} \hat{a}_{0t}^2 - \hat{a}_{0t} \hat{a}_{0t}^2) / \hat{a}_{0t}^2.$$

Note that $\hat{\mathcal{Q}}$, $\hat{\mathcal{R}}$, $\hat{\mathcal{T}}$, $\hat{\mathcal{B}}$, etc. are evaluated at the estimates $\hat{w}_t$ and $\hat{\pi}_{it}$. Finally we will show how to obtain $\hat{p}_{ii}$ and $\hat{p}_{ij}$, the differential coefficients of the duplicated probabilities $\hat{p}_{ii}$ and $\hat{p}_{ij}$ at $\varepsilon = 0$, respectively. Using (1) and (6), we have

$$\hat{p}_{ii}(1) = 2 \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it} \hat{a}_{0t} \hat{a}_{0t}^2 + 2 \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it} (\hat{a}_{0t} \hat{a}_{0t}^2 - \hat{a}_{0t} \hat{a}_{0t}^2) / \hat{a}_{0t}^2,$$

$$\hat{p}_{ij}(1) = 2 \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it} \hat{a}_{0t} \hat{a}_{0t}^2 + 2 \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it} \hat{a}_{0t} \hat{a}_{0t} (\hat{a}_{0t} \hat{a}_{0t}^2 - \hat{a}_{0t} \hat{a}_{0t}^2) / \hat{a}_{0t}^2 + \sum_{t=1}^{m} \hat{w}_t \hat{\pi}_{it} (\hat{a}_{0t} \hat{a}_{0t}^2 - \hat{a}_{0t} \hat{a}_{0t}^2) / \hat{a}_{0t}^2.$$

Since Equation (5) indicates that $\hat{\mathcal{A}}(1)$ is a function of $\hat{\mathcal{Q}}(1)$ and $\hat{\mathcal{R}}(1)$, we replace the undetermined differential coefficients of the duplicated probabilities appearing in $\hat{\mathcal{Q}}(1)$ and $\hat{\mathcal{R}}(1)$ with $\hat{p}_{ii}(1)$ and $\hat{p}_{ij}(1)$ and solve the system of linear equations (7) and (8) with respect to $\hat{p}_{ii}(1)$ and $\hat{p}_{ij}(1)$. In summary, the procedure proposed is as follows:
Step 1. Obtain the convergent solutions $\hat{w}_t$ and $\hat{\pi}_t$ on the basis of the procedure described in Section 2.

Step 2. Obtain $Q^{(1)}$, $R^{(1)}$, $B^{(1)}$, $T^{(1)}$, $F^{(1)}$, and $A^{(1)}$, which contain unknown quantities $\hat{p}_i^{(1)}$ and $\hat{p}_{ij}^{(1)}$.

Step 3. Solve the equations (7) and (8) and obtain $\hat{p}_i^{(1)}$ and $\hat{p}_{ij}^{(1)}$.

Step 4. By using (6), obtain the sensitivity $w_i^{(1)}$ and $\pi_i^{(1)}$.

As the simultaneous equations (7) and (8) are somewhat complicated to solve, in our numerical example in Section 4 we used the convergent values of $\hat{p}_i^{(1)}$ and $\hat{p}_{ij}^{(1)}$ obtained by the iteration with the initial values set to 0.

4. Numerical example

To illustrate our procedure, we assumed a model shown in Table 3 and generated 1000 data as shown in Table 4 on the basis of the model. We applied LCA to the data set and evaluated the sensitivity by our proposed method. The results of LCA are given in Table 5. To investigate the influence of perturbation in individuals, we considered two types of perturbation and calculated $w_i^{(1)}$ and $\pi_i^{(1)}$ by using the proposed procedure. One type of perturbation was to perturb observations of certain proportion in each response pattern while the other was to perturb a certain number of observations in each response pattern. Let $p$ denote the proportion of individuals perturbed in each pattern. We considered the cases where $p = 0.1, 0.3$ and $0.5$, and the case where 50 individuals are perturbed. The results of their sensitivity $||w_i^{(1)}||$ and $||\pi_i^{(1)}||$ are given in Table 6, where $||w_i^{(1)}|| = |w_i^{(1)}|$ and $||\pi_i^{(1)}|| = (\sum_{i=1}^{m} \sum_{t=1}^{m} \pi_i^{(1,t)})^{1/2}$.

<table>
<thead>
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<th>Class</th>
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<td>.7</td>
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<table>
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For any \( \rho \) in the case of the first type of perturbation, \( \hat{w}_1^{(1)} \) and \( \hat{\pi}_1^{(1)} \) were most influenced by perturbation in Pattern 2 \( (n=124) \) and in Pattern 1 \( (n=251) \), respectively. On the other hand, perturbation in Pattern 8 \( (n=141) \) was least influential on the whole. In the case where a fixed number (=50) of observations was perturbed, perturbation in Pattern 6 \( (n=63) \) had the most influence and this was probably due to the small number of observations in this pattern.

5. Concluding remarks

In the present paper we proposed a method of sensitivity analysis in the Green’s solution of latent class analysis. We treated two types of perturbation. One is to perturb the weights for a constant proportion of individuals and the other to perturb those for a constant number of individuals in each response pattern. The first order differential coefficients of the model parameters with respect to the perturbation parameter are derived in the form of a system of simultaneous equations, and it is solved by an iterative procedure. In a numerical study we found that in the case of the second type of perturbation the most influential response pattern is the pattern with the smallest number of individuals as is expected. In the case of the first type of perturbation, the amounts of influence differ considerably among the eight
response patterns, i.e., 4.6 times between the largest and the smallest for \( w \) and 1.4 times for \( \pi \). So far we have not yet been successful in interpreting these results. To do this will require more extended numerical studies. In what circumstances does the solution become unstable? How does the instability discussed in the present paper relate to the so-called improper solution? These are the topics we wish to study in the future.

Acknowledgements

The authors are indebted to the referees for their useful comments.

REFERENCES


