A MEASURE OF ASYMMETRY FOR MULTI-WAY CONTINGENCY TABLES WITH ORDERED CATEGORIES

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ABSTRACT

For square contingency tables with ordered categories, Tomizawa, Miyamoto and Hatanaka (2001) considered a measure that represents the degree of departure from symmetry. This paper extends the measure to multi-way tables with ordered categories. The measure proposed is expressed by using the Cressie-Read power-divergence or the Patil-Taillie diversity index. The measure could be useful for comparing the degrees of departure from symmetry in several multi-way tables with ordered categories. Examples are given.

1. Introduction

Consider an $r \times r$ square contingency table with the same ordinal row and column classifications. Let p_{ij} denote the probability that an observation will fall in the i -th row and j -th column of the table ($i = 1, \dots, r; j = 1, \dots, r$). The symmetry model is defined by

$$p_{ij} = p_{ji} \quad (i = 1, \dots, r; j = 1, \dots, r; i \neq j);$$

see, e.g., Bowker (1948) and Bishop, Fienberg and Holland (1975, p.282). The symmetry model may also be considered for multi-way r^T ($T \geq 3$) contingency tables (see Bishop *et al.*, 1975, p.300, and Agresti, 2002, p.440); see also Sections 2 and 3.

For square contingency tables, denote the row and column variables by X_1 and X_2 , respectively, and let

$$G_{ij} = \sum_{s=1}^i \sum_{t=j}^r p_{st} = P(X_1 \leq i, X_2 \geq j) \quad \text{when } i < j,$$

and

$$G_{ij} = \sum_{s=i}^r \sum_{t=1}^j p_{st} = P(X_1 \geq i, X_2 \leq j) \quad \text{when } i > j.$$

Then the symmetry model can be expressed as

$$G_{ij} = G_{ji} \quad (i = 1, \dots, r; j = 1, \dots, r; i \neq j). \tag{1}$$

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When the symmetry model does not hold, we may be interested in whether other structure exists, such as extended symmetry (e.g., quasi-symmetry (Causinus, 1965), or marginal homogeneity (Stuart, 1955); see also Bishop *et al.*, 1975, Chapter 8) or for ordinal data some particular form of asymmetry (e.g., conditional symmetry (McCullagh, 1978), diagonals-parameter symmetry (Goodman, 1979), and so on; see also Agresti, 1984, p.202). When there is a structure of asymmetry in a square contingency table, we are also interested in measuring the degree of departure from symmetry. For square contingency tables with *nominal* categories, Tomizawa (1994) and Tomizawa, Seo and Yamamoto (1998) considered measures which represent the degree of departure from symmetry. For square contingency tables with *ordered* categories, Tomizawa, Miyamoto and Hatanaka (2001) considered a measure, $\phi^{(\lambda)}$, which represents the degree of departure from symmetry (see Appendix 1). For multi-way tables with *nominal* categories, Yamamoto (2004) considered a measure, $\Phi^{(\lambda)}$, to represent it (see Appendix 2), which includes the measures of Tomizawa (1994) and Tomizawa *et al.* (1998).

We can test goodness-of-fit of the symmetry in multi-way contingency tables using the test statistic, e.g., the likelihood ratio chi-squared statistic or Pearson's chi-squared statistic. However, the test statistics would not be suitable for *comparing* the degrees of asymmetry in several tables (see Section 7). Since the symmetry model for square tables with ordered categories can be expressed as (1), the measure $\phi^{(\lambda)}$ would be useful to see to what degree the cumulative probabilities $\{G_{ij}\}$ are distant from those with a symmetric structure. We are also interested in a measure that represents the degree of departure from symmetry for a multi-way table with *ordered* categories.

The purpose of this paper is (i) to give another expression of the symmetry model based on the cumulative probabilities (instead of cell probabilities) for multi-way contingency tables and (ii) to propose a measure, which represents the degree of departure from symmetry in multi-way contingency tables with ordered categories. Section 2 proposes the measure for the case of three-way contingency tables, and Section 3 extends it to T -way ($T \geq 3$) tables. The measures proposed are useful for comparing the degrees of departure from symmetry. The proposed measure is an extension of the measure $\phi^{(\lambda)}$ (in Appendix 1) to the multi-way table.

2. Case of three-way tables

Consider an $r \times r \times r$ contingency table with ordered categories. Let p_{ijk} denote the probability that an observation will fall in the (i, j, k) -th cell of the table for $1 \leq i, j, k \leq r$. Denote the s th variable by X_s ($s = 1, 2, 3$).

2.1. Symmetry model

The symmetry (S) model (Bishop *et al.*, 1975, p.300) is defined by

$$p_{ijk} = p_{ikj} = p_{jik} = p_{jki} = p_{kij} = p_{kji} \quad (1 \leq i, j, k \leq r).$$

Note that for a $3 \times 3 \times 3$ table, the S model can be expressed as follows:

$$p_{123} = p_{132} = p_{213} = p_{231} = p_{312} = p_{321},$$

$$p_{112} = p_{121} = p_{211}, \quad p_{113} = p_{131} = p_{311}, \quad p_{223} = p_{232} = p_{322},$$

$$p_{122} = p_{212} = p_{221}, \quad p_{133} = p_{313} = p_{331}, \quad p_{233} = p_{323} = p_{332}.$$

We shall consider another expression of the S model based on the cumulative probabilities, which are defined in a similar way to the cumulative probabilities $\{G_{ij}\}$ in two-way tables. First, we shall define some sets about the cell's index. Let

$$E_1 = \{(i, j, k) \mid 1 \leq i < j < k \leq r\},$$

$$E_2 = \{(i, j, k) \mid 1 \leq i = j < k \leq r\},$$

$$E_3 = \{(i, j, k) \mid 1 \leq i < j = k \leq r\},$$

and

$$D(i, j, k) = \{(l, m, n) \mid (l, m, n) \text{ is any permutation of } (i, j, k)\}.$$

For example, when $r = 3$, $E_1 = \{(1, 2, 3)\}$, $E_2 = \{(1, 1, 2), (1, 1, 3), (2, 2, 3)\}$ and $E_3 = \{(1, 2, 2), (1, 3, 3), (2, 3, 3)\}$.

Secondly, we shall consider the cumulative probability for each set E_t ($t = 1, 2, 3$) defined by, for $(k_1, k_2, k_3) \in D(1, 2, 3)$,

$$G_{k_1 k_2 k_3(i, j, k)}^{(1)} = P(X_{k_1} \leq i < X_{k_2} \leq j, k \leq X_{k_3}) \quad \text{for } (i, j, k) \in E_1,$$

$$G_{k_1 k_2 k_3(i, j, k)}^{(2)} = P(X_{k_1} \leq i, j \leq X_{k_2} < k \leq X_{k_3}) \quad \text{for } (i, j, k) \in E_1,$$

$$G_{k_1 k_2 k_3(i, j, k)}^{(3)} = P(X_{k_1}, X_{k_2} \leq j, k \leq X_{k_3}) \quad \text{for } (i, j, k) \in E_2,$$

$$G_{k_1 k_2 k_3(i, j, k)}^{(4)} = P(X_{k_1} \leq i, j \leq X_{k_2}, X_{k_3}) \quad \text{for } (i, j, k) \in E_3.$$

Note that for E_1 , we define the cumulative probabilities with two types. For example,

$$G_{123(i, j, k)}^{(1)} = P(X_1 \leq i < X_2 \leq j, k \leq X_3),$$

$$G_{123(i, j, k)}^{(2)} = P(X_1 \leq i, j \leq X_2 < k \leq X_3).$$

When $r = 5$ and $(i, j, k) = (1, 2, 4)$,

$$\begin{aligned} G_{123(1, 2, 4)}^{(1)} &= P(X_1 \leq 1 < X_2 \leq 2, 4 \leq X_3) \\ &= p_{124} + p_{125}, \end{aligned}$$

$$\begin{aligned} G_{123(1, 2, 4)}^{(2)} &= P(X_1 \leq 1, 2 \leq X_2 < 4 \leq X_3) \\ &= p_{124} + p_{125} + p_{134} + p_{135}. \end{aligned}$$

When $r = 5$ and $(i, j, k) = (2, 2, 4)$,

$$\begin{aligned} G_{123(2, 2, 4)}^{(3)} &= P(X_1, X_2 \leq 2, 4 \leq X_3) \\ &= p_{114} + p_{124} + p_{214} + p_{224} + p_{115} + p_{125} + p_{215} + p_{225}. \end{aligned}$$

When $r = 5$ and $(i, j, k) = (2, 4, 4)$,

$$\begin{aligned} G_{123(2, 4, 4)}^{(4)} &= P(X_1 \leq 2, 4 \leq X_2, X_3) \\ &= p_{144} + p_{145} + p_{154} + p_{155} + p_{244} + p_{245} + p_{254} + p_{255}. \end{aligned}$$

Moreover, we consider the following four models based on the cumulative probabilities described above: the S_1 model is defined by

$$G_{123(i,j,k)}^{(1)} = G_{k_1 k_2 k_3(i,j,k)}^{(1)} \quad \text{for } (k_1, k_2, k_3) \in D(1, 2, 3), (i, j, k) \in E_1,$$

the S_2 model is defined by

$$G_{123(i,j,k)}^{(2)} = G_{k_1 k_2 k_3(i,j,k)}^{(2)} \quad \text{for } (k_1, k_2, k_3) \in D(1, 2, 3), (i, j, k) \in E_1,$$

the S_3 model is defined by

$$G_{123(i,j,k)}^{(3)} = G_{k_1 k_2 k_3(i,j,k)}^{(3)} \quad \text{for } (k_1, k_2, k_3) \in D(1, 2, 3), (i, j, k) \in E_2,$$

and the S_4 model is defined by

$$G_{123(i,j,k)}^{(4)} = G_{k_1 k_2 k_3(i,j,k)}^{(4)} \quad \text{for } (k_1, k_2, k_3) \in D(1, 2, 3), (i, j, k) \in E_3.$$

