Determining Stress Components from Isopachic Contours Using Airy Stress Function

Satoru YONEYAMA and Keigo OKUNO

Department of Mechanical Engineering, Aoyama Gakuin University, Sagamihara 252-5258, Japan

(Received 8 November 2010; received in revised form 5 April 2011; accepted 5 June 2011)

Abstract
A method for determining stresses from isopachic contours obtained by interferometry or thermoelasticity is proposed in this paper. A Poisson equation that represents the relationship between the sum of principal stresses and an Airy stress function is solved using a finite element method. Therefore, the distribution of the Airy stress function can be obtained from the isopachic contours. The stresses are obtained from the Airy stress function. Boundary conditions are required for solving the Poisson equation. In the present stage, tractions along the boundary are used for obtaining the Neumann boundary condition. Results of simulation and experiment indicate that stress components can be obtained from isopachics by the proposed method.

Key words
Stress Separation, Stress Components, Isopachics, Sum of Principal Stresses, Airy Stress Function, Finite Element Method, Interferometry

1. Introduction
It is known that the distributions of stress or strain components are usually obtained by the measurement of displacements. Interferometric techniques such as moiré, speckle, and holography have long been used for the measurement of in-plane displacements for the study of deformation and fracture of solids [1]. In these techniques, two independent measurements are usually required for in-plane displacements. In other words, an optical setup for two different directions is needed for obtaining two displacement components. Furthermore, an optical setup for the in-plane displacement measurement is complicated compared with other interferometers such as Twyman-Green or Mach-Zehnder interferometry. When Mach-Zehnder fringe pattern is observed under plane stress condition, isopachic contours, that is, the sum of principal stresses can be obtained [2,3]. The sum of principal stresses is also obtainable by holographic interferometry [4] and thermoelasticity [5]. Therefore, it is convenient if stress components can be obtained from the sum of principal stresses.

Various techniques have been proposed for stress separation of isopachics in thermoelasticity. Ryall et al [6], and Rowlands and coworkers [7,8] developed hybrid methods of thermoelasticity and the classical theory of elasticity. In their techniques, unknown coefficients of a stress function are determined from isopachic contours. Then, stresses are obtained from the stress function. Murakami and Yoshimura [9], Kishimoto et al. [10], and Machida and Katsuma [11] proposed inverse techniques for estimating unknown boundary conditions from isopachic contours. After determining the boundary conditions, stress distributions are obtained by finite element or boundary element direct analysis. However, inverse boundary value problems are often ill-posed. Therefore, various techniques should be introduced to the inverse analysis for obtaining stable and accurate results [9-13].

On the other hand, Hori and Kameda [14] proposed an inverse method for determining stresses from strains with unknown constitutive relations. In their method, an Airy stress function is determined from measured strains, elastic modulus, and tractions along boundaries using a finite element method. Because a Poisson equation of an Airy stress function is related to the sum of principal stresses, it is expected that the Hori and Kameda’s method can be extended to the stress separation.

In the present paper, the first step approach to the development of an alternative method for determining stress components from the sum of principal stresses is described. The sum of normal stresses is equivalent to the sum of the second order derivatives of an Airy stress function. This relationship forms Poisson equation that can be solved by numerical techniques such as a finite element method. Therefore, the distribution of the Airy stress function can be obtained from the measured isopachic contours. Then, the stresses are obtained from the Airy stress function. Boundary conditions are required for solving the Poisson equation. In the present stage, tractions along the boundary are used for obtaining the Neumann boundary condition input into the algorithm by the proposed method. Results of simulation and experiment indicate that stress components can be obtained from isopachics by the proposed method. Further investigation on the determination of the boundary condition for solving the Poisson equation is required. However, it is expected that stress components can be obtained easily from the measurement of isopachics by further improvement of the proposed method.

2. Basic Principle and Formulation
It is well known that the solution for stresses can be expressed in terms of Airy stress function for a two-dimensional isotropic material under elastic deformation. The stresses are obtained as [15]

\[
\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y},
\]

where \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) are the stress components and \( U \) is the Airy stress function. The sum of normal stresses is known as a stress invariant and is equivalent to the sum of
principal stresses. From Eq. (1), the sum of principal stresses are related to the Airy stress function as

\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \sigma_1 + \sigma_2,
\]

(2)

where \(\sigma_1\) and \(\sigma_2\) are the principal stresses. It is recognized that Eq. (2) is Poisson equation. Therefore, Eq. (2) can be solved provided that boundary conditions are given [14].

