A STATISTIC FOR TESTING THE EQUALITY OF EIGENVALUE OF COVARIANCE MATRIX ON MULTIPOPULATION

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ABSTRACT

A test statistic for the equality of the j -th largest eigenvalues of the covariance matrix in a multipopulation is proposed. Asymptotic distribution of the statistic is derived under the normal population when the sample sizes are equal. By simulation studies, we investigate the power of a test using the suggested statistic for normal, contaminated normal and skew normal populations, and compare it with two nonparametric tests.

1. Introduction

Principal component analysis (PCA) is an extremely important technique in multivariate analysis from both practical and theoretical viewpoints. Recent developments in computerized measurement technology have permitted the accumulation of multivariate data. The analysis of multidimensional data is particularly important in studies of DNA microarrays, finance, environmental threats, image analysis, functional data, and other similar data. There have been many books and papers concerning PCA (Anderson, 2003; Jackson, 2003; Jolliffe, 2002). However, under the non-normal population family such as contaminated normal populations, lognormal populations and skew normal populations, it is very difficult to obtain the exact distribution of eigenvalues of a covariance matrix. Therefore, there have not been many papers on testing the hypothesis that the j -th largest eigenvalues are equal in bipopulations and in multipopulations. Therefore we are interested in testing the equality of the j -th largest eigenvalues of the covariance matrix, and we treat the contaminated normal and skew normal populations as examples of non-normal populations in this paper. For bipopulations, Sugiyama and Ushizawa (1998) propose a testing procedure for the equality of the j -th largest eigenvalues by applying the Ansari-Bradley test (1960). In addition, Takeda (2001) proposed a test statistic and derived the exact distribution of the statistic under normality. Recently, Murakami, Hino and Tsukada (2007) extended the testing procedure, which uses two nonparametric tests, from a bipopulation to a multipopulation. However, we need large sample sizes to obtain high power when we use a nonparametric test. Therefore the purpose of this paper is to propose a more effective statistic for testing the equality of the j -th largest eigenvalues in multipopulation. To examine the equality of the j -th largest eigenvalues, we test the equality of the variance of the principal component scores.

Suppose that $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_{N_i}^{(i)}$ are random observations from a p -dimensional distribution with mean $\boldsymbol{\mu}_i$ and covariance matrix $\boldsymbol{\Sigma}_i$, where $i = 1, \dots, k$. Let $\lambda_j^{(i)}$ be the j -th largest

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eigenvalue of the covariance matrix Σ_i in the i -th population. For fixed j , we consider testing the hypothesis

$$H_0 : \lambda_j^{(1)} = \lambda_j^{(2)} = \dots = \lambda_j^{(k)} \quad \text{against} \quad H_1 : \text{not } H_0.$$

Let $\mathbf{h}_j^{(i)}$ be the sample eigenvector corresponding to the j -th largest sample eigenvalue $l_j^{(i)}$ of the i -th sample covariance matrix given by

$$\mathbf{S}^{(i)} = \frac{1}{N_i - 1} \sum_{\alpha=1}^{N_i} \left(\mathbf{x}_{\alpha}^{(i)} - \bar{\mathbf{x}}^{(i)} \right) \left(\mathbf{x}_{\alpha}^{(i)} - \bar{\mathbf{x}}^{(i)} \right)',$$

where $\bar{\mathbf{x}}^{(i)}$ is the sample mean vector of the i -th population. We express the j -th principal component score as

$$y_{j\alpha}^{(i)} = \mathbf{h}_j^{(i)'} \left(\mathbf{x}_{\alpha}^{(i)} - \bar{\mathbf{x}}^{(i)} \right), \quad \alpha = 1, \dots, N_i.$$

In Section 2, we develop a testing procedure in multipopulation and propose a new test statistic, namely T_j . We also derive the asymptotic distribution of the statistic T_j under a normal population when sample sizes are equal. In Section 3, we simulate to investigate the type I error and the power of a test using the statistic T_j under various populations and to compare the power of the test with two nonparametric tests. In this paper, we treat the normal population, the contaminated normal population as a symmetric non-normal population and the skew normal population as an asymmetric non-normal population. Finally, concluding remarks are given in Section 4.

