THE ORTHOGONAL PROJECTION MATRIX FOR THE UNBALANCED FACTORIAL DESIGN

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ABSTRACT

The design matrix X for an unbalanced factorial design can be expressed as a product of two matrices T and X_o , where T is replication matrix and X_o is the corresponding balanced design matrix. We suggest an efficient method for computing the orthogonal projection matrix $P_X = X(X'X)^-X'$ for an unbalanced model $\mathbf{y} = X\boldsymbol{\beta} + \mathbf{e}$ using the nonzero eigenvalues and eigenvectors of $X_o'X_o$. Also we can easily find a regular rule for the nonzero eigenvalues and the corresponding eigenvectors.

1. Introduction

We consider the model

$$\mathbf{y} = X\boldsymbol{\beta} + \mathbf{e} \quad (1.1)$$

where $\mathbf{y}$ is an $N \times 1$ vector of observations, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, X is an $N \times p$ design matrix of 0's and 1's ($N \geq p$), and $\mathbf{e}$ is a vector of random error terms. In general, when each cell does not contain the same number of observations or when there are some empty cells, the model is called an unbalanced model. The analysis of this case is not easy and requires much computation. To avoid this difficulty, one needs to easily find the orthogonal projection matrix that is necessary for analyzing the model. The total sum of squares for (1.1) can be decomposed into

$$\mathbf{y}'\mathbf{y} = \mathbf{y}'P_X\mathbf{y} + \mathbf{y}'(I - P_X)\mathbf{y} \quad (1.2)$$

where $P_X = X(X'X)^-X'$ and $(X'X)^-$ is a generalized inverse of $X'X$. In view of (1.2) P_X is important to compute various sums of squares in the analysis of variance.

The matrix $X(X'X)^-X$, denoted by P_X , is called the orthogonal projection matrix and is unique for any choice of generalized inverse of $X'X$. Because of $P_X = XX^+$ where X^+ is the Moore-Penrose inverse of a matrix X , it is useful to find the Moore-Penrose inverse of design matrix in order to compute the orthogonal projection matrix. There are many methods to compute the Moore-Penrose inverse of a matrix. Shinozaki et al.(1972a, 1972b) surveyed the methods for computing the Moore-Penrose inverse into two subjects: direct methods and iterative methods. In order to find the Moore-Penrose inverse of an arbitrary matrix, these methods are very useful. But in most cases the design matrix has the special form, actually the design matrix consists of regular arrangement of 0's and 1's. Hence we can find X^+ using the Moore-Penrose conditions without using general computational methods. Kim and Sunwoo(1989) found the explicit form of the Moore-Penrose inverse of design

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matrix for the balanced model with no interactions. Also Kim and Sunwoo(1990) derived the iterative method for computing the Moore-Penrose of design matrix for balanced factorial design with interactions and from this the orthogonal projection matrix is computed. But it is more important to compute P_X than X^+ from the point of view of statistics. Also when the number of levels of main effects is large or the model has many main and interaction effects, it requires much computation to find X^+ , especially for an unbalanced model. Because of this difficulty, we shall find $(X'X)^+$ instead of X^+ in obtaining P_X . Because $X'X$ is a symmetric positive semi-definite and hence it is easy to compute $(X'X)^+$ using the nonzero eigenvalues and eigenvectors of X'_oX_o for the corresponding balanced model.

We will propose an efficient algorithm for computing P_X for an unbalanced model using the nonzero eigenvalues and eigenvectors of X'_oX_o as a direct procedure. In section 2 we derive a concrete form of the orthogonal projection matrix for an unbalanced model from the spectral decomposition of X'_oX_o . In section 3 we will find the nonzero eigenvalues of X'_oX_o and the corresponding eigenvectors in cases with and without interaction effects. Also some examples will be given.