Now we consider a $3 \times 3 \times 3$ table. If the S_1 , S_2 , S_3 and S_4 models hold, the following equations hold:

$$G_{123(1,2,3)}^{(1)} = G_{132(1,2,3)}^{(1)} = G_{213(1,2,3)}^{(1)} = G_{231(1,2,3)}^{(1)} = G_{312(1,2,3)}^{(1)} = G_{321(1,2,3)}^{(1)}, \quad (2)$$

$$G_{123(1,2,3)}^{(2)} = G_{132(1,2,3)}^{(2)} = G_{213(1,2,3)}^{(2)} = G_{231(1,2,3)}^{(2)} = G_{312(1,2,3)}^{(2)} = G_{321(1,2,3)}^{(2)}, \quad (3)$$

$$G_{123(1,1,3)}^{(3)} = G_{132(1,1,3)}^{(3)} = G_{231(1,1,3)}^{(3)}, \quad (4)$$

$$G_{123(1,1,2)}^{(3)} = G_{132(1,1,2)}^{(3)} = G_{231(1,1,2)}^{(3)}, \quad (5)$$

$$G_{123(2,2,3)}^{(3)} = G_{132(2,2,3)}^{(3)} = G_{231(2,2,3)}^{(3)}, \quad (6)$$

$$G_{123(1,3,3)}^{(4)} = G_{213(1,3,3)}^{(4)} = G_{312(1,3,3)}^{(4)}, \quad (7)$$

$$G_{123(1,2,2)}^{(4)} = G_{213(1,2,2)}^{(4)} = G_{312(1,2,2)}^{(4)}, \quad (8)$$

$$G_{123(2,3,3)}^{(4)} = G_{213(2,3,3)}^{(4)} = G_{312(2,3,3)}^{(4)}. \quad (9)$$

From (2) and (3), we can obtain

$$p_{123} = p_{132} = p_{213} = p_{312} = p_{231} = p_{321}.$$

[Note that when $r = 3$, (2) is identical to (3), however, when $r \geq 4$, (2) is different from (3).] Also, from (4) and (7), we can obtain

$$p_{113} = p_{131} = p_{311} \quad \text{and} \quad p_{133} = p_{313} = p_{331}.$$

Moreover, from (5) and (9), we can obtain

$$p_{112} = p_{121} = p_{211} \quad \text{and} \quad p_{233} = p_{323} = p_{332}.$$

Finally, from (6) and (8), we can obtain

$$p_{223} = p_{232} = p_{322} \quad \text{and} \quad p_{122} = p_{212} = p_{221}.$$

Therefore, in a $3 \times 3 \times 3$ table, the S model holds if and only if the S_1 , S_2 , S_3 and S_4 models hold. Generally, in an $r \times r \times r$ table, the S model holds if and only if the S_1 , S_2 , S_3 and S_4 models hold.

2.2. A measure of symmetry

First, we shall consider a measure to represent the degree of departure from each S_t ($t = 1, 2, 3, 4$). Secondly, from Section 2.1, we consider a measure to represent the degree of departure from S by considering the weighted average of the four measures which represent the degree of departure from S_1 , S_2 , S_3 and S_4 . Let for $t = 1, 2$,

$$\delta_t = \sum_{(j_1, j_2, j_3) \in E_1} \sum \sum \sum \left(\sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} G_{k_1 k_2 k_3}^{(t)}(j_1, j_2, j_3) \right),$$

for $t = 3$,

$$\delta_t = \sum_{(j_1, j_2, j_3) \in E_2} \sum \sum \sum \left(\sum_{\substack{(k_1, k_2, k_3) \in D(1, 2, 3) \\ k_1 < k_2}} G_{k_1 k_2 k_3}^{(t)}(j_1, j_2, j_3) \right),$$

and for $t = 4$,

$$\delta_t = \sum_{(j_1, j_2, j_3) \in E_3} \sum \sum \sum \left(\sum_{\substack{(k_1, k_2, k_3) \in D(1, 2, 3) \\ k_2 < k_3}} G_{k_1 k_2 k_3}^{(t)}(j_1, j_2, j_3) \right).$$

For example, for a $3 \times 3 \times 3$ table, we see

$$\delta_1 = \delta_2 = p_{123} + p_{132} + p_{213} + p_{312} + p_{231} + p_{321},$$

$$\delta_3 = (p_{113} + p_{131} + p_{311}) + (p_{112} + p_{121} + p_{211} + p_{113} + p_{131} + p_{311}) + (p_{113} + p_{131} + p_{311} + p_{223} + p_{232} + p_{322} + p_{123} + p_{132} + p_{213} + p_{312} + p_{231} + p_{321}),$$

and

$$\delta_4 = (p_{133} + p_{313} + p_{331}) + (p_{233} + p_{323} + p_{332} + p_{133} + p_{313} + p_{331}) + (p_{122} + p_{212} + p_{221} + p_{133} + p_{313} + p_{331} + p_{123} + p_{132} + p_{213} + p_{312} + p_{231} + p_{321}).$$

Consider a measure to represent the degree of departure from S_t ($t = 1, 2, 3, 4$) as follows:

$$\Psi_t^{(\lambda)} = \frac{\lambda(\lambda+1)}{6^\lambda - 1} I_t^{(\lambda)} \quad (t = 1, 2),$$

and

$$\Psi_t^{(\lambda)} = \frac{\lambda(\lambda+1)}{3^\lambda - 1} I_t^{(\lambda)} \quad (t = 3, 4),$$

where for $t = 1, 2$,

$$I_t^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{(j_1, j_2, j_3) \in E_1} \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \times \left\{ \left(\frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}}{G_{(j_1, j_2, j_3)}^{S[t]}} \right)^\lambda - 1 \right\},$$

for $t = 3$,

$$I_t^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{(j_1, j_2, j_3) \in E_2} \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \times \left\{ \left(\frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}}{G_{(j_1, j_2, j_3)}^{S[t]}} \right)^\lambda - 1 \right\},$$

for $t = 4$,

$$I_t^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{(j_1, j_2, j_3) \in E_3} \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_2 < k_3} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \times \left\{ \left(\frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}}{G_{(j_1, j_2, j_3)}^{S[t]}} \right)^\lambda - 1 \right\},$$

and

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} = \frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(t)}}{\delta_t} \quad (t = 1, 2, 3, 4),$$

$$G_{(j_1, j_2, j_3)}^{S[t]} = \frac{1}{6} \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \quad (t = 1, 2, 3, 4),$$

where the value at $\lambda = 0$ is taken to be the continuous limit as $\lambda \rightarrow 0$. Therefore, consider the measure to represent the degree of departure from S (i.e., from S_1 , S_2 , S_3 and S_4 holding simultaneously) as follows:

$$\Psi_S^{(\lambda)} = \frac{\delta_1 \Psi_1^{(\lambda)} + \delta_2 \Psi_2^{(\lambda)} + \delta_3 \Psi_3^{(\lambda)} + \delta_4 \Psi_4^{(\lambda)}}{\delta_1 + \delta_2 + \delta_3 + \delta_4} \quad \text{for } \lambda > -1.$$

The measure $\Psi_S^{(\lambda)}$ would represent the weighted average of $\Psi_1^{(\lambda)}$, $\Psi_2^{(\lambda)}$, $\Psi_3^{(\lambda)}$ and $\Psi_4^{(\lambda)}$, which represent the degree of departure from S_1 , S_2 , S_3 and S_4 , respectively. When $\lambda = 0$, for $t = 1, 2$,

$$\Psi_t^{(0)} = \frac{1}{\log 6} I_t^{(0)},$$

for $t = 3, 4$,

$$\Psi_t^{(0)} = \frac{1}{\log 3} I_t^{(0)},$$

with for $t = 1, 2$,

$$I_t^{(0)} = \sum_{(j_1, j_2, j_3) \in E_1} \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \log \left(\frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}}{G_{(j_1, j_2, j_3)}^{S[t]}} \right),$$

for $t = 3$,

$$I_t^{(0)} = \sum_{(j_1, j_2, j_3) \in E_2} \sum_{\substack{(k_1, k_2, k_3) \in D(1, 2, 3) \\ k_1 < k_2}} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \log \left(\frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}}{G_{(j_1, j_2, j_3)}^{S[t]}} \right),$$

and for $t = 4$,

$$I_t^{(0)} = \sum_{(j_1, j_2, j_3) \in E_3} \sum_{\substack{(k_1, k_2, k_3) \in D(1, 2, 3) \\ k_2 < k_3}} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)} \log \left(\frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}}{G_{(j_1, j_2, j_3)}^{S[t]}} \right).$$

Note that $I_t^{(\lambda)}$ ($t = 1, 2, 3, 4$) is the power-divergence between $\{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{*(t)}\}$ and $\{G_{(j_1, j_2, j_3)}^{S[t]}\}$, and especially when $\lambda = 0$, $I_t^{(0)}$ is the Kullback-Leibler information between them. For details of power-divergence, see Cressie and Read (1984) and Read and Cressie (1988, p.15). Note that the value of λ should be chosen by the user.