2. Test statistic and asymptotic distribution

In this section, we suggest a new test statistic and derive its asymptotic distribution. To test the equality of the j -th largest eigenvalues in the k -population, we use the principal component scores Y_i as

$$Y_1 = \left\{ y_{j1}^{(1)}, y_{j2}^{(1)}, \dots, y_{jN_1}^{(1)} \right\}, Y_2 = \left\{ y_{j1}^{(2)}, y_{j2}^{(2)}, \dots, y_{jN_2}^{(2)} \right\}, \dots, Y_k = \left\{ y_{j1}^{(k)}, y_{j2}^{(k)}, \dots, y_{jN_k}^{(k)} \right\}.$$

Under normal populations, the eigenvalues of the covariance matrices are equal to the variances of the principal component scores. For non-normal populations, Murakami *et al.* (2007) give the variance of the principal component score $y_{j\alpha}^{(i)}$ as follows:

$$\text{Var}[y_{j\alpha}^{(i)}] = \lambda_j^{(i)} - \frac{2}{N_i - 1} \sum_{q \neq j}^p \frac{\text{E}[x_{q\alpha}^{(i)2} x_{j\alpha}^{(i)2}]}{\lambda_q^{(i)} - \lambda_j^{(i)}} + O(N_i^{-2}), \quad (1)$$

where $x_{j\alpha}^{(i)}$ is the j -th element of $\mathbf{x}_{\alpha}^{(i)}$. From equation (1), we may treat testing the null hypothesis as testing the equality of variances of principal component scores when the sample sizes N_i are sufficiently large. When the differences between the j -th largest eigenvalue and other eigenvalues in each population are sufficiently large, the term of $O(N_i^{-1})$ may not affect the test of equality of the j -th largest eigenvalues.

Now we propose a statistic T_j for testing the hypothesis as follows:

$$T_j = \frac{N}{2k} \sum_{1 \leq \omega < \beta \leq k} \sum \left(\log \frac{l_j^{(\beta)}}{l_j^{(\omega)}} \right)^2,$$

where

$$N = \sum_{i=1}^k N_i, \quad l_j^{(\omega)} = \frac{1}{N_\omega} \sum_{i=1}^{N_\omega} \left(y_{ji}^{(\omega)} - \bar{y}_j^{(\omega)} \right)^2 \quad \text{and} \quad \bar{y}_j^{(\omega)} = \frac{1}{N_\omega} \sum_{i=1}^{N_\omega} y_{ji}^{(\omega)}.$$

Assuming normal populations, we derive the limiting distribution of the statistic T_j as the following theorem.

Theorem 1 *When $N_1 = N_2 = \dots = N_k \equiv n + 1$, the limiting distribution of the statistic T_j is the χ^2 distribution with $(k - 1)$ degrees of freedom under normal populations.*

Proof. The limiting distribution of

$$\mathbf{z}_j = \sqrt{n} \begin{pmatrix} \log l_j^{(1)} - \log \lambda_j^{(1)} \\ \log l_j^{(2)} - \log \lambda_j^{(2)} \\ \vdots \\ \log l_j^{(k)} - \log \lambda_j^{(k)} \end{pmatrix}$$

is $N(\mathbf{0}, 2I_k)$ when the population is normal (Anderson, 2003). Under the null hypothesis,

$$A\mathbf{z}_j = \sqrt{n} \begin{pmatrix} \log l_j^{(1)} - \log l_j^{(2)} \\ \log l_j^{(2)} - \log l_j^{(3)} \\ \vdots \\ \log l_j^{(k-1)} - \log l_j^{(k)} \end{pmatrix}$$

is asymptotically distributed as $N(\mathbf{0}, 2AA')$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & & \\ 0 & 1 & -1 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}.$$

Therefore, we may obtain that

$$(A\mathbf{z}_j)'(2AA')^{-1}(A\mathbf{z}_j) = \frac{1}{2} \mathbf{z}_j' \{A'(AA')^{-1}A\} \mathbf{z}_j$$

is asymptotically distributed as a χ^2 distribution with $(k - 1)$ degrees of freedom. Noting that

$$kA'(AA')^{-1}A = \begin{pmatrix} k-1 & -1 & \cdots & -1 \\ -1 & k-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & k-1 \end{pmatrix},$$

we have

$$k\mathbf{z}_j' \{A'(AA')^{-1}A\} \mathbf{z}_j = n \sum_{1 \leq \omega < \beta \leq k} \left(\log l_j^{(\beta)} - \log l_j^{(\omega)} \right)^2.$$