2. Orthogonal projection matrix

For a given unbalanced linear model $\mathbf{y} = X\beta + \mathbf{e}$, the corresponding balanced model is defined as the model that contains exactly one observation in each cell and design matrix of the corresponding balanced model is denoted by X_o . It is well known that design matrix X can be expressed as a product $X = TX_o$ where T is replication matrix. Also the matrix of cell frequencies D can be obtained as $D = T'T$. For example, in a two-way additive model

$$y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk}, \quad i = 1, \dots, a; \quad j = 1, \dots, b; \quad k = 1, \dots, n_{ij}$$

with $a = 2, b = 2, n_{11} = 2, n_{12} = 3, n_{21} = n_{22} = 1$, we would have

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad X_o = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Put $r = \text{rank}(X_o)$. Let $X'_oX_o = Q_r \Lambda_r Q'_r$ be a spectral decomposition of X'_oX_o where Λ_r is a diagonal matrix of positive eigenvalues of X'_oX_o and Q_r is a $p \times r$ matrix whose columns are orthonormalized eigenvectors corresponding to positive eigenvalues.

The following lemma gives us the useful facts in obtaining the simple form of the orthogonal projection matrix for an unbalanced model.

Lemma 2.1. Letting $K_r = \Lambda_r^{-1/2} Q_r' X' X Q_r \Lambda_r^{-1/2}$

$$X'X = X_o' D X_o = Q_r \Lambda_r^{1/2} K_r \Lambda_r^{1/2} Q_r', \quad (2.1)$$

$$(X'X)^+ = (X_o' D X_o)^+ = (Q_r \Lambda_r^{1/2} K_r \Lambda_r^{1/2} Q_r')^+ = Q_r \Lambda_r^{-1/2} K_r^{-1} \Lambda_r^{-1/2} Q_r'. \quad (2.2)$$

Proof. Since $X_o' X_o$ is symmetric positive semi-definite, by spectral decomposition theorem we can write $X_o' X_o = Q \Lambda Q'$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$ and $Q = [Q_r \ Q_o]$ is an orthogonal matrix partitioned so that Q_r is a $p \times r$ matrix. Then $Q_o' Q_o = I_{p-r}$, $Q_r' Q_r = I_r$ and $Q_o Q_o' + Q_r Q_r' = I_p$ since Q is orthogonal matrix. From $K_r = \Lambda_r^{-1/2} Q_r' X' X Q_r \Lambda_r^{-1/2}$,

$$Q_r \Lambda_r^{1/2} K_r \Lambda_r^{1/2} Q_r' = X_o' D X_o = X' X$$

since $Q_o' X_o' X_o Q_o = 0$ and also $X_o Q_o = 0$. Next the proof of (2.2) can be easily verified. $\square$

Using these properties we can find the simple form of the orthogonal projection matrix for an unbalanced model. The form is as follows.

Theorem 2.1. For matrices defined above

$$P_X = T Z K_r^{-1} Z' T' \quad (2.3)$$

where $Z = X_o Q_r \Lambda_r^{-1/2}$ and $K_r = Z' D Z$.

In general the size of P_X is very large. But as is shown in (2.3) the prime part of computations are accomplished by handling matrices with smaller size. The remaining computations are merely matrix multiplications. Since K_r is an $r \times r$ symmetric positive definite, by Cholesky decomposition, we can write $K_r = L L'$ where L is a lower triangular matrix with positive diagonal elements. Furthermore, $K_r^{-1} = (L^{-1})' L^{-1}$. Hence it is important to find Z in order to compute P_X because of $K_r = Z' D Z$. Also it is interesting to note that $Z Z' = P_{X_o}$ and $Z' Z = I_r$ hold. A concrete form of $Z = X_o Q_r \Lambda_r^{-1/2}$ is described in the next section. If Z is known for each classification model, it is useful to compute the reduction sum of squares. For example, partitioning main effects α as $\alpha' = [\alpha'_1, \alpha'_2, \dots, \alpha'_k]$ for k -way factorial design, Type II sum of squares is as follows:

$$\begin{aligned} & R(\alpha_i | \mu, \alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k) \\ &= R(\mu, \alpha_1, \dots, \alpha_k) - R(\mu, \alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k) \\ &= \mathbf{y}' P_X \mathbf{y} - \mathbf{y}' P_{X_{-i}} \mathbf{y} \end{aligned}$$

where X_{-i} is the design matrix for $(k-1)$ -way factorial design that is removing i th main effect from the original model.