Next, we shall consider another expression of measure $\Psi_S^{(\lambda)}$ below. For $(k_1, k_2, k_3) \in D(1, 2, 3)$, let for $(j_1, j_2, j_3) \in E_1$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} = \frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(t)}}{\sum_{(l_1, l_2, l_3) \in D(1, 2, 3)} G_{l_1 l_2 l_3(j_1, j_2, j_3)}^{(t)}} \quad (t = 1, 2),$$

for $(j_1, j_2, j_3) \in E_2$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} = \frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(t)}}{\sum_{\substack{(l_1, l_2, l_3) \in D(1, 2, 3) \\ l_1 < l_2}} G_{l_1 l_2 l_3(j_1, j_2, j_3)}^{(t)}} \quad (t = 3),$$

and for $(j_1, j_2, j_3) \in E_3$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} = \frac{G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(t)}}{\sum_{\substack{(l_1, l_2, l_3) \in D(1, 2, 3) \\ l_2 < l_3}} G_{l_1 l_2 l_3(j_1, j_2, j_3)}^{(t)}} \quad (t = 4).$$

For example, for a $3 \times 3 \times 3$ table, we see

$$G_{123(123)}^{c(1)} = \frac{G_{123(123)}^{(1)}}{G_{123(123)}^{(1)} + G_{132(123)}^{(1)} + G_{213(123)}^{(1)} + G_{312(123)}^{(1)} + G_{231(123)}^{(1)} + G_{321(123)}^{(1)}},$$

$$G_{123(223)}^{c(3)} = \frac{G_{123(223)}^{(3)}}{G_{123(223)}^{(3)} + G_{132(223)}^{(3)} + G_{231(223)}^{(3)}},$$

and

$$G_{123(233)}^{c(4)} = \frac{G_{123(233)}^{(4)}}{G_{123(233)}^{(4)} + G_{213(233)}^{(4)} + G_{312(233)}^{(4)}}.$$

Then the $\Psi_t^{(\lambda)}$ may be expressed as for $t = 1, 2$,

$$\Psi_t^{(\lambda)} = \sum_{(j_1, j_2, j_3) \in E_1} \sum_{(l_1, l_2, l_3) \in D(1, 2, 3)} \sum_{l_1 < l_2} \sum_{l_1 < l_3} G_{l_1 l_2 l_3(j_1, j_2, j_3)}^{*(t)} \left(1 - \frac{\lambda 6^\lambda}{6^\lambda - 1} H_{t(j_1, j_2, j_3)}^{(\lambda)} \right),$$

for $t = 3$,

$$\Psi_t^{(\lambda)} = \sum_{(j_1, j_2, j_3) \in E_2} \sum_{(l_1, l_2, l_3) \in D(1, 2, 3)} \sum_{l_1 < l_2} \sum_{l_1 < l_3} G_{l_1 l_2 l_3(j_1, j_2, j_3)}^{*(t)} \left(1 - \frac{\lambda 3^\lambda}{3^\lambda - 1} H_{t(j_1, j_2, j_3)}^{(\lambda)} \right),$$

and for $t = 4$,

$$\Psi_t^{(\lambda)} = \sum_{(j_1, j_2, j_3) \in E_3} \sum_{(l_1, l_2, l_3) \in D(1, 2, 3)} \sum_{l_2 < l_3} \sum_{l_1 < l_2} G_{l_1 l_2 l_3(j_1, j_2, j_3)}^{*(t)} \left(1 - \frac{\lambda 3^\lambda}{3^\lambda - 1} H_{t(j_1, j_2, j_3)}^{(\lambda)} \right),$$

where the value at $\lambda = 0$ is taken to be the continuous limit as $\lambda \rightarrow 0$, and with for $t = 1, 2$,

$$H_{t(j_1, j_2, j_3)}^{(\lambda)} = \frac{1}{\lambda} \left\{ 1 - \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} \sum_{k_1 < k_3} \left(G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \right)^{\lambda+1} \right\} \quad (\lambda \neq 0),$$

$$H_{t(j_1, j_2, j_3)}^{(0)} = - \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} \sum_{k_1 < k_3} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \log \left(G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \right),$$

for $t = 3$,

$$H_{t(j_1, j_2, j_3)}^{(\lambda)} = \frac{1}{\lambda} \left\{ 1 - \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} \sum_{k_1 < k_3} \left(G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \right)^{\lambda+1} \right\} \quad (\lambda \neq 0),$$

$$H_{t(j_1, j_2, j_3)}^{(0)} = - \sum_{(k_1, k_2, k_3) \in D(1, 2, 3)} \sum_{k_1 < k_2} \sum_{k_1 < k_3} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \log \left(G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \right),$$

and for $t = 4$,

$$H_{t(j_1, j_2, j_3)}^{(\lambda)} = \frac{1}{\lambda} \left\{ 1 - \sum_{\substack{(k_1, k_2, k_3) \in D(1, 2, 3) \\ k_2 < k_3}} \left(G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \right)^{\lambda+1} \right\} \quad (\lambda \neq 0),$$

$$H_{t(j_1, j_2, j_3)}^{(0)} = - \sum_{\substack{(k_1, k_2, k_3) \in D(1, 2, 3) \\ k_2 < k_3}} G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \log \left(G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} \right).$$

Note that $H_{t(j_1, j_2, j_3)}^{(\lambda)}$ is the Patil and Taillie (1982) diversity index for $G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)}$, $(k_1, k_2, k_3) \in D(1, 2, 3)$, including the Shannon entropy when $\lambda = 0$. For a $3 \times 3 \times 3$ table, for example,

$$H_{1(123)}^{(\lambda)} = \frac{1}{\lambda} \left[1 - \left(G_{123(123)}^{c(1)} \right)^{\lambda+1} - \left(G_{132(123)}^{c(1)} \right)^{\lambda+1} - \left(G_{213(123)}^{c(1)} \right)^{\lambda+1} \right. \\ \left. - \left(G_{312(123)}^{c(1)} \right)^{\lambda+1} - \left(G_{231(123)}^{c(1)} \right)^{\lambda+1} - \left(G_{321(123)}^{c(1)} \right)^{\lambda+1} \right] \quad (\lambda \neq 0),$$

$$H_{1(123)}^{(0)} = -G_{123(123)}^{c(1)} \log G_{123(123)}^{c(1)} - G_{132(123)}^{c(1)} \log G_{132(123)}^{c(1)} \\ - G_{213(123)}^{c(1)} \log G_{213(123)}^{c(1)} - G_{312(123)}^{c(1)} \log G_{312(123)}^{c(1)} \\ - G_{231(123)}^{c(1)} \log G_{231(123)}^{c(1)} - G_{321(123)}^{c(1)} \log G_{321(123)}^{c(1)}.$$

$$H_{3(223)}^{(\lambda)} = \frac{1}{\lambda} \left[1 - \left(G_{123(223)}^{c(3)} \right)^{\lambda+1} - \left(G_{132(223)}^{c(3)} \right)^{\lambda+1} - \left(G_{231(223)}^{c(3)} \right)^{\lambda+1} \right] \quad (\lambda \neq 0),$$

$$H_{3(223)}^{(0)} = -G_{123(223)}^{c(3)} \log G_{123(223)}^{c(3)} - G_{132(223)}^{c(3)} \log G_{132(223)}^{c(3)} - G_{231(223)}^{c(3)} \log G_{231(223)}^{c(3)}.$$

We note that

$$0 \leq H_{t(j_1, j_2, j_3)}^{(\lambda)} \leq M^{(\lambda)},$$

where

$$M^{(\lambda)} = \frac{6^\lambda - 1}{\lambda 6^\lambda} \quad \text{for any } (j_1, j_2, j_3) \in E_1 \quad \text{when } t = 1, 2,$$

$$M^{(\lambda)} = \frac{3^\lambda - 1}{\lambda 3^\lambda} \quad \text{for any } (j_1, j_2, j_3) \in E_{t-1} \quad \text{when } t = 3, 4.$$

The S model holds if and only if $H_{t(j_1, j_2, j_3)}^{(\lambda)} = M^{(\lambda)}$ for any (j_1, j_2, j_3) and t ($t = 1, 2, 3, 4$) and the degree of departure from S is the largest if and only if $H_{t(j_1, j_2, j_3)}^{(\lambda)} = 0$ for any (j_1, j_2, j_3) and t ($t = 1, 2, 3, 4$). When $\lambda = 0$,

$$0 \leq H_{t(j_1, j_2, j_3)}^{(0)} \leq M^{(0)},$$

where

$$M^{(0)} = \log 6 \quad \text{for any } (j_1, j_2, j_3) \in E_1 \quad \text{when } t = 1, 2,$$

$$M^{(0)} = \log 3 \quad \text{for any } (j_1, j_2, j_3) \in E_{t-1} \quad \text{when } t = 3, 4.$$

For a $3 \times 3 \times 3$ table, for example,

$$0 \leq H_{1(123)}^{(0)} \leq \log 6,$$

$$0 \leq H_{3(223)}^{(0)} \leq \log 3.$$

When the S model holds, $G_{123(123)}^{c(1)} = G_{132(123)}^{c(1)} = G_{213(123)}^{c(1)} = G_{312(123)}^{c(1)} = G_{231(123)}^{c(1)} = G_{321(123)}^{c(1)} = 1/6$, and $G_{123(223)}^{c(3)} = G_{132(223)}^{c(3)} = G_{231(223)}^{c(3)} = 1/3$, and so on; thus $H_{1(123)}^{(0)} = \log 6$ and $H_{3(223)}^{(0)} = \log 3$. When the degree of departure from S is the largest, $H_{1(123)}^{(0)} = 0$ and $H_{3(223)}^{(0)} = 0$, and so on; namely, e.g., $G_{123(123)}^{c(1)} = 1$ and $G_{132(123)}^{c(1)} = G_{213(123)}^{c(1)} = G_{312(123)}^{c(1)} = G_{231(123)}^{c(1)} = G_{321(123)}^{c(1)} = 0$, and $G_{123(223)}^{c(3)} = 1$ and $G_{132(223)}^{c(3)} = G_{231(223)}^{c(3)} = 0$, and so on.