As shown in Fig. 1, it is considered that the sums of principal stresses are obtained by the measurement in the analysis region \(\Omega\). The boundary of the analysis region is referred as \(\Gamma\) and is composed of Dirichlet boundary \(\Gamma_1\) and Neumann boundary \(\Gamma_2\). The Airy stress function is obtained by solving the following boundary value problem. The Poisson equation is

\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x, y) \text{ in } \Omega,
\]

(3)

where \(f(x, y) = \sigma_1 + \sigma_2\). The Dirichlet and Neumann boundary conditions for the Poisson equation (3) are expressed respectively as

\[
U = U_\text{d} \text{ along } \Gamma_1,
\]

\[
q = \frac{\partial U}{\partial n} = \frac{\partial U}{\partial x} n_x + \frac{\partial U}{\partial y} n_y = q_\text{d} \text{ along } \Gamma_2.
\]

(4)

Here, a bar on \(U\) and \(q\) indicates that the quantities are given as the boundary condition. The outer normal along the boundary is expressed as \(n\), its components are \(n_x\) and \(n_y\), and \(\partial/\partial n\) is the symbol for the derivative to the normal direction. Equations (3) and (4) form a boundary value problem for Airy stress function. In Eq. (3), \(f(x, y) = \sigma_1 + \sigma_2\) is measured by interferometry or thermoelasticity. The Dirichlet boundary condition, that is, the Airy stress function along boundary cannot be known. However, in the present stage, the Neumann boundary condition is determined from the traction. The determination of the boundary condition is one of the difficulties in the proposed method and will be discussed in the future. The resultant forces, \(r_x\), and \(r_y\), along the boundary are

\[
r_x = \int t_x \, d\ell = \int n_x \sigma_x + n_y \tau_{xy} \, d\ell = \frac{\partial U}{\partial y},
\]

\[
r_y = \int t_y \, d\ell = \int n_y \tau_{xy} + n_x \sigma_y \, d\ell = \frac{\partial U}{\partial x},
\]

(5)

where \(t_x\) and \(t_y\) are the tractions, \(\ell\) expresses the distance along the boundary from a reference point. These resultant forces prescribe the Neumann boundary condition as

\[
q = -r_x n_x + r_y n_y \text{ along } \Gamma_2,
\]

(6)

The existence and uniqueness are guaranteed for the solution of Eq. (3) because the governing equation is the Poisson equation and the Neumann boundary condition is provided.

In this study, the boundary value problems described above is solved using a finite element method. The analysis region is digitized by finite elements. The weak form of Eq. (3) for a single finite element can be written as

\[
\int_{\Omega} \left( \frac{\partial U}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial v}{\partial y} \right) \, d\Omega = -\int_{\Gamma} fv \, d\Gamma + \int_{\Gamma} qv \, d\Gamma,
\]

(7)

where \(v\) is a test function, \(e\) expresses the region of a single element and \(\partial e\) is its boundary.

The unknown function \(U\) of an element is expressed as

\[
U = N^e U^e,
\]

(8)

where \(N^e\) is the shape function of the element and \(U^e\) is the values at its nodes. In the case of 8-noded isoparametric element, they are

\[
N^e = \begin{pmatrix} N_1^e & N_2^e & N_3^e & N_4^e & N_5^e & N_6^e & N_7^e & N_8^e \end{pmatrix},
\]

\[
U^e = \begin{pmatrix} U_1^e & U_2^e & U_3^e & U_4^e & U_5^e & U_6^e & U_7^e & U_8^e \end{pmatrix}^T,
\]

where \(N_1^e\sim N_8^e\) can be represented using a local coordinate system. The left side of Eq. (7) can be rewritten as

\[
\int_{\Omega} \left( \frac{\partial U}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial v}{\partial y} \right) \, d\Omega = (v^e)^T \int_{\Omega} \left( \frac{\partial N^e}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial N^e}{\partial y} \frac{\partial v}{\partial y} \right) \, d\Omega U^e.
\]

(9)

On the other hand, the right hand side is

\[
\int_{\Gamma} fv \, d\Gamma = \int_{\Gamma} (v^T) \left( \int N^e \, d\Omega \right) f \, d\Gamma = (v^e)^T \Gamma^e,
\]

(10)
Then, the discrete representation of the weak form for an element can be expressed as

$$
K_e U_e = F_e,
$$

(12)

where

$$
K' = (v')^T \int_x \left( \frac{\partial N'}{\partial x} \right)^T \left( \frac{\partial N'}{\partial x} \right) + \left( \frac{\partial N'}{\partial y} \right)^T \left( \frac{\partial N'}{\partial y} \right) d\Omega,
$$

$$
F' = q' - f'.
$$

Equation (12) is a finite element equation for an element. A finite element equation for the whole model can be obtained by the superposition of Eq. (12) for the entire elements. By solving the finite element equation, the values of the Airy stress function at all the nodes of the model are obtained. Then, the stresses inside the elements are obtained by differentiating the Airy stress function twice.

3. Verification by Simulation

3.1 Procedure for simulation

In order to validate the effectiveness of the proposed method, the stresses are computed by the proposed method from the sum of principal stresses obtained by the finite element direct analysis. A perforated plate under tension shown in Fig. 2 is analyzed to generate the distribution of the sum of principal stresses to be used in the simulation. In Fig. 2, the model is subjected to the tensile load $P$ in the $x$-direction. The material in this simulation is assumed to be polymethylmethacrylate (PMMA); the elastic modulus is $E = 3$ GPa and the Poisson’s ratio is $\nu = 0.3$. The distribution of the sum of principal stresses is calculated under the condition of the load of $P = 216$ N. In Fig. 2, the 10 mm $\times$ 10 mm region around a hole indicated as ABCD is used for subsequent analysis. Figure 3 shows the distribution of the stress components in the region of ABCD in Fig. 2 obtained by finite element direct analysis. These stresses are treated as the exact stress distributions in this study.