3. Eigenvalues and eigenvectors

Let X_o be the corresponding balanced design matrix. With a help of matrix algebra we know that positive eigenvalues and eigenvectors of $X_o' X_o$ depend on only the number of levels of main effects by a regular rule. From this we can find an explicit form of the corresponding eigenvectors. For the regular rule of positive eigenvalues and eigenvectors of $X_o' X_o$ we will deal with two cases: with and without interaction effects.

We shall use boldface lower case letters to denote column vectors and the following notation: $\mathbf{1}_m$ is an m -dimensional column vector of unities and $\mathbf{0}_m$ is an m -dimensional column vector of zeros.

3.1. Case without interaction effects

Since X_o has special form of 0's and 1's, positive eigenvalues of $X_o'X_o$ and the corresponding eigenvectors can be easily found as shown in the following theorem.

Theorem 3.1.1. *Let $X_o = [1_m \ X_1 \ \dots \ X_k]$ be the corresponding balanced design matrix for the k -way model*

$$y = 1_m\mu + X_1\alpha_1 + \dots + X_k\alpha_k + e \quad (3.1.1)$$

where $m = \prod_{j=1}^k m_j$ and m_j is the number of levels of j th main effect. Then for positive eigenvalues λ_i 's of $X_o'X_o$

$$u_0 = [m \ \frac{m}{m_1}1'_{m_1} \ \dots \ \frac{m}{m_k}1'_{m_k}]' \quad (3.1.2)$$

is the eigenvector corresponding to $\lambda_0 = m + \sum_{j=1}^k (m/m_j)$; for $j = 1, \dots, k$

$$u_i = [0 \ 0'_{m_1} \ \dots (e_1^{(j)} - e_{i+1}^{(j)})' \ \dots 0'_{m_k}]', \quad i = 1, \dots, m_j - 1 \quad (3.1.3)$$

are the eigenvectors corresponding to $\lambda_j = m/m_j$, where $e_i^{(j)}$ is an $m_j \times 1$ vector having 1 as i -th entry and zero elsewhere, i.e., $e_i^{(j)} = (0 \dots 0 \ 1 \ 0 \dots 0)'$.

Proof. Since the incidence matrices X_i 's can be expressed using Kronecker products of identity matrix and vectors with all elements equal to unity, that is,

$$X_i = 1_{m_1} \otimes \dots \otimes I_{m_i} \otimes \dots \otimes 1_{m_k}$$

it is easy to see that the following relations hold:

$$X_i'X_i = \frac{m}{m_i}I_{m_i}, \quad X_i'1_m = \frac{m}{m_i}1_{m_i}, \quad X_i1_{m_i} = 1_m \quad \text{for } i = 1, \dots, k$$

$$X_i'X_j = \frac{m}{m_i m_j}J_{m_i \times m_j} \quad \text{for } i \neq j,$$

where $J_{m_i \times m_j} = 1_{m_i}1'_{m_j}$. Then

$$X_o'X_o u_0 = \begin{bmatrix} m^2 + \sum_{j=1}^k \lambda_j 1'_m X_j 1_{m_j} \\ mX_1'1_m + \sum_{j=1}^k \lambda_j X_1'X_j 1_{m_j} \\ \vdots \\ mX_k'1_m + \sum_{j=1}^k \lambda_j X_k'X_j 1_{m_j} \end{bmatrix} = (m + \sum_{j=1}^k \lambda_j) \begin{bmatrix} m \\ \lambda_1 1_{m_1} \\ \vdots \\ \lambda_k 1_{m_k} \end{bmatrix}.$$

Hence u_0 is the eigenvector of $X_o'X_o$ corresponding to $\lambda_0 = m + \sum_{j=1}^k \lambda_j$. Similarly, the remaining part follows from the relations described above. $\square$

As special cases of theorem 3.1.1, positive eigenvalues of $X_o'X_o$ for two- and three-way factorial designs are as follows. The numbers in brackets are multiplicities of the corresponding eigenvalues.