Thus for $t = 1, 2, 3, 4$, $\Psi_t^{(\lambda)}$ must lie between 0 and 1, namely, $\Psi_S^{(\lambda)}$ must lie between 0 and 1. We note that (i) $\Psi_t^{(\lambda)} = 0$ (every $t = 1, 2, 3, 4$), i.e., $\Psi_S^{(\lambda)} = 0$ if and only if the S model holds, and (ii) $\Psi_t^{(\lambda)} = 1$ (every $t = 1, 2, 3, 4$), i.e., $\Psi_S^{(\lambda)} = 1$ if and only if the degree of departure from S is the largest in the sense that, for $t = 1, 2$ and $(j_1, j_2, j_3) \in E_1$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{c(t)} = \begin{cases} 1 & \text{for } (k_1, k_2, k_3) = (l_0, m_0, n_0) \\ & \text{with one } (l_0, m_0, n_0) \in D(1, 2, 3), \\ 0 & \text{for otherwise,} \end{cases}$$

namely,

$$G_{l_0 m_0 n_0(j_1, j_2, j_3)}^{(t)} \neq 0 \quad \text{and} \quad G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(t)} = 0 \quad \text{for } (k_1, k_2, k_3) \neq (l_0, m_0, n_0),$$

and it is defined in a similar way for $t = 3, 4$ and $(j_1, j_2, j_3) \in E_{t-1}$. For example, for a $3 \times 3 \times 3$ table, (i) $\Psi_1^{(\lambda)} = 1$ indicates that one of $G_{123(123)}^{(1)}$, $G_{132(123)}^{(1)}$, $G_{213(123)}^{(1)}$, $G_{312(123)}^{(1)}$, $G_{231(123)}^{(1)}$ and $G_{321(123)}^{(1)}$ is not zero and the others are all zeros, and (ii) $\Psi_3^{(\lambda)} = 1$ indicates that one of $G_{123(112)}^{(3)}$, $G_{132(112)}^{(3)}$ and $G_{231(112)}^{(3)}$ is not zero and the others are zeros, one of $G_{123(113)}^{(3)}$, $G_{132(113)}^{(3)}$ and $G_{231(113)}^{(3)}$ is not zero and the others are zeros, and one of $G_{123(223)}^{(3)}$, $G_{132(223)}^{(3)}$ and $G_{231(223)}^{(3)}$ is not zero and the others are zeros.

According to the Patil-Taillie diversity index or the power-divergence, the degree of departure from S increases as the value of $\Psi_S^{(\lambda)}$ increases.

3. Case of multi-way tables

Consider an r^T ($T \geq 3$) contingency table with ordered categories. Let $p_{i_1 \dots i_T}$ denote the probability that an observation will fall in the $(i_1, \dots, i_T)$ -th cell of the table ($i_t = 1, \dots, r; t = 1, \dots, T$). Denote the s th variable by X_s ($s = 1, \dots, T$).

3.1. Symmetry model for multi-way tables

The symmetry (S) model (Agresti, 2002, p.440) is defined by

$$p_{i_1 \dots i_T} = p_{j_1 \dots j_T}$$

for $i_t = 1, \dots, r; t = 1, \dots, T$, and $(j_1, \dots, j_T) \in D(i_1, \dots, i_T)$, where

$$D(i_1, \dots, i_T) = \{(j_1, \dots, j_T) \mid (j_1, \dots, j_T) \text{ is any permutation of } (i_1, \dots, i_T)\}.$$

For a fixed s ($s = 2, \dots, T$), let

$$E(r_1, \dots, r_s) = \{(j_1, \dots, j_T) \mid 1 \leq j_1 \leq \dots \leq j_T \leq r; \\ j_1 = \dots = j_{r_1} < j_{r_1+1} = \dots = j_{r_1+r_2} < \dots < j_{r_1+\dots+r_{s-1}+1} = \dots = j_T\},$$

with

$$r_1 + \dots + r_s = T \text{ and } 1 \leq r_i \leq T - 1.$$

For example, when $T = 3$, for $s = 3$,

$$E(1, 1, 1) = \{(j_1, j_2, j_3) \mid 1 \leq j_1 < j_2 < j_3 \leq r\},$$

and for $s = 2$,

$$E(2, 1) = \{(j_1, j_2, j_3) \mid 1 \leq j_1 = j_2 < j_3 \leq r\},$$

$$E(1, 2) = \{(j_1, j_2, j_3) \mid 1 \leq j_1 < j_2 = j_3 \leq r\}.$$

Note that $E(1, 1, 1)$, $E(2, 1)$ and $E(1, 2)$ are identical to E_1 , E_2 and E_3 defined in Section 2.1, respectively. We shall consider an extension of the cumulative probabilities defined in Section 2.1 to multi-way tables.

For $(k_1, \dots, k_T) \in D(1, \dots, T)$, and a fixed s ($s = 2, \dots, T$), l ($l = 1, \dots, s - 1$), $(j_1, \dots, j_T) \in E(r_1, \dots, r_s)$, let

$$G_{k_1 \dots k_T(j_1, \dots, j_T)}^{(r_1 + \dots + r_l)} = P(X_{k_1}, \dots, X_{k_{r_1}} \leq j_{r_1} < X_{k_{r_1+1}}, \dots, X_{k_{r_1+r_2}} \leq j_{r_1+r_2} < \\ X_{k_{r_1+r_2+1}}, \dots, X_{k_{r_1+r_2+r_3}} \leq j_{r_1+r_2+r_3} < \dots, X_{k_{r_1+\dots+r_l}} \leq j_{r_1+\dots+r_l}, \\ j_{r_1+\dots+r_{l+1}} \leq X_{k_{r_1+\dots+r_{l+1}}}, \dots, X_{k_{r_1+\dots+r_{l+1}}} < j_{r_1+\dots+r_{l+2}} \leq \\ X_{k_{r_1+\dots+r_{l+1}+1}}, \dots < j_T \leq X_{k_{r_1+\dots+r_{s-1}+1}}, \dots, X_{k_T}).$$

Note that the $(r_1 + \dots + r_l)$ variables $X_{k_1}, X_{k_2}, \dots, X_{k_{r_1+\dots+r_l}}$ are less than or equal to $j_{r_1+\dots+r_l}$, and $T - (r_1 + \dots + r_l)$ variables $X_{k_{r_1+\dots+r_{l+1}}}, \dots, X_{k_T}$ are greater than or equal to $j_{r_1+\dots+r_{l+1}}$. For example, when $T = 3$, (i) for $s = 3, l = 2, (j_1, j_2, j_3) \in E(1, 1, 1)$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(2)} = P(X_{k_1} \leq j_1 < X_{k_2} \leq j_2, j_3 \leq X_{k_3}), \quad (10)$$

(ii) for $s = 3, l = 1, (j_1, j_2, j_3) \in E(1, 1, 1)$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(1)} = P(X_{k_1} \leq j_1, j_2 \leq X_{k_2} < j_3 \leq X_{k_3}), \quad (11)$$

(iii) for $s = 2, l = 1, (j_1, j_2, j_3) \in E(2, 1)$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(2)} = P(X_{k_1}, X_{k_2} \leq j_2, j_3 \leq X_{k_3}), \quad (12)$$

and (iv) for $s = 2, l = 1, (j_1, j_2, j_3) \in E(1, 2)$,

$$G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(1)} = P(X_{k_1} \leq j_1, j_2 \leq X_{k_2}, X_{k_3}). \quad (13)$$

Note that (10), (11), (12) and (13) are identical to $G_{k_1 k_2 k_3(j_1, j_2, j_3)}^{(t)}$ ($t = 1, 2, 3, 4$) defined in Section 2.1, respectively.

In a similar way to Section 2.1, we consider the following model: for fixed s ($s = 2, \dots, T$), l ($l = 1, \dots, s-1$) and $(j_1, \dots, j_T) \in E(r_1, \dots, r_s)$,

$$G_{1\dots T(j_1, \dots, j_T)}^{(r_1+\dots+r_l)} = G_{k_1\dots k_T(j_1, \dots, j_T)}^{(r_1+\dots+r_l)}, \quad (14)$$

where $(k_1, \dots, k_T) \in D(1, \dots, T)$. We shall refer to (14) as the $S_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]}$ model. Note that when $T = 3$, $S_{(1,1,1)}^{[2]}$, $S_{(1,1,1)}^{[1]}$, $S_{(2,1)}^{[2]}$ and $S_{(1,2)}^{[1]}$ models are identical to S_1 , S_2 , S_3 and S_4 defined in Section 2.1, respectively. In a similar way to Section 2.1, we point out that the S model holds if and only if any of the $S_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]}$ models hold.

3.2. A measure of symmetry for multi-way tables

We shall extend the measure proposed in Section 2.2 to multi-way tables. In a similar way to Section 2.2, we consider the measure to represent the degree of departure from S by considering the weighted average of the measures, which represent the degree of departure from each $S_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]}$.

Let

$$\delta_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]} = \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} \dots \sum G_{k_1\dots k_T(j_1, \dots, j_T)}^{[r_1+\dots+r_l]},$$

$$\delta = \sum_{(r_1, \dots, r_s) \in S^*} \sum_{l=1}^{s-1} \delta_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]},$$

$$D^*(r_1, \dots, r_s) = \{(k_1, \dots, k_T) \mid (k_1, \dots, k_T) \text{ is any permutation } (1, \dots, T) \text{ with } k_1 < \dots < k_{r_1}, k_{r_1+1} < \dots < k_{r_1+r_2}, \dots, k_{r_1+\dots+r_{s-1}} < \dots < k_T\},$$

and

$$S^* = \{(r_1, \dots, r_s) \mid s = 2, \dots, T, r_1 + \dots + r_s = T, 1 \leq r_i \leq T-1 \ (i = 1, \dots, s)\}.$$