Therefore the limiting distribution of

$$\frac{1}{2} z_j' \{A'(AA')^{-1}A\} z_j = \frac{n}{2k} \sum_{1 \leq \omega < \beta \leq k} \left(\log l_j^{(\beta)} - \log l_j^{(\omega)} \right)^2 = T_j$$

is a χ^2 distribution with $(k-1)$ degrees of freedom. $\square$

For the case of unequal sample sizes and non-normal populations, it is difficult to derive the limiting distribution of the statistic T_j because the weight is different for each $\log l_j^{(\beta)} - \log l_j^{(\omega)}$. Then the limiting distribution of the statistic T_j may not be a simple expression such as a χ^2 distribution with $(k-1)$ degrees of freedom. Therefore, we adopt a permutation test for the statistic T_j to examine the equality of eigenvalues in next section.

3. Simulation study

In this section, we investigate the type I error and the power of tests for the equality of the j -th largest eigenvalues at a significance level of 5%. We carry out simulations for normal populations, contaminated normal populations as an example of symmetric non-normal populations and skew normal populations as an example of asymmetric non-normal populations. Let the number of populations be three. The simulations are repeated 100,000 times with 10,000 permutations for the statistic T_j . Therefore, the type I error and the power maintain precision down to two decimal places.

Now we give the generalized Ansari-Bradley statistic. The Ansari-Bradley test is known as a method of testing the variances. One of the assumptions for the Ansari-Bradley test is that the samples are independent. Sugiyama and Ushizawa (1998) prove that the degree of dependence between each principal component score is weak when the sample size is sufficiently large under normality. They show the Ansari-Bradley test could be applicable to test the equality of the eigenvalues using principal component scores. Let $R_{jm}^{(i)}$ be the increasing order rank of $y_{jm}^{(i)}$ in the combined N observations. The Ansari-Bradley statistic for k -sample is given by

$$AB_{jk} = \begin{cases} \frac{48(N-1)}{N(N^2-4)} \sum_{i=1}^k N_i \left(\bar{A}_j^{(i)} - \frac{N+2}{4} \right)^2 & (N \text{ is even}) \\ \frac{48N^2}{N(N+1)(N^2+3)} \sum_{i=1}^k N_i \left(\bar{A}_j^{(i)} - \frac{(N+1)^2}{4N} \right)^2 & (N \text{ is odd}), \end{cases}$$

where

$$\bar{A}_j^{(i)} = \frac{1}{N_i} \sum_{m=1}^{N_i} \left(\frac{N+1}{2} - \left| R_{jm}^{(i)} - \frac{N+1}{2} \right| \right).$$

The limiting distribution of the Ansari-Bradley statistic for a k -population is a χ^2 distribution with $(k-1)$ degrees of freedom under the null hypothesis (Tsai *et al.*, 1975). Gibbons and Chakraborti (2003) indicate that the asymptotic relative efficiency of the test by Mood (1954), the Mood test, is higher than that of the Ansari-Bradley test. The Mood statistic for a k -sample is known as

$$M_{jk} = \frac{180}{N(N+1)(N^2-4)} \sum_{i=1}^k N_i \left(\bar{M}_j^{(i)} - \frac{N^2-1}{12} \right)^2,$$

where

$$\bar{M}_j^{(i)} = \frac{1}{N_i} \sum_{m=1}^{N_i} \left(R_{jm}^{(i)} - \frac{N+1}{2} \right)^2.$$

The limiting distribution of the Mood statistic M_{jk} is also a χ^2 distribution with $(k-1)$ degrees of freedom (Tsai *et al.*, 1975). Therefore we set the critical value of both test as 5.991 for $k = 3$.