(i) Two-way model: setting $m_1 = a$, $m_2 = b$

$$\text{eigenvalues: } ab + b + a[1], \ b[a - 1], \ a[b - 1]$$

(ii) Three-way model: setting $m_1 = a$, $m_2 = b$, $m_3 = c$

eigenvalues : $abc + bc + ac + ab[1]$, $bc[a - 1]$, $ac[b - 1]$, $ab[c - 1]$

By theorem 3.1.1 we can construct Q_r for the k -way model from Gram-Schmidt orthogonalization. Using Q_r and Λ_r we also can easily find $Z = X_o Q_r \Lambda_r^{-1/2}$. Put $Z = [z_0 \ Z_1 \ \cdots \ Z_k]$ where $Z_j = [z_1 \ \cdots \ z_{m_j-1}]$, $j = 1, \dots, k$. Then

$$z_0 = \frac{1}{\sqrt{m}} \mathbf{1}_m \quad (3.1.4)$$

$$z_i^{(j)} = \frac{\sqrt{m_j}}{\sqrt{m} \sqrt{i(i+1)}} \left[\mathbf{1}_{m_1} \otimes \cdots \otimes \left(\sum_{t=1}^i \mathbf{e}_t^{(j)} - i \mathbf{e}_{i+1}^{(j)} \right) \otimes \cdots \otimes \mathbf{1}_{m_k} \right] \quad (3.1.5)$$

for all $j = 1, \dots, k$ and $i = 1, \dots, m_j - 1$.

From (3.1.4) and (3.1.5) we know that the computation of Z requires only the number of levels of main effects. Also note that $\sum_{t=1}^i \mathbf{e}_t^{(j)} - i \mathbf{e}_{i+1}^{(j)}$ in (3.1.5) is an $m_j \times 1$ vector having 1 as first through i -th components, $-i$ as $i+1$ -th component and zero elsewhere, that is, $\sum_{t=1}^i \mathbf{e}_t^{(j)} - i \mathbf{e}_{i+1}^{(j)} = (1 \cdots 1 \ -i \ 0 \cdots 0)'$.

We introduce a simple example to find the orthogonal projection matrix P_X for the three-way model

$$\mathbf{y} = \mathbf{1}_m \mu + X_1 \alpha + X_2 \beta + X_3 \gamma + \mathbf{e} \quad (3.1.6)$$

with $m_1 = m_2 = m_3 = 2$.

Example 1. For model (3.1.6) with cell frequencies $n_{111} = n_{122} = n_{212} = n_{221} = 1$, $n_{112} = n_{121} = n_{211} = n_{222} = 2$, we would get

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad X_o = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & & & & & & & \\ & 2 & & & & & & \\ & & 2 & & & & & \\ & & & 1 & & & & \\ & & & & 2 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 2 \end{bmatrix}$$

Then from (3.1.4) and (3.1.5) appropriate matrices are as follows:

$$\Lambda_r = \begin{bmatrix} 20 & & & \\ & 4 & & \\ & & 4 & \\ & & & 4 \end{bmatrix}, \quad Z = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix},$$

$$K_r = \frac{3}{2}I_4.$$

Hence from (2.3) P_X becomes

$$P_X = \frac{1}{6} \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 2 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \end{bmatrix}$$

3.2. Case with interaction effects

Before we describe the general procedure we consider the representations of the model equation. We denote the incidence matrix of interaction effects between i -th and j -th factor by $V_{\{i,j\}}$ and between i -th, j -th, and k -th factor by $V_{\{i,j,k\}}$, and so on. For the purpose of explaining our results consider the set of integers $T = \{1, 2, \dots, k\}$ and let S_j be a subset of T which contains at least two elements of T such that $n(S_1) \leq n(S_2) \leq \dots \leq n(S_c)$, where $n(S_j)$ is the size of set S_j , $j = 1, 2, \dots, c$ for some integer c . $V_j = V_{S_j}$ denotes the