Consider a measure to represent the degree of departure from $S_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]}$ model as follows: for $(r_1, \dots, r_s) \in S^*$ and l ($l = 1, \dots, s-1$),

$$\Psi_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l](\lambda)} = \frac{\lambda(\lambda+1)}{(T C_{r_1\dots r_s})^\lambda - 1} I_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l](\lambda)},$$

where

$$I_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l](\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} \dots \sum G_{k_1\dots k_T(j_1, \dots, j_T)}^{*[r_1+\dots+r_l]} \times \left\{ \left(\frac{G_{k_1\dots k_T(j_1, \dots, j_T)}^{[r_1+\dots+r_l]}}{G_{(j_1, \dots, j_T)}^{S[r_1+\dots+r_l]}} \right)^\lambda - 1 \right\},$$

with

$$G_{k_1\dots k_T(j_1, \dots, j_T)}^{*[r_1+\dots+r_l]} = \frac{G_{k_1\dots k_T(j_1, \dots, j_T)}^{[r_1+\dots+r_l]}}{\delta_{(r_1, \dots, r_s)}^{[r_1+\dots+r_l]}},$$

$$G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]} = \frac{1}{T!} \left(\sum_{(k_1, \dots, k_T) \in D(1, \dots, T)} \dots \sum_{k_1 \dots k_T(j_1, \dots, j_T)} G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]} \right),$$

$${}^T C_{r_1 \dots r_s} = \frac{T!}{r_1! \dots r_s!},$$

and the value at $\lambda = 0$ is taken to be the continuous limit as $\lambda \rightarrow 0$. Consider the measure to represent the degree of departure from S (i.e., from $\{S_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]}\}$ models) as follows:

$$\Psi_S^{(\lambda)} = \frac{1}{\delta} \sum_{(r_1, \dots, r_s) \in S^*} \dots \sum_{l=1}^{s-1} \left(\sum_{(r_1, \dots, r_s)} \delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]} \Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)} \right) \quad \text{for } \lambda > -1.$$

Note that when $T = 3$,

$$\Psi_S^{(\lambda)} = \frac{1}{\delta} \left(\delta_{(1,1,1)}^{[2]} \Psi_{(1,1,1)}^{[2](\lambda)} + \delta_{(1,1,1)}^{[1]} \Psi_{(1,1,1)}^{[1](\lambda)} + \delta_{(2,1)}^{[2]} \Psi_{(2,1)}^{[2](\lambda)} + \delta_{(1,2)}^{[1]} \Psi_{(1,2)}^{[1](\lambda)} \right)$$

corresponds to $\Psi_S^{(\lambda)}$ defined in Section 2.2. When $\lambda = 0$, $\Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)}$ is

$$\begin{aligned} \Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](0)} &= \frac{1}{\log({}^T C_{r_1 \dots r_s})} \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \dots \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]} \\ &\quad \times \log \left(\frac{G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]}}{G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]}} \right). \end{aligned}$$

Next, we shall consider another expression of measure. Let for $(j_1, \dots, j_T) \in E(r_1, \dots, r_s)$ and $(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)$,

$$G_{k_1 \dots k_T(j_1, \dots, j_T)}^{e[r_1 + \dots + r_l]} = \frac{G_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]}}{\sum_{(l_1, \dots, l_T) \in D^*(r_1, \dots, r_s)} \dots \sum_{l_1 \dots l_T(j_1, \dots, j_T)} G_{l_1 \dots l_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]}}.$$

Then, for $(r_1, \dots, r_s) \in S^*$, we also see

$$\begin{aligned} \Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)} &= \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \dots \sum_{(l_1, \dots, l_T) \in D^*(r_1, \dots, r_s)} \left(\sum_{l_1 \dots l_T(j_1, \dots, j_T)} G_{l_1 \dots l_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]} \right) \\ &\quad \times \left[1 - \frac{\lambda ({}^T C_{r_1 \dots r_s})^\lambda}{({}^T C_{r_1 \dots r_s})^\lambda - 1} H_{(j_1, \dots, j_T)}^{[r_1 + \dots + r_l](\lambda)} \right], \end{aligned}$$

where

$$H_{(j_1, \dots, j_T)}^{[r_1 + \dots + r_l](\lambda)} = \frac{1}{\lambda} \left\{ 1 - \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} \dots \sum_{k_1 \dots k_T(j_1, \dots, j_T)} \left(G_{k_1 \dots k_T(j_1, \dots, j_T)}^{e[r_1 + \dots + r_l]} \right)^{\lambda+1} \right\}.$$

For $(r_1, \dots, r_s) \in S^*$, $\Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)}$ ($l = 1, \dots, s-1$) must lie between 0 and 1, and therefore $\Psi_S^{(\lambda)}$ must lie between 0 and 1. Also, (i) there is a structure of S in the r^T table if

and only if $\Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)} = 0$ for all $(r_1, \dots, r_s) \in S^*$, $l = 1, \dots, s-1$, i.e., $\Psi_S^{(\lambda)} = 0$, and (ii) the degree of departure from S is the largest in the sense that, for $(j_1, \dots, j_T) \in E(r_1, \dots, r_s)$, $(r_1, \dots, r_s) \in S^*$, $l = 1, \dots, s-1$, and $(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)$,

$$G_{k_1 \dots k_T(j_1, \dots, j_T)}^{c[r_1 + \dots + r_l]} = \begin{cases} 1 & \text{for } (k_1, \dots, k_T) = (k_1^0, \dots, k_T^0) \\ & \text{with one } (k_1^0, \dots, k_T^0) \in D^*(r_1, \dots, r_s), \\ 0 & \text{for } (k_1, \dots, k_T) \neq (k_1^0, \dots, k_T^0), \end{cases}$$

if and only if $\Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)} = 1$ for all $(r_1, \dots, r_s) \in S^*$, $l = 1, \dots, s-1$, i.e., $\Psi_S^{(\lambda)} = 1$.

4. Approximate confidence interval for measure

Let $n_{i_1 \dots i_T}$ denote the observed frequency in the $(i_1, \dots, i_T)$ -th cell of the r^T table ($i_t = 1, \dots, r$; $t = 1, \dots, T$) with $T \geq 3$. Assuming that the $\{n_{i_1 \dots i_T}\}$ result from full multinomial sampling, we consider an approximate standard error and large-sample confidence interval for $\Psi_S^{(\lambda)}$, using the delta method, descriptions of which are given by Bishop *et al.* (1975, Sec.14.6) and Agresti (1984, p.185). The sample version of $\Psi_S^{(\lambda)}$, $\hat{\Psi}_S^{(\lambda)}$, is given by $\Psi_S^{(\lambda)}$ with $\{p_{i_1 \dots i_T}\}$ replaced by $\{\hat{p}_{i_1 \dots i_T}\}$, where $\hat{p}_{i_1 \dots i_T} = n_{i_1 \dots i_T}/n$ and $n = \sum \sum \dots \sum n_{i_1 \dots i_T}$. Using the delta method, $\sqrt{n}(\hat{\Psi}_S^{(\lambda)} - \Psi_S^{(\lambda)})$ has asymptotically (as $n \rightarrow \infty$) a normal distribution with mean zero and variance $\sigma^2[\Psi_S^{(\lambda)}]$. See Appendix 3 for the expression for $\sigma^2[\Psi_S^{(\lambda)}]$.

Let $\hat{\sigma}^2[\Psi_S^{(\lambda)}]$ denote $\sigma^2[\Psi_S^{(\lambda)}]$ with $\{p_{i_1 \dots i_T}\}$ replaced by $\{\hat{p}_{i_1 \dots i_T}\}$. Then $\hat{\sigma}[\Psi_S^{(\lambda)}]/\sqrt{n}$ is an estimated standard error of $\hat{\Psi}_S^{(\lambda)}$, and $\hat{\Psi}_S^{(\lambda)} \pm z_{p/2}\hat{\sigma}[\Psi_S^{(\lambda)}]/\sqrt{n}$ is an approximate $100(1-p)\%$ confidence interval for $\Psi_S^{(\lambda)}$, where $z_{p/2}$ is the percentage point from the standard normal distribution corresponding to a two-tail probability equal to p .

5. Comparison between measures

We shall compare our measure $\Psi_S^{(\lambda)}$ with $\Phi^{(\lambda)}$ which is considered by Yamamoto (2004) (see Appendix 2 for $\Phi^{(\lambda)}$).

First, consider the artificial data in Table 1a, and the modified data in Table 1b which are obtained by interchanging categories 1 and 2. We see from Table 1c that (i) for each λ , the value of $\hat{\Psi}_S^{(\lambda)}$ for Table 1a is different from that for Table 1b, but (ii) for each λ , the value of $\hat{\Phi}^{(\lambda)}$ for Table 1a is equal to that for Table 1b. Generally, (i) the measure $\Psi_S^{(\lambda)}$ is not invariant under arbitrary similar permutations of row and column categories except the reverse order, but (ii) the measure $\Phi^{(\lambda)}$ is invariant under them. Thus $\Psi_S^{(\lambda)}$ should be used for the data on an ordinal scale and $\Phi^{(\lambda)}$ should be used for the data on a nominal scale.

Secondly, we shall compare the cases of $\Psi_S^{(\lambda)} = 1$ and $\Phi^{(\lambda)} = 1$ in three-way tables. As shown in Section 2.2, $\Psi_S^{(\lambda)} = 1$ indicates that the degree of departure from S is the largest. On the other hand, $\Phi^{(\lambda)} = 1$ indicates that, for $(l, m, n) \in D(i, j, k)$, $i, j, k = 1, \dots, r$, $t = 1, 2$,

$$p_{lmn[t]}^c = \begin{cases} 1 & \text{for } (l, m, n) = (l_0, m_0, n_0) \\ & \text{with a cell } (l_0, m_0, n_0) \in D(i, j, k), \\ 0 & \text{for } (l, m, n) \neq (l_0, m_0, n_0), \end{cases}$$

where

$$p_{ijk[1]}^c = \frac{p_{ijk}}{p_{ijk} + p_{ikj} + p_{jik} + p_{jki} + p_{kij} + p_{kji}} \quad \text{for } (i, j, k) \in E_1,$$

and

$$p_{ijk[2]}^c = \frac{2p_{ijk}}{p_{ijk} + p_{ikj} + p_{jik} + p_{jki} + p_{kij} + p_{kji}} \quad \text{for } (i, j, k) \in E_2 \text{ or } E_3.$$

Assuming that for each (i, j, k) with $1 \leq i \leq j \leq k \leq r$, $\sum \sum \sum_{(l, m, n) \in D(i, j, k)} p_{lmn} > 0$, $\Psi_S^{(\lambda)} = 1$ implies $\Phi^{(\lambda)} = 1$; however, $\Phi^{(\lambda)} = 1$ does not always imply $\Psi_S^{(\lambda)} = 1$. The definition of the maximum departure from S (i.e. complete asymmetry) for the measure $\Psi_S^{(\lambda)}$, generally, depends on the ordering of the categories. However, the definition of complete asymmetry for the measure $\Phi^{(\lambda)}$ does not depend on the ordering of the categories. For example, consider the artificial data in Table 2a, and the modified data in Table 2b which are obtained by interchanging categories 2 and 3. Then we see from Table 2c that (i) for each λ , the value of $\hat{\Psi}_S^{(\lambda)}$ for Table 2a equals 1 (being the maximum value) but that for Table 2b it is less than 1, and (ii) for each λ , the value of $\hat{\Phi}^{(\lambda)}$ for Table 2a equals 1 (being the maximum value) and it also equals 1 for Table 2b. These indicate that the definition of the maximum departure from S for $\Psi_S^{(\lambda)}$ depends on the order of listing the categories, but for $\Phi^{(\lambda)}$ the definition does not depend on it.