Murakami *et al.* (2007) show how the statistic AB_{jk} and the statistic M_{jk} are used to test the hypothesis. We simulate 100,000 times for the statistics AB_{jk} and M_{jk} , and investigate the type I error and the power of the test using the statistics T_j , AB_{jk} and M_{jk} . We set the populations as normal populations $N_3(\mathbf{0}, \Sigma^{(i)})$ and contaminated normal populations $0.95 \times N_3(\mathbf{0}, \Sigma^{(i)}) + 0.05 \times N_3(\mathbf{0}, 3\Sigma^{(i)})$, and the sample sizes are $N_1 = N_2 = N_3 = 50$, $N_1 = N_2 = N_3 = 100$ and $N_1 = 200$, $N_2 = 150$, $N_3 = 100$. We treat the same cases as Murakami *et al.* (2007) and add the case of $N_1 = 150$, $N_2 = 100$, $N_3 = 50$ to compare the type I error and the power for the case of $N_1 = N_2 = N_3 = 100$. Since the distribution of the eigenvalues may not depend on the eigenvector, it is sufficient to examine only the case that the population covariance matrices are diagonal without loss of generality. We set the population covariance matrices as

$$\Sigma^{(i)} = \text{diag}(\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)})$$

and treat the following four cases.

Case 1:

$$\lambda_1^{(1)} = 6, \lambda_2^{(1)} = 3, \lambda_3^{(1)} = 1; \quad \lambda_1^{(2)} = 6, \lambda_2^{(2)} = 3, \lambda_3^{(2)} = 1; \quad \lambda_1^{(3)} = 6, \lambda_2^{(3)} = 3, \lambda_3^{(3)} = 1,$$

Case 2:

$$\lambda_1^{(1)} = 10, \lambda_2^{(1)} = 3, \lambda_3^{(1)} = 1; \quad \lambda_1^{(2)} = 8, \lambda_2^{(2)} = 3, \lambda_3^{(2)} = 1; \quad \lambda_1^{(3)} = 6, \lambda_2^{(3)} = 3, \lambda_3^{(3)} = 1,$$

Case 3:

$$\lambda_1^{(1)} = 9, \lambda_2^{(1)} = 5, \lambda_3^{(1)} = 2; \quad \lambda_1^{(2)} = 6, \lambda_2^{(2)} = 3, \lambda_3^{(2)} = 1; \quad \lambda_1^{(3)} = 6, \lambda_2^{(3)} = 3, \lambda_3^{(3)} = 1,$$

Case 4:

$$\lambda_1^{(1)} = 9, \lambda_2^{(1)} = 5, \lambda_3^{(1)} = 2; \quad \lambda_1^{(2)} = 7.5, \lambda_2^{(2)} = 4, \lambda_3^{(2)} = 1.5; \quad \lambda_1^{(3)} = 6, \lambda_2^{(3)} = 3, \lambda_3^{(3)} = 1.$$

Case 1 is under the null hypothesis for all eigenvalues. Case 2 is that only the largest eigenvalues are different and other eigenvalues are under the null hypothesis. Case 3 and Case 4 is that all eigenvalues are different. In the all of Tables, the results on the left of the dividing lines shows the type I error of the statistics. In addition, the results on the right of the dividing lines reveal the power of tests using the M_{jk} , AB_{jk} and T_j statistics. Table 1 and Table 2 represent the results for the normal population.

Table 1: Normal populations

		$N_1 = N_2 = N_3 = 50$				$N_1 = N_2 = N_3 = 100$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.041	0.242	0.224	0.169	0.046	0.478	0.428	0.327
	AB_{1k}	0.042	0.201	0.184	0.144	0.047	0.395	0.348	0.269
	T_1	0.040	0.302	0.269	0.206	0.045	0.593	0.531	0.415
$j = 2$	M_{2k}	0.048	0.049	0.335	0.253	0.049	0.050	0.615	0.488
	AB_{2k}	0.048	0.049	0.271	0.211	0.049	0.050	0.511	0.403
	T_2	0.047	0.049	0.408	0.315	0.048	0.050	0.733	0.601
$j = 3$	M_{3k}	0.056	0.056	0.564	0.445	0.053	0.053	0.867	0.763
	AB_{3k}	0.055	0.055	0.465	0.370	0.052	0.052	0.776	0.664
	T_3	0.057	0.056	0.666	0.550	0.054	0.054	0.942	0.871