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incidence matrix of interaction effects and their indices are members of S_j . It is well known that for k -way factorial design the incidence matrices can be expressed using Kronecker products (Rogers(1984)):

$$V_j = V_{S_j} = \otimes_{i=1}^k A_i \quad (3.2.1)$$

$$\text{where } A_i = \begin{cases} \mathbf{1}_{m_i} & \text{if } i \notin S_j, \\ I_{m_i} & \text{if } i \in S_j. \end{cases} \quad (3.2.2)$$

Now let $X_o = [\mathbf{1}_m \ U_o \ V_o]$ be the balanced design matrix for the model with k main effects and c interaction effects

$$\mathbf{y} = \mathbf{1}_m \mu + U_o \alpha + V_o \beta + \mathbf{e} \quad (3.2.3)$$

where U_o is the incidence matrix of main effects and $V_o = [V_1 \ V_2 \ \dots \ V_c]$, V_j is an $m \times d_j$ matrix of known values ($d_j = \prod_{s \in S_j} m_s$), and other terms are defined as usual. Then for positive eigenvalues and eigenvectors of $X_o' X_o$ we can obtain the following theorem based on the case without interaction effects.

Theorem 3.2.1. For model (3.2.3) positive eigenvalues of $X_o' X_o$ are as follows:

(i)

$$\lambda = \lambda_0^* [1], \quad \lambda_i^* [m_i - 1] \ (i = 1, \dots, k), \quad \lambda_{k+j}^* \left[\prod_{s \in S_j} (m_s - 1) \right] \ (j = 1, \dots, c)$$

$$\lambda_0^* = \lambda_0 + \sum_{j=1}^c \frac{m}{d_j}, \quad \text{where } \lambda_0 = m + \sum_{i=1}^k \frac{m}{m_i}, \quad (3.2.4)$$

$$\lambda_i^* = \lambda_i + \sum_{j=1}^c h_j, \quad i = 1, \dots, k, \quad (3.2.5)$$

$$\text{where } \lambda_i = \frac{m}{m_i} \quad \text{and} \quad h_j = \begin{cases} m/d_j & \text{if } i \in S_j \\ 0 & \text{if } i \notin S_j, \end{cases}$$

$$\lambda_{k+j}^* = \lambda_{k+j} + \sum_{n=j+1}^c t_n, \quad j = 1, \dots, c, \quad (3.2.6)$$

$$\text{where } \lambda_{k+j} = \frac{m}{d_j} \quad \text{and} \quad t_n = \begin{cases} m/d_n & \text{if } S_j \subset S_n \\ 0 & \text{if } S_j \not\subset S_n. \end{cases}$$

The numbers in brackets are multiplicities of the corresponding eigenvalues.

(ii)

$$\mathbf{u}_0^* = [m \ \lambda_1 \mathbf{1}'_{m_1} \ \dots \ \lambda_k \mathbf{1}'_{m_k} \ \lambda_{k+1} \mathbf{1}'_{d_1} \ \dots \ \lambda_{k+c} \mathbf{1}'_{d_c}]' \quad (3.2.7)$$

is the eigenvector of $X_o' X_o$ corresponding to $\lambda = \lambda_0^*$. And for $i = 1, \dots, k$

$$\mathbf{u}_l^* = [0 \ 0'_{m_1} \ \dots \ \lambda_i (\mathbf{e}_{li} - \mathbf{e}_{l+1i})'_{m_i} \ \dots \ 0'_{m_k} \ \lambda_{k+1} \mathbf{p}'_1 \ \dots \ \lambda_{k+c} \mathbf{p}'_c]' , \quad l = 1, \dots, m_i - 1 \quad (3.2.8)$$

are the eigenvectors of $X_o' X_o$ corresponding to $\lambda = \lambda_i^*$, where

$$\mathbf{p}_j = \begin{cases} \otimes_{s \in S_j} \mathbf{b}_s & \text{if } i \in S_j \\ \mathbf{0}_{d_j} & \text{if } i \notin S_j \end{cases}, \quad \mathbf{b}_s = \begin{cases} \mathbf{e}_1^{(i)} - \mathbf{e}_{l+1}^{(i)} & \text{if } s = i \\ \mathbf{1}_{m_s} & \text{if } s \neq i \end{cases}$$