When the measures are used for analyzing the data on an ordinal scale, $\Phi^{(\lambda)}$ does not use the information about the category ordering. Therefore, for the data on an ordinal scale, the measure $\Psi_S^{(\lambda)}$ rather than $\Phi^{(\lambda)}$ should be used when one wants to use the information about that ordering.

6. Examples

6.1. Example 1

Consider the data in Table 3 from the Meteorological Agency in Japan. These are obtained from the daily temperature at three places, Sapporo, Tokyo and Naha in Japan in 2001 and in 2005, using three levels, (1) below normals, (2) normals and (3) above normals. Table 4 gives the estimates of the measure $\Psi_S^{(\lambda)}$, the estimated approximate standard errors for $\hat{\Psi}_S^{(\lambda)}$ and the approximate 95% confidence intervals for $\Psi_S^{(\lambda)}$. Because the confidence intervals for $\Psi_S^{(\lambda)}$ applied to each of Tables 3a and 3b do not include zero for any λ (> -1) (see Table 4), these would indicate that there is not a structure of symmetry in each of Tables 3a and 3b.

In each of Tables 3a and 3b, the symmetry model indicates that the probability that the temperatures in (Sapporo, Tokyo, Naha) are (1, 2, 3) equals the probability that they are (1, 3, 2), the probability that they are (2, 1, 3), the probability that they are (3, 1, 2), the probability that they are (2, 3, 1), and the probability that they are (3, 2, 1), and so on. So, in such a sense, there is the structure of symmetry among the temperatures at Sapporo, Tokyo and Naha if the symmetry model holds; however, there is a difference among the temperatures at three points if the symmetry model does not hold. Now since there is not a structure of symmetry in each of Tables 3a and 3b, there is a difference between the temperatures at Sapporo, Tokyo and Naha in each of 2001 and 2005.

When the degrees of asymmetry in Tables 3a and 3b are compared using the confidence interval for $\Psi_S^{(\lambda)}$, that is greater for Table 3a than Table 3b. Therefore, we could estimate that the difference among the temperatures at three places in 2001 is greater than that in 2005.

Let $W^{(\lambda)}$ denote the power-divergence statistic for testing goodness-of-fit of the symmetry model for a three-way contingency table (see Appendix 4 for $W^{(\lambda)}$, and see Cressie

Table 1: (a) Artificial data; (b) modified data obtained by interchanging categories 1 and 2; and (c) corresponding values of $\hat{\Psi}_S^{(\lambda)}$ and $\hat{\Phi}^{(\lambda)}$.

(a) Artificial data

X_1	X_3								
	(1)			(2)			(3)		
	X_2			X_2			X_2		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	83	52	37	77	15	32	52	44	39
(2)	38	21	58	68	81	40	30	35	57
(3)	59	43	10	25	29	17	40	82	70

(b) Modified data

X_1	X_3								
	(1)			(2)			(3)		
	X_2			X_2			X_2		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	81	68	40	21	38	58	35	30	57
(2)	15	77	32	52	83	37	44	52	39
(3)	29	25	17	43	59	10	82	40	70

(c) Values of $\hat{\Psi}_S^{(\lambda)}$ and $\hat{\Phi}^{(\lambda)}$

λ	For Table 1a		For Table 1b	
	$\hat{\Psi}_S^{(\lambda)}$	$\hat{\Phi}^{(\lambda)}$	$\hat{\Psi}_S^{(\lambda)}$	$\hat{\Phi}^{(\lambda)}$
-0.4	0.023	0.053	0.030	0.053
0	0.028	0.067	0.038	0.067
0.6	0.028	0.072	0.040	0.072
1.0	0.026	0.069	0.037	0.069
1.6	0.021	0.059	0.031	0.059

and Read, 1984, and Read and Cressie, 1988, p.15, for details of the power-divergence test statistic). From the values of $W^{(\lambda)}$ in Table 5, we see that the symmetry model fits each of Tables 3a and 3b very poorly.

6.2. Example 2

The data in Table 6, taken directly from Bishop *et al.* (1975, p.305), give the $3 \times 3 \times 3$ array of counts of stationary two-step transitions in the panel survey of potential voters in Erie County, Ohio, 1940, which summarize the voting intentions of the 1940 presidential elections. Although the voter's supportive political party was classified into Republican, Democrat, and Undecided, we regard the voters with 'Undecided' as the middle class which could not decide Republican or Democrat, and give an order like Republican, Undecided, and Democrat.

Table 7 gives the estimates of the measure $\Psi_S^{(\lambda)}$, the estimated approximate standard errors for $\hat{\Psi}_S^{(\lambda)}$ and the approximate 95% confidence intervals for $\Psi_S^{(\lambda)}$. Because the confidence intervals for $\Psi_S^{(\lambda)}$ applied to the data in Table 6 do not include zero for any λ (> -1)

Table 2: (a) Artificial data; (b) modified data obtained by interchanging categories 2 and 3; and (c) the corresponding values of $\hat{\Psi}_S^{(\lambda)}$ and $\hat{\Phi}^{(\lambda)}$.

(a) Artificial data

X_1	X_3								
	(1)			(2)			(3)		
	X_2			X_2			X_2		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	50	0	0	15	30	0	40	70	25
(2)	0	0	0	0	45	0	0	20	35
(3)	0	0	0	0	0	0	0	0	55

(b) Modified data

X_1	X_3								
	(1)			(2)			(3)		
	X_2			X_2			X_2		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	50	0	0	40	25	70	15	0	30
(2)	0	0	0	0	55	0	0	0	0
(3)	0	0	0	0	35	20	0	0	45

 (c) Values of $\hat{\Psi}_S^{(\lambda)}$ and $\hat{\Phi}^{(\lambda)}$

λ	For Table 2a		For Table 2b	
	$\hat{\Psi}_S^{(\lambda)}$	$\hat{\Phi}^{(\lambda)}$	$\hat{\Psi}_S^{(\lambda)}$	$\hat{\Phi}^{(\lambda)}$
-0.4	1.0	1.0	0.748	1.0
0	1.0	1.0	0.752	1.0
0.6	1.0	1.0	0.749	1.0
1.0	1.0	1.0	0.745	1.0
1.6	1.0	1.0	0.735	1.0

(see Table 7), these would indicate that there is not a structure of symmetry in the data in Table 6.

For the data in Table 6, denote ‘Republican’, ‘Undecided’ and ‘Democrat’ by ‘R’, ‘U’ and ‘D’, respectively. Let $P(A, B, C)$ denote the probability that an individual has opinion A in time $t - 2$, B in time $t - 1$, and C in time t . Then the symmetry model indicates that (i) $P(R, U, D) = P(R, D, U) = P(U, R, D) = P(D, R, U) = P(U, D, R) = P(D, U, R)$, (ii) $P(R, R, D) = P(R, D, R) = P(D, R, R)$, (iii) $P(D, D, R) = P(D, R, D) = P(R, D, D)$, and so on. Now since there is not a structure of symmetry in these data, we are interested in measuring to what degree the departure from symmetry is toward the maximum asymmetry by using the measure $\hat{\Psi}_S^{(\lambda)}$. Note that the maximum asymmetry means that, e.g., (i) $P(R, U, D) \neq 0$, $P(R, D, U) = P(U, R, D) = P(D, R, U) = P(U, D, R) = P(D, U, R) = 0$, (ii) $P(R, R, D) \neq 0$, $P(R, D, R) = P(D, R, R) = 0$, (iii) $P(D, D, R) \neq 0$, $P(D, R, D) = P(R, D, D) = 0$, and so on.

We shall investigate the degree of departure from symmetry in greater detail. For

Table 3: The daily temperature at Sapporo, Tokyo and Naha in Japan in 2001 and 2005.

(a) in 2001

at Sapporo	at Naha								
	(1)			(2)			(3)		
	at Tokyo			at Tokyo			at Tokyo		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	17	6	1	17	19	12	18	25	8
(2)	4	9	2	16	34	28	8	19	16
(3)	0	3	3	3	25	39	1	8	24

(b) in 2005

at Sapporo	at Naha								
	(1)			(2)			(3)		
	at Tokyo			at Tokyo			at Tokyo		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	15	9	1	25	22	6	6	7	8
(2)	11	10	4	24	37	18	2	23	10
(3)	1	3	3	5	39	34	0	21	21

Table 4: Estimates of $\Psi_S^{(\lambda)}$, estimated approximate standard errors for $\hat{\Psi}_S^{(\lambda)}$ and approximate 95% confidence intervals for $\Psi_S^{(\lambda)}$, applied to Table 3.