Table 2: Normal populations

		$N_1 = 150, N_2 = 100, N_3 = 50$				$N_1 = 200, N_2 = 150, N_3 = 100$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.045	0.367	0.453	0.241	0.047	0.613	0.641	0.420
	AB_{1k}	0.046	0.309	0.375	0.207	0.048	0.518	0.538	0.349
	T_1	0.043	0.458	0.408	0.307	0.046	0.737	0.687	0.529
$j = 2$	M_{2k}	0.049	0.050	0.695	0.428	0.049	0.050	0.852	0.641
	AB_{2k}	0.049	0.050	0.594	0.361	0.049	0.050	0.761	0.545
	T_2	0.049	0.050	0.680	0.534	0.049	0.050	0.898	0.768
$j = 3$	M_{3k}	0.055	0.055	0.928	0.703	0.052	0.052	0.985	0.898
	AB_{3k}	0.055	0.055	0.861	0.613	0.051	0.052	0.956	0.823
	T_3	0.060	0.060	0.933	0.815	0.055	0.055	0.994	0.962

The simulation results show the validity of the permutation test and nonparametric tests. Every test may be conservative for $j = 1, 2$ under the null hypothesis. The nonparametric tests are more robust than the test using the statistic T_j . However, the power of the test using the statistic T_j is the greatest in both cases regardless of whether the sample sizes are equal or unequal, except only for Case 3 in Table 2. The power of Mood's test is greatest for $j = 1$ and 2 for Case 3 in the left of Table 2. The total sample sizes are equal in the right of Table 1 and the left of Table 2. The power of the test using the statistic T_j tends to decrease when the sample sizes are unequal. Further when we apply the permutation test for the statistic M_{jk} and AB_{jk} , their powers are almost equivalent to the power of the Mood test and Ansari-Bradley test.

In the study of environmental problems, the contaminated normal distribution is used a lot. Therefore we treat the contaminated normal population as a symmetric non-normal population in Table 3 and Table 4. Table 3 indicates the results for the cases of equal sample sizes. We show the results of tests for unequal sample sizes in Table 4.

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Table 3: Contaminated normal populations

		$N_1 = N_2 = N_3 = 50$				$N_1 = N_2 = N_3 = 100$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.040	0.233	0.215	0.163	0.044	0.462	0.411	0.311
	AB_{1k}	0.042	0.198	0.178	0.141	0.045	0.386	0.337	0.260
	T_1	0.033	0.261	0.228	0.174	0.037	0.512	0.452	0.344
$j = 2$	M_{2k}	0.048	0.048	0.329	0.248	0.047	0.048	0.601	0.473
	AB_{2k}	0.048	0.048	0.268	0.208	0.048	0.048	0.501	0.395
	T_2	0.046	0.047	0.379	0.293	0.043	0.044	0.672	0.541
$j = 3$	M_{3k}	0.055	0.055	0.555	0.438	0.051	0.050	0.858	0.751
	AB_{3k}	0.054	0.054	0.460	0.366	0.051	0.050	0.769	0.655
	T_3	0.056	0.058	0.639	0.524	0.049	0.050	0.913	0.824

Table 4: Contaminated normal populations

		$N_1 = 150, N_2 = 100, N_3 = 50$				$N_1 = 200, N_2 = 150, N_3 = 100$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.042	0.373	0.452	0.247	0.045	0.593	0.617	0.401
	AB_{1k}	0.044	0.318	0.377	0.214	0.046	0.505	0.521	0.338
	T_1	0.034	0.425	0.355	0.279	0.038	0.653	0.587	0.447
$j = 2$	M_{2k}	0.048	0.049	0.691	0.431	0.047	0.047	0.836	0.621
	AB_{2k}	0.050	0.050	0.593	0.365	0.048	0.048	0.745	0.531
	T_2	0.046	0.047	0.638	0.514	0.045	0.043	0.841	0.700
$j = 3$	M_{3k}	0.055	0.055	0.926	0.705	0.050	0.050	0.982	0.887
	AB_{3k}	0.055	0.055	0.860	0.616	0.050	0.050	0.951	0.812
	T_3	0.058	0.058	0.911	0.798	0.049	0.049	0.986	0.935