(iii) for $j = 1, \dots, c$

$$\mathbf{u}_{l(s)_j} = [0 \ \mathbf{0}'_{m_1} \cdots \mathbf{0}'_{m_k} \ \mathbf{0}'_{d_1} \cdots \lambda_{k+j} \mathbf{r}'_j \ \lambda_{k+j+1} \mathbf{r}'_{j+1} \cdots \lambda_{k+c} \mathbf{r}'_c]' \quad (3.2.9)$$

are the eigenvectors of $X'_o X_o$ corresponding to $\lambda = \lambda_{k+j}^*$, where the index notation $l(s)_j$ denotes a possible combination of $\{1, \dots, m_s - 1\}$ for each $s \in S_j$, and

$$\mathbf{r}_j = \otimes_{s \in S_j} (\mathbf{e}_1^{(s)} - \mathbf{e}_{l(s)+1}^{(s)})$$

and for $n = j + 1, \dots, c$

$$\mathbf{r}_n = \begin{cases} \otimes_{s \in S_n} \mathbf{c}_s & \text{if } S_j \subset S_n \\ \mathbf{0}_{d_n} & \text{if } S_j \not\subset S_n \end{cases}, \quad \mathbf{c}_s = \begin{cases} \mathbf{e}_1^{(s)} - \mathbf{e}_{l(s)+1}^{(s)} & \text{if } s \in S_j \\ \mathbf{1}_{m_s} & \text{if } s \notin S_j \end{cases}.$$

Proof. Similarly as in theorem 3.1.1, the theorem can be easily proved. $\square$

As example for (c), when a three-way model with $\alpha\beta$, $\alpha\beta\gamma$ interaction effects has $m_1 = 2$, $m_2 = m_3 = 3$, it does $S_1 = \{1, 2\}$, $S_2 = \{1, 2, 3\}$, and $\{l(s)_1 | s \in S_1\} = \{11, 12\}$, $\{l(s)_2 | s \in S_2\} = \{111, 112, 121, 122\}$.

From theorem 3.2.1 we can find a regular rule that changes the eigenvalues for the resulting model when the interaction effects are added the original model. Also this rule can be applied to the corresponding eigenvectors. For example, positive eigenvalues correspond to each effect one by one:

$$\mu \rightarrow \lambda_o = m + \sum_{i=1}^k (m/m_i)$$

$$\alpha_i \rightarrow \lambda_i = m/m_i$$

$$\beta_j \rightarrow \lambda_{k+j} = m/d_j.$$

Then the new eigenvalues λ_i^* and λ_{k+j}^* are changed by adding an eigenvalue corresponding to the effect whenever there is interaction effect containing i th main effect and j th interaction effect, respectively. Also $\lambda_o^* = \lambda_o + \sum_{j=1}^c \lambda_{k+j}$.

As in the case without interaction effects, we can easily compute Z by Gram-Schmidt orthogonalization. Now let $Z = [z_0 \ z_1 \ \cdots \ z_k \ z_{k+1} \ \cdots \ z_{k+c}]$ where $Z_i = [z_{i1} \ \cdots \ z_{i m_i - 1}]$, and Z_{k+j} is a matrix whose columns are $\mathbf{z}_{l(s)_j}$'s. Then for all $i = 1, \dots, k$ and $l = 1, \dots, m_i - 1$

$$\mathbf{z}_0 = \frac{1}{\sqrt{m}} \mathbf{1}_m \quad (3.2.10)$$

$$\mathbf{z}_l = \frac{\sqrt{m_i}}{\sqrt{m} \sqrt{l(l+1)}} \left[\mathbf{1}_{m_1} \otimes \cdots \otimes \left(\sum_{k=1}^l \mathbf{e}_k^{(i)} - l \mathbf{e}_{l+1}^{(i)} \right) \otimes \cdots \otimes \mathbf{1}_{m_k} \right] \quad (3.2.11)$$