Tables	Value of λ	Estimated measure	Standard error	Confidence interval
For Table 3a	-0.4	0.141	0.025	(0.092, 0.189)
	0	0.174	0.030	(0.116, 0.233)
	0.6	0.186	0.032	(0.124, 0.249)
	1.0	0.180	0.031	(0.118, 0.241)
	1.6	0.161	0.029	(0.103, 0.218)
For Table 3b	-0.4	0.051	0.013	(0.026, 0.077)
	0	0.064	0.017	(0.031, 0.096)
	0.6	0.069	0.019	(0.032, 0.105)
	1.0	0.067	0.019	(0.030, 0.103)
	1.6	0.059	0.017	(0.025, 0.093)

instance, when $\lambda = 0$, the estimated measure $\hat{\Psi}_S^{(0)}$ equals 0.189 (see Table 7). Thus for these data, the degree of departure from symmetry is estimated to be 18.9 percent of the maximum degree of departure from symmetry.

Table 8 gives the values of power-divergence statistic $W^{(\lambda)}$ for testing goodness-of-fit of the symmetry model. We see that the symmetry model fits the data in Table 6 very poorly. Note that we cannot see to what degree the departure from symmetry is toward the maximum asymmetry by using the test statistic $W^{(\lambda)}$.

Table 5: Values of power-divergence statistic $W^{(\lambda)}$ (with 17 degrees of freedom) for testing goodness-of-fit of the symmetry model, applied to Tables 3a and 3b.

λ	For Table 3a	For Table 3b
-0.4	128.83	65.33
0	119.43	60.05
0.6	118.71	57.69
1.0	123.50	57.67
1.6	137.97	59.20

Table 6: Stationary two-step transitions in a panel study of potential voters in Erie County, Ohio, 1940.

Time $t - 2$	Time t								
	R			U			D		
	Time $t - 1$			Time $t - 1$			Time $t - 1$		
	R	U	D	R	U	D	R	U	D
R	557	17	3	16	21	0	6	5	8
U	71	62	6	11	346	6	1	54	63
D	18	4	9	5	24	22	0	10	435

Taken directly from Bishop *et al.* (1975, p.305):
 (R : Republican, U : Undecided, D : Democrat).

 Table 7: Estimates of $\Psi_S^{(\lambda)}$, estimated approximate standard errors for $\hat{\Psi}_S^{(\lambda)}$ and approximate 95% confidence intervals for $\Psi_S^{(\lambda)}$, applied to Table 6.

Values of λ	Estimated measure	Standard error	Confidence interval
-0.4	0.153	0.018	(0.118, 0.188)
0	0.189	0.022	(0.145, 0.233)
0.6	0.203	0.024	(0.155, 0.250)
1.0	0.196	0.024	(0.148, 0.244)
1.6	0.175	0.023	(0.130, 0.220)

7. Concluding Remarks

The measure $\hat{\Psi}_S^{(\lambda)}$ could be useful for *measuring* to what degree the departure from symmetry is toward the maximum asymmetry (described in Sections 2.2, 3.2 and 6.2) for *ordinal* data (although the test statistic $W^{(\lambda)}$ is not useful for measuring it).

The measure $\hat{\Psi}_S^{(\lambda)}$ could be useful for *comparing* the degrees of departure from symmetry in several tables with ordered categories.

Table 8: Values of power-divergence statistic $W^{(\lambda)}$ (with 17 degrees of freedom) for testing goodness-of-fit of the symmetry model, applied to Table 6.

λ	$W^{(\lambda)}$
-0.4	247.29
0	229.76
0.6	222.98
1.0	225.55
1.6	237.72

For the measure $\Psi_S^{(\lambda)}$, the analyst may be interested in which value of λ should be preferred. However, in comparing tables, it seems best to compare values of $\hat{\Psi}_S^{(\lambda)}$ for a range of values of λ . For example, consider the artificial data in Tables 9a and 9b. We see from Table 9c that the value of $\hat{\Psi}_S^{(0)}$ is greater for Table 9a than Table 9b, but the value of $\hat{\Psi}_S^{(1)}$ is less for Table 9a than Table 9b (although the differences are slight in these cases). So, for these cases, it may be impossible to decide (by using $\hat{\Psi}_S^{(\lambda)}$) whether the degree of asymmetry is greater for Table 9a or Table 9b. But generally, for the comparison between two tables, it would be possible to draw a conclusion if $\Psi_S^{(\lambda)}$ (for every λ) is always greater (or always less) for one table than the other table.

Consider two tables, A and B. When one wants to compare the degrees of departure from symmetry, it may be dangerous to use the test statistic $W^{(\lambda)}$ because, for a fixed λ , the value of measure $\hat{\Psi}_S^{(\lambda)}$ may be less for Table A than Table B, but the value of $W^{(\lambda)}$ is greater for Table A than Table B. For example, consider the artificial data in Tables 10a and 10b. From Table 10c, it seems natural to conclude that the degree of departure from symmetry is greater for Table 10a than Table 10b. We see from Table 11 that the values of $\hat{\Psi}_S^{(\lambda)}$ are greater for Table 10a than Table 10b, but the values of $W^{(\lambda)}$ are less for Table 10a than Table 10b. Therefore, the measure $\hat{\Psi}_S^{(\lambda)}$ may be useful for comparing the degree of departure from symmetry in several multi-way tables.

The test statistic $W^{(\lambda)}$ does not depend on the order of listing the categories because it is invariant under arbitrary similar permutation of categories for each variable (although the measure $\Psi_S^{(\lambda)}$ is not invariant under them). Thus $W^{(\lambda)}$ is not suitable for measuring the degree of departure from symmetry for the data on an *ordinal* scale when one wants to use the information about the category ordering. It seems that the test statistic $W^{(\lambda)}$ should be used only for testing goodness-of-fit of symmetry.

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Appendix 1

For an $r \times r$ table with ordered categories, the measure of departure from symmetry considered by Tomizawa *et al.* (2001) is given as follows: assuming that $\{G_{ij} + G_{ji} > 0\}$ for

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Table 9: (a), (b) Artificial data (n is sample size) and (c) the corresponding values of $\hat{\Psi}_S^{(\lambda)}$.

(a) $n = 1325$ (sample size)

X_1	X_2			X_3			X_2		
	(1)			(2)			(3)		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	95	80	50	45	90	40	70	15	15
(2)	60	10	20	70	80	80	35	10	10
(3)	50	55	30	40	55	35	60	40	85

(b) $n = 1220$

X_1	X_2			X_3			X_2		
	(1)			(2)			(3)		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	80	40	60	30	25	40	30	65	25
(2)	35	20	50	65	40	65	55	55	45
(3)	30	20	20	45	60	25	65	35	95

(c) Values of $\hat{\Psi}_S^{(\lambda)}$

λ	For Table 9a	For Table 9b
-0.4	0.0271	0.0259
0	0.0343	0.0336
0.6	0.0357*	0.0362
1.0	0.0332*	0.0344
1.6	0.0274*	0.0294

* indicates that $\hat{\Psi}_S^{(\lambda)}$ is less for Table 9a than for Table 9b.

$i \neq j$,

$$\phi^{(\lambda)} = \frac{\lambda(\lambda + 1)}{2^\lambda - 1} I^{(\lambda)} \quad \text{for } \lambda > -1,$$

where $\Delta = \sum \sum_{s \neq t} G_{st}$, $G_{ij}^* = G_{ij} / \Delta$ and $Q_{ij}^* = (G_{ij}^* + G_{ji}^*) / 2$,

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda + 1)} \sum_{i=1}^r \sum_{\substack{j=1 \\ i \neq j}}^r G_{ij}^* \left\{ \left(\frac{G_{ij}^*}{Q_{ij}^*} \right)^\lambda - 1 \right\},$$

and the value at $\lambda = 0$ is taken to be the continuous limit as $\lambda \rightarrow 0$.

Table 10: Artificial $3 \times 3 \times 3$ data.(a) $n = 375$ (sample size)

X_1	X_3								
	(1)			(2)			(3)		
	X_2			X_2			X_2		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	5	5	25	5	10	10	25	60	25
(2)	5	15	10	35	5	15	10	10	30
(3)	10	10	5	10	15	5	5	5	5

(b) $n = 1875$

X_1	X_3								
	(1)			(2)			(3)		
	X_2			X_2			X_2		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(1)	25	25	125	25	50	50	125	200	125
(2)	100	75	50	175	25	75	50	50	25
(3)	50	50	25	50	75	100	25	100	25

(c) the values of $\hat{G}_{k_1 k_2 k_3(j_1, j_2, j_3)}^{[r_1 + \dots + r_l]}$ for Tables 10a and 10b

	(j_1, j_2, j_3)	$r_1 + \dots + r_l$	$\hat{G}_{k_1 k_2 k_3(j_1, j_2, j_3)}^{[r_1 + \dots + r_l]}$ for various $(k_1, k_2, k_3) \in D(1, 2, 3)$
For Table 10a	(1, 1, 2)	2	(0.08, 0.08, 0.04)
	(1, 1, 3)	2	(0.07, 0.07, 0.03)
	(2, 2, 3)	2	(0.28, 0.16, 0.12)
	(1, 2, 3)	1	(0.16, 0.03, 0.03, 0.03, 0.03, 0.03)
	(1, 2, 3)	2	(0.16, 0.03, 0.03, 0.03, 0.03, 0.03)
	(1, 2, 2)	1	(0.28, 0.16, 0.11)
	(1, 3, 3)	1	(0.07, 0.01, 0.01)
	(2, 3, 3)	1	(0.15, 0.03, 0.03)
For Table 10b	(1, 1, 2)	2	(0.08, 0.08, 0.08)
	(1, 1, 3)	2	(0.07, 0.07, 0.03)
	(2, 2, 3)	2	(0.23, 0.16, 0.12)
	(1, 2, 3)	1	(0.11, 0.03, 0.03, 0.03, 0.03, 0.03)
	(1, 2, 3)	2	(0.11, 0.03, 0.03, 0.03, 0.03, 0.03)
	(1, 2, 2)	1	(0.23, 0.16, 0.11)
	(1, 3, 3)	1	(0.07, 0.01, 0.01)
	(2, 3, 3)	1	(0.08, 0.07, 0.07)

Appendix 2

For an r^T ($T \geq 3$) table with nominal categories, the measure of departure from symmetry considered by Yamamoto (2004) is given as follows:

$$\Phi^{(\lambda)} = \frac{1}{\delta^*} \sum_{(r_1, \dots, r_T) \in R^*} \sum \dots \sum \delta_{r_1 \dots r_T}^* \Phi_{r_1 \dots r_T}^{(\lambda)} \quad \text{for } \lambda > -1,$$

Table 11: Values of $\hat{\Psi}_S^{(\lambda)}$ and the power-divergence statistic $W^{(\lambda)}$ (with 17 degrees of freedom) for testing goodness-of-fit of the symmetry model, applied to Tables 10a and 10b.