The sum of the principal stress is calculated from the stresses in Fig. 3. The values of the sum of principal stresses are obtained at the points on the two-dimensional grating of 0.2 mm in interval. Figure 4 shows the three-dimensional representation of the distribution of the sum of the principal stresses for the analysis region of ABCD.
The values in Fig. 4 are used as the data input into the algorithm by the proposed method.

Figure 5 shows the finite element model of the 10 mm × 10 mm region used for the proposed method. In this model, 8-noded isoparametric elements are used. The numbers of the elements and the nodes are 200 and 680, respectively. The nodal values of the sum of the principal stresses are determined by fitting the interpolation function of the elements to the distribution of the sum of the principal stresses by the method of least-squares, like the method for smoothing displacements [16].

In this study, two kinds of the Neumann boundary condition are examined to investigate the effect of the boundary condition on the results. One is determined from the tractions along the boundary of the analysis region obtained by the finite element direct analysis (BC 1). The boundary condition BC 1 is considered as the exact condition. Another is determined from the average stress along the left and right boundaries of the analysis region (BC 2). That is, the Neumann boundary condition is determined from the tensile stress of 3.6 MPa on the left and right edges. This condition BC 2 is considered as the approximated condition. The Neumann boundary conditions BC 1 and BC 2 along the boundaries of AB and BC are shown in Fig. 6. The boundary conditions along CD and DA are identical with those along AB and BC, respectively. As shown in this figure, because the tractions on the boundary of the model, i.e., in this case stresses around the hole is different from the average stress due to the stress concentration, the different boundary conditions BC 1 and BC 2 are obtained.

3.2 Results and discussion
The Airy stress functions are computed from the sum of the principal stresses shown in Fig. 4. Then, the stresses are computed by differentiating the Airy stress function twice. The stress distributions obtained with the exact boundary condition BC 1 are shown in Fig. 7. Because the stresses are obtained as the second order derivatives of the Airy stress function, they are obtained as constant values to the differentiation direction within the element. Therefore, the stresses in Fig. 7 show the discontinuous distributions. From this figure, however, it is recognized that the stresses are obtained from the sum of the principal stresses by solving the Poisson equation.

Figure 8 shows the stresses computed with the boundary condition obtained from the approximated tractions BC 2. Almost same stress distributions with Fig. 7 are obtained except near the boundaries even if the approximated boundary condition is used. In Eq. (12), $F_e$ is composed of the sum of the principal stresses on the whole field of the model and the Neumann boundary condition along the boundary. It is considered that because the effect of the boundary condition term $q_e'$ on the term of $F_e$ is small, almost same stresses are obtained even if the approximated boundary condition is used.

4. Verification by Experiment
The proposed method is also validated by a simple experiment. The specimen and the loading condition are same with the simulation described in previous section. That is, a perforated plate specimen shown in Fig. 2 is subjected to the tensile load of $P = 216$ N in the $x$-direction. The sum of principal stresses is measured using...
a Mach-Zehnder interferometer with a laser light of the wavelength of 633 nm.

Figure 9 shows the interference fringes around the hole before and after the loading. Not only the fringes related to the deformation around the hole but also the carrier fringes are observed. Therefore, the sum of the principal stresses is obtained from the phase difference between the phases of the interference fringes before and after the loading. Here, a phase-stepping method for random phase-steps [17] is used for the phase analysis of the fringe pattern. Figure 10(a) shows the phase difference distribution. The phase unwrapping is performed to obtain the absolute phase values of the isopachics. Figure 10(b) shows the unwrapped phase difference distribution that represents the sum of the principal stresses. The stress separation is performed from the values in Fig. 10(b) by the proposed method. The approximate boundary condition BC 2 is given for solving the Poisson equation.

Figure 11 shows the stress distributions obtained by the proposed method. As shown in this figure, almost the same results with the simulation are obtained. That is, the stress separation method proposed in this paper is also validated by the experiment.

In the present study, the Neumann boundary condition that is determined from the traction along the boundary is given for solving the Poisson equation. In general, however, the Neumann boundary condition cannot be given because the traction along the boundary cannot be known in advance. The determination method of not only the Neumann but also the Dirichlet boundary conditions from
the measured values of the sum of the principal stresses is being developed and reported in the future.

5. Conclusions

The first step approach to the development of a method for determining stress components from the isopachic contours is described in this paper. The Poisson equation that represents the relationship between the sum of the principal stresses and the Airy stress function is solved using the finite element method. In the present stage, the boundary condition is determined from the traction along the boundary. Results of the simulation and the experiment indicate that the stress components can be obtained from the isopachics by the proposed method. It is expected that the stress components can be obtained easily from the measurement of the isopachics by further improvement of the proposed method.

References