and for all $j = 1, \dots, c$ and $l(s) = 1, \dots, m_s - 1$ for all $s \in S_j$

$$\mathbf{z}_{l(s)_j} = \frac{\sqrt{d_j}}{\sqrt{m} \sqrt{\sum_{s \in S_j} l(s)(l(s)+1)}} [\otimes_{i=1}^k B_i] \quad (3.2.12)$$

where

$$B_i = \begin{cases} \sum_{t=1}^l \mathbf{e}_t^{(i)} - l\mathbf{e}_{l+1}^{(i)} & \text{if } i \in S_j \\ \mathbf{1}_{m_i} & \text{if } i \notin S_j \end{cases}.$$

Example 2. Consider the three-way model with $\alpha\beta$ interaction effect

$$\mathbf{y} = \mathbf{1}_m\mu + X_1\alpha + X_2\beta + X_3\gamma + V(\alpha\beta) + \mathbf{e} \quad (3.2.13)$$

having $m_1 = m_2 = m_3 = 2$, $S_1 = \{1, 2\}$, $d_1 = m_1m_2 = 4$, and cell frequencies $n_{111} = n_{122} = n_{212} = n_{221} = 1$, $n_{112} = n_{121} = n_{211} = n_{222} = 2$.

In this case, the positive eigenvalues of $X_o'X_o$ are as follows:

$$\lambda_0^* = 22[1], \quad \lambda_1^* = \lambda_2^* = 6[1], \quad \lambda_3^* = 4[1], \quad \lambda_4^* = 2[1]$$

where the numbers in brackets are multiplicities of the corresponding eigenvalues.

Also as in example 1 appropriate matrices are given by

$$Z = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix}, \quad K_r = \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

from (3.2.10), (3.2.11) and (3.2.12). By (2.3) P_X becomes

$$P_X = \frac{1}{24} \begin{bmatrix} 12 & 6 & 6 & 2 & 2 & -4 & 2 & 2 & -4 & 4 & -2 & -2 \\ 6 & 9 & 9 & -1 & -1 & 2 & -1 & -1 & 2 & -2 & 1 & 1 \\ 6 & 9 & 9 & -1 & -1 & 2 & -1 & -1 & 2 & -2 & 1 & 1 \\ 2 & -1 & -1 & 9 & 9 & 6 & 1 & 1 & -2 & 2 & -1 & -1 \\ 2 & -1 & -1 & 9 & 9 & 6 & 1 & 1 & -2 & 2 & -1 & -1 \\ -4 & 2 & 2 & 6 & 6 & 12 & -2 & -2 & 4 & -4 & 2 & 2 \\ 2 & -1 & -1 & 1 & 1 & -2 & 9 & 9 & 6 & 2 & -1 & -1 \\ 2 & -1 & -1 & 1 & 1 & -2 & 9 & 9 & 6 & 2 & -1 & -1 \\ -4 & 2 & 2 & -2 & -2 & 4 & 6 & 6 & 12 & -4 & 2 & 2 \\ 4 & -2 & -2 & 2 & 2 & -4 & 2 & 2 & -4 & 12 & 6 & 6 \\ -2 & 1 & 1 & -1 & -1 & 2 & -1 & -1 & 2 & 6 & 9 & 9 \\ -2 & 1 & 1 & -1 & -1 & 2 & -1 & -1 & 2 & 6 & 9 & 9 \end{bmatrix}$$

4. Concluding remarks

From (2.3) we can easily obtain the orthogonal projection matrix for a given unbalanced model by computing the inverse matrix and then multiplying the resulting matrices. Although the notation in theorem 3.1.1 and theorem 3.2.1 seems to be complicated, the computing procedure for Z as shown in (3.1.5), (3.2.11) and (3.2.12) is simple and efficient whether or not there are interactions because the expressions depend only on the number of levels of main effects. By using Z we also can reduce much of the computational effort and computational storage since computing P_X relies upon Z . It is useful to compute various sums of squares in the analysis of variance. As well, it can be applied to computing F -statistic for testing a general linear hypothesis of the form $H : K'\beta = \mathbf{m}$.

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