λ	For Table 10a		For Table 10b	
	$\hat{\Psi}_S^{(\lambda)}$	$W^{(\lambda)}$	$\hat{\Psi}_S^{(\lambda)}$	$W^{(\lambda)}$
-0.4	0.096	151.77	0.045	549.13
0	0.127	158.31	0.059	557.08
0.6	0.141	175.80	0.064	587.03
1.0	0.137	193.99	0.061	620.54
1.6	0.121	234.97	0.052	695.65

where

$$\delta_{r_1 \dots r_T}^* = \sum_{(i_1, \dots, i_T) \in E_{(r_1, \dots, r_T)}^*} \sum \dots \sum p_{i_1 \dots i_T},$$

$$\delta^* = \sum_{(r_1, \dots, r_T) \in R^*} \sum \dots \sum \delta_{r_1 \dots r_T}^* \left(= 1 - \sum_{k=1}^r p_{k \dots k} \right),$$

$$E_{(r_1, \dots, r_T)}^* = \{(i_1, \dots, i_T) \mid i_t = 1, \dots, r; t = 1, \dots, T, \text{ where the } r_1 \text{'s elements of } (i_1, \dots, i_T) \text{ are equal, } r_2 \text{'s elements are equal, } \dots, \text{ and } r_T \text{'s elements are equal}\},$$

for $(r_1, \dots, r_T)$ which satisfies

$$r_1 + \dots + r_T = T \quad \text{and} \quad T - 1 \geq r_1 \geq \dots \geq r_T \geq 0,$$

$$R^* = \{(r_1, \dots, r_T) \mid r_1 + \dots + r_T = T, T - 1 \geq r_1 \geq \dots \geq r_T \geq 0\},$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \rightarrow 0$. For $(r_1, \dots, r_T) \in R^*$, $\Phi_{r_1 \dots r_T}^{(\lambda)}$ in $\Phi^{(\lambda)}$ is defined by

$$\Phi_{r_1 \dots r_T}^{(\lambda)} = \frac{\lambda(\lambda + 1)}{({}_T C_{r_1 \dots r_T})^\lambda - 1} I_{r_1 \dots r_T}^{(\lambda)},$$

where

$$I_{r_1 \dots r_T}^{(\lambda)} = \frac{1}{\lambda(\lambda + 1)} \sum_{(i_1, \dots, i_T) \in E_{(r_1, \dots, r_T)}^*} \sum \dots \sum p_{i_1 \dots i_T [r_1 \dots r_T]}^* \left\{ \left(\frac{p_{i_1 \dots i_T [r_1 \dots r_T]}^*}{p_{i_1 \dots i_T [r_1 \dots r_T]}^s} \right)^\lambda - 1 \right\},$$

$$p_{i_1 \dots i_T [r_1 \dots r_T]}^* = p_{i_1 \dots i_T} / \delta_{r_1 \dots r_T}^*,$$

$$p_{i_1 \dots i_T [r_1 \dots r_T]}^s = \sum_{(j_1, \dots, j_T) \in D(i_1, \dots, i_T)} \sum \dots \sum p_{j_1 \dots j_T [r_1 \dots r_T]}^* / ({}_T C_{r_1 \dots r_T}),$$

$${}_T C_{r_1 \dots r_T} = \frac{T!}{r_1! \dots r_T!}.$$

Appendix 3

Consider the r^T table with $T \geq 3$. Using the delta method, $\sqrt{n}(\hat{\Psi}_S^{(\lambda)} - \Psi_S^{(\lambda)})$ has asymptotically (as $n \rightarrow \infty$) a normal distribution with mean zero and variance $\sigma^2[\Psi_S^{(\lambda)}]$, where for $\lambda > -1$ ($\lambda \neq 0$),

$$\sigma^2[\Psi_S^{(\lambda)}] = \sum_{\text{not } i_1 \dots i_T} \sum \dots \sum (w_{i_1 \dots i_T}^{(\lambda)})^2 p_{i_1 \dots i_T},$$

where

$$w_{i_1 \dots i_T}^{(\lambda)} = \frac{1}{\delta} \left(R_{i_1 \dots i_T}^{(\lambda)} - \Psi_S^{(\lambda)} V_{i_1 \dots i_T} \right),$$

$$R_{i_1 \dots i_T}^{(\lambda)} = \sum_{(r_1, \dots, r_s) \in S^*} \sum \dots \sum \left\{ \sum_{l=1}^{s-1} \delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]} \Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](\lambda)} + \delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]} \Psi_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l](\lambda)} \right\},$$

$$V_{i_1 \dots i_T} = \sum_{(r_1, \dots, r_s) \in S^*} \sum \dots \sum \left(\sum_{l=1}^{s-1} \delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]} \right),$$

$$\delta_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l]} = \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \sum \dots \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} I_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]},$$

$$\begin{aligned} I_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} &= I(i_{k_1}, \dots, i_{k_{r_1}} \leq j_{r_1} < i_{k_{r_1+1}}, \dots, i_{k_{r_1+r_2}} \leq j_{r_1+r_2} < \\ &\quad i_{k_{r_1+r_2+1}}, \dots, i_{k_{r_1+r_2+r_3}} \leq j_{r_1+r_2+r_3} < \dots, i_{k_{r_1+\dots+r_l}} \leq j_{r_1+\dots+r_l}, \\ &\quad j_{r_1+\dots+r_{l+1}} \leq i_{k_{r_1+\dots+r_{l+1}}}, \dots, i_{k_{r_1+\dots+r_{l+1}}} < j_{r_1+\dots+r_{l+2}} \leq i_{k_{r_1+\dots+r_{l+1}+1}}, \\ &\quad \dots < j_T \leq i_{k_{r_1+\dots+r_{s-1}+1}}, \dots, i_{k_T}), \end{aligned}$$

with

$$I(A) = \begin{cases} 1 & \text{when } A \text{ holds,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \Psi_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l](\lambda)} &= \frac{1}{(T C_{r_1 \dots r_s})^\lambda - 1} \\ &\times \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \sum \dots \sum \left\{ \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} S_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \left(\frac{G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]}}{G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]}} \right)^\lambda \right. \\ &\quad \left. + \lambda G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]} \left(\frac{G_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]}}{G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]}} \right)^{\lambda-1} T_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \right\}, \end{aligned}$$

$$S_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} = \frac{I_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]} - G_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \delta_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l]}}{(\delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]})^2},$$

$$T_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} = \frac{S_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]} - G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]} U_{(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]}}{(G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]})^2},$$

$$U_{(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} = \frac{1}{T!} \left(\sum_{(k_1, \dots, k_T) \in D(1, \dots, T)} \sum \dots \sum S_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \right),$$

and for $\lambda = 0$,

$$\sigma^2[\Psi_S^{(0)}] = \sum_{\text{not}} \sum_{i_1 = \dots = i_T} \dots \sum (w_{i_1 \dots i_T}^{(0)})^2 p_{i_1 \dots i_T},$$

where

$$w_{i_1 \dots i_T}^{(0)} = \frac{1}{\delta} \left(R_{i_1 \dots i_T}^{(0)} - \Psi_S^{(0)} V_{i_1 \dots i_T} \right),$$

$$R_{i_1 \dots i_T}^{(0)} = \sum_{(r_1, \dots, r_s) \in S^*} \sum \dots \sum \left\{ \sum_{l=1}^{s-1} \delta_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l]} \Psi_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l](0)} + \delta_{(r_1, \dots, r_s)}^{[r_1 + \dots + r_l]} \Psi_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l](0)} \right\},$$

$$\begin{aligned} \Psi_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l](0)} &= \frac{1}{\log(T C_{r_1 \dots r_s})} \\ &\times \sum_{(j_1, \dots, j_T) \in E(r_1, \dots, r_s)} \sum \dots \sum \left\{ \sum_{(k_1, \dots, k_T) \in D^*(r_1, \dots, r_s)} S_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \log \left(\frac{G_{k_1 \dots k_T(j_1, \dots, j_T)}^{*[r_1 + \dots + r_l]}}{G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]}} \right) \right. \\ &\quad \left. + G_{(j_1, \dots, j_T)}^{S[r_1 + \dots + r_l]} T_{k_1 \dots k_T(j_1, \dots, j_T)}^{[r_1 + \dots + r_l]} \right\}. \end{aligned}$$

Note that $\Psi_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l](0)}$ is equal to the limit of $\Psi_{(r_1, \dots, r_s)}'^{[r_1 + \dots + r_l](\lambda)}$ as $\lambda \rightarrow 0$, namely, $\sigma^2[\Psi_S^{(0)}]$ is equal to the limit of $\sigma^2[\Psi_S^{(\lambda)}]$ as $\lambda \rightarrow 0$.

Appendix 4

For an $r \times r \times r$ table, the power-divergence chi-squared statistic for testing goodness-of-fit of the symmetry model is given by, for $-\infty < \lambda < \infty$,

$$W^{(\lambda)} = \frac{2}{\lambda(\lambda + 1)} \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r n_{ijk} \left[\left(\frac{6n_{ijk}}{n_{ijk} + n_{ikj} + n_{jik} + n_{jki} + n_{kij} + n_{kji}} \right)^\lambda - 1 \right],$$

and the values at $\lambda = -1$ and $\lambda = 0$ are taken to be the limits as $\lambda \rightarrow -1$ and $\lambda \rightarrow 0$, respectively. The number of degrees of freedom is $r(r-1)(5r+2)/6$.

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