The simulation results permit that we may use the T_j statistic for testing the equality of the j -th largest eigenvalues for the contaminated normal population. All tests may be conservative for $j = 1, 2$ under the null hypothesis. If any N_i is greater than 100, all tests are also conservative for $j = 3$. In addition, the nonparametric tests are more robust than the test using the statistic T_j , especially $j = 1$. But the power of the test using the statistic T_j is the greatest in these cases regardless of whether the sample sizes are equal. For Case 3 on unequal sample sizes, the Mood test is more suitable than the test using the statistic T_j . However, the power of the test using the statistic T_j is higher than that of the nonparametric tests for Case 2 and Case 4. We further applied the permutation test for the M_{jk} and AB_{jk} statistics. Their powers are almost equivalent to the power of the original tests. The total sample sizes are equal in the right of Table 3 and the left of Table 4, but the power of the test using the statistic T_j tends to decrease when the sample sizes are unequal.

Finally, we simulate the skew normal populations $MSN(\nu^{(i)}, \mathbf{0}, \Sigma^{(i)})$ for an asymmetric non-normal population, where $\nu^{(i)} = (\nu_1, \nu_2, \nu_3)'$ denotes the shape parameter (Genton, 2004, Chap. 1, Sec. 5). We set $\nu_1 = \nu_2 = \nu_3 = 0.5$. The reason why we treat this population is that we sometimes assume this population in financial markets, property markets, labor markets *etc.*

Table 5 to Table 8 represents the results for the skew normal populations. In Table 5 and Table 6, we give the type I error and the power for the case of equal sample sizes. For

the case of the unequal sample sizes, we show the type I error and the power in Table 7 and Table 8. Let the correlation matrix be

$$\begin{pmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_3 \\ r_2 & r_3 & 1 \end{pmatrix}.$$

In the case of $r = r_1 = r_2 = r_3$, the results are listed in the left of Table 5 - Table 8. When $r_1 \neq r_2 \neq r_3$, we show the results in the right of Table 5 - Table 8.

Table 5: Skew normal populations ($N_1 = N_2 = N_3 = 50$)

		$r = 0.5$				$r_1 = 0.5, r_2 = 0.3, r_3 = 0.2$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.049	0.180	0.270	0.207	0.049	0.190	0.258	0.198
	AB_{1k}	0.049	0.155	0.226	0.174	0.049	0.161	0.211	0.167
	T_1	0.049	0.220	0.326	0.254	0.049	0.232	0.312	0.243
$j = 2$	M_{2k}	0.050	0.057	0.319	0.243	0.045	0.051	0.321	0.241
	AB_{2k}	0.050	0.055	0.259	0.203	0.046	0.050	0.261	0.201
	T_2	0.051	0.060	0.389	0.302	0.045	0.051	0.391	0.299
$j = 3$	M_{3k}	0.056	0.056	0.545	0.429	0.054	0.054	0.548	0.428
	AB_{3k}	0.055	0.055	0.449	0.356	0.053	0.053	0.450	0.356
	T_3	0.058	0.058	0.645	0.529	0.054	0.055	0.650	0.530

Table 6: Skew normal populations ($N_1 = N_2 = N_3 = 100$)

		$r = 0.5$				$r_1 = 0.5, r_2 = 0.3, r_3 = 0.2$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.050	0.336	0.500	0.387	0.050	0.356	0.479	0.371
	AB_{1k}	0.050	0.277	0.409	0.319	0.050	0.293	0.391	0.305
	T_1	0.050	0.423	0.609	0.486	0.049	0.448	0.586	0.466
$j = 2$	M_{2k}	0.050	0.063	0.578	0.456	0.048	0.058	0.585	0.459
	AB_{2k}	0.050	0.061	0.478	0.376	0.049	0.056	0.482	0.379
	T_2	0.050	0.066	0.695	0.566	0.048	0.059	0.703	0.570
$j = 3$	M_{3k}	0.053	0.053	0.851	0.743	0.052	0.053	0.859	0.753
	AB_{3k}	0.052	0.053	0.757	0.643	0.052	0.052	0.766	0.653
	T_3	0.054	0.054	0.930	0.851	0.053	0.053	0.936	0.861

For equal sample sizes, the simulation results reveal that all tests maintain a significant level for Case 1. The nonparametric tests are more robust than the test using the statistic T_j . However, the power of the test using the statistic T_j is higher than the power of the Ansari-Bradley test or Mood test under the alternative hypothesis. Next we show the results for unequal sample sizes.

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Table 7: Skew normal populations ($N_1 = 150, N_2 = 100, N_3 = 50$)

		$r = 0.5$				$r_1 = 0.5, r_2 = 0.3, r_3 = 0.2$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.049	0.268	0.545	0.306	0.049	0.282	0.522	0.291
	AB_{1k}	0.049	0.229	0.455	0.260	0.049	0.241	0.435	0.248
	T_1	0.049	0.336	0.511	0.385	0.048	0.354	0.486	0.367
$j = 2$	M_{2k}	0.051	0.062	0.647	0.385	0.047	0.054	0.645	0.377
	AB_{2k}	0.051	0.061	0.548	0.326	0.048	0.055	0.545	0.318
	T_2	0.052	0.073	0.625	0.484	0.047	0.062	0.621	0.476
$j = 3$	M_{3k}	0.055	0.056	0.915	0.679	0.055	0.055	0.925	0.700
	AB_{3k}	0.055	0.055	0.843	0.589	0.054	0.055	0.856	0.609
	T_3	0.060	0.062	0.919	0.793	0.060	0.062	0.930	0.814

Table 8: Skew normal populations ($N_1 = 200, N_2 = 150, N_3 = 100$)

		$r = 0.5$				$r_1 = 0.5, r_2 = 0.3, r_3 = 0.2$			
		Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
$j = 1$	M_{1k}	0.050	0.443	0.731	0.505	0.050	0.468	0.708	0.483
	AB_{1k}	0.050	0.369	0.628	0.422	0.050	0.390	0.605	0.403
	T_1	0.049	0.552	0.781	0.624	0.049	0.582	0.757	0.599
$j = 2$	M_{2k}	0.050	0.068	0.814	0.594	0.049	0.061	0.816	0.593
	AB_{2k}	0.050	0.065	0.716	0.502	0.049	0.059	0.718	0.500
	T_2	0.050	0.077	0.863	0.719	0.048	0.068	0.865	0.719
$j = 3$	M_{3k}	0.053	0.053	0.981	0.882	0.052	0.053	0.984	0.894
	AB_{3k}	0.052	0.053	0.947	0.802	0.052	0.053	0.953	0.812
	T_3	0.055	0.055	0.992	0.952	0.054	0.055	0.993	0.959

For the cases of unequal sample sizes, the simulation results reveal that all tests maintain the significance level for $j = 1$ and 2 on the Case 1. However, in Case 2 for $j = 2$, the Mood test, Ansari-Bradley test and our test don't keep the significance level well. For $j = 3$ in Case 1 and Case 2, the nonparametric tests maintain the significance level. Therefore the Mood test and Ansari-Bradley test are more robust than our test. In Case 3 of Table 7, the Mood test is more powerful than our test, but the power of our test is higher than the nonparametric statistics under other alternative hypothesis.

4. Conclusion

We develop a testing procedure for the statistic T_j in a multipopulation and compare the power of the test using the statistic T_j and two nonparametric tests. The limiting distribution of the proposed statistic is derived when the sample sizes are equal. In the case of unequal sample sizes, it is difficult to derive the limiting distribution of the T_j statistic. Therefore we use the permutation test for the statistic T_j . In addition, we apply the testing procedure with the normal population, the contaminated normal population and the skew normal population and show that the estimated significance level of our test procedure, the Mood test and the Ansari-Bradley test converge for the significance level. Through the simulation studies, every test may be conservative for $j = 1, 2$ under the null hypothesis and the nonparametric tests are more robust than the test using the statistic T_j . However, the

power of the test using the statistic T_j is the greatest in both cases regardless of whether the sample sizes are equal or unequal. Therefore, we expect that the proposed test procedure is suitable for a multipopulation and this is shown by simulation.

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