Theoretical Analysis for the Fast Birefringence Measurement System using Dual Electro-optic Crystal Modulators

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Abstract
Two Y-cut Z-propagation (Y-Z) LiNbO₃ (LN) electro-optic (EO) crystals have been considered as transverse phase modulators for the fast birefringence measurement. To obtain birefringent parameters from modulated polarization status, the LN crystals were driven at different frequencies. This work aims to analyze the effect of parameters such as modulation amplitude and the order of Bessel functions on decision of the frequency ratio of drivers. According to our theoretical analysis, it was found that the modulation amplitude greatly influences the selection of the driving frequency.

Key words
Birefringence, LiNbO₃ Crystal, Phase Modulation, Electro-optic Effect, Fourier Transform, Bessel Function

1. Introduction
The polarimetric optical approaches of measurement are widely used in various fields such as industry, medicine, biology, and astronomy, because they can attain non-contact, non-invasive and fast measurement. The speed is critical in measuring birefringence resulting from dynamical photo-elastic transformation. Several techniques have been reported for real-time measurement of birefringence for different purposes. These techniques can be sorted into 3 types: one is using left and right and circularly polarized laser beams with different frequencies, which are generated by an axial Zeeman laser [1]; the second is using photo-electric effect modulator (PEM) or electro-optic (EO) effect modulator to produce the desired polarization-state modulation [2], [3], [4]; the last uses four photo detectors [5].

To obtain both birefringence and azimuth angle of samples from the detected intensity, the DC element of intensity signal is often included in the algorithms [1]-[5]. Scattering caused by optical elements and sample itself depolarizes the polarization state of incident light and remains as noise in this DC component. To avoid the effect of these unexpected elements on the measurement accuracy, we propose to use two phase modulators driven at different frequency to obtain birefringent parameters from modulated polarization states only. An approach using two PEMs has been applied to ellipsometry [6], [7]. The PEM is typically driven at the resonance frequency 20-80 kHz of the fused silica. Since the data process was carried out with an application software, it limits the Fourier coefficient measurement speed to some extend.

When a stress is applied to the isotropic elastic material, it manifests birefringent property. For the stress under the elastic limit, the relationship between the stress and the strain, which causes the birefringence of the material, is characterized as linear. As the stress is above the elastic limit, the relationship becomes nonlinear. Our research aims to obtain the characteristics of stress-birefringence of the material while the stress is continuously applied to it. This measurement requires high speed and high accuracy. To realize high speed and high accuracy measurement, we propose to use two Y-Z LiNbO₃ EO crystals as transverse phase modulators. When light propagates through Z axis of LN crystal, the natural birefringence is zero. In the presence of an electric field parallel to Y axis, according to EO effect, the LN crystal will have the retardance proportional to parameters such as EO coefficient r₂₂ and driving voltage. The operating amplitude of Y-Z LN was examined very stable, although it is somewhat dependent on temperature [8]. The advantage in using the EO crystal is that the retardance of the crystal can be theoretically modulated with the frequency up to several MHz. Our work in the current step is to carry out the theoretical analysis on the two Y-Z LN modulator measurement system.

In the following Sections, we describe the measurement principle of our technique, and analyze the effect of parameters such as modulation amplitude and the order of Bessel functions on decision of the frequency ratio of the EO crystal drivers.

2. Theory

2.1 Y-Z LN crystal
The configuration of Y-Z LN crystal is depicted in Fig. 1. Z axis is the optic axis of the crystal, therefore, when the light propagates through the LN crystal along it, with zero applied field, the natural birefringence is zero; with the presence of an alternating electric field parallel to Y axis, according to the EO effect, the phase change is induced for two mutually orthogonal eigen modes of incident light polarized along directions X and Y. When an AC voltage with angular frequency ω or Vsin(ωt) is applied to the LN modulator, the induced retardance Γ is given by

\[ \Gamma = \theta \sin \omega t, \]  

(1)
where the retardance amplitude $\theta$ is given by

$$\theta = \frac{2 \pi n_o r_{22} l}{\lambda d},$$  \hspace{0.5cm} (2)

where $n_o$ and $r_{22}$ are the refractive index and EO coefficient of the LN crystal, $l$ and $d$ are the length and the thickness of the LN crystal, respectively.

2.2 Modulated intensities

Figure 2 shows the configuration of dual EO crystal phase modulator based birefringence measurement system. A polarizer and two EO crystals are aligned in front of the sample followed by a quarter-wave plate and the analyzer. The orientation of two EO crystals is at a 45º angle from each other. Denoting retardance and azimuth angle of the sample follows by a quarter-wave plate, $\Gamma_1$ and $\Gamma_2$, respectively, the Mueller matrix of this system $\textbf{M}_{\text{system}}$ is given by

$$\textbf{M}_{\text{system}} = \textbf{P} \frac{\pi}{4} \textbf{R}_{\text{QWP}}(0) \textbf{S}(\Delta, \phi) \textbf{R}_{\text{pm}}(\Gamma_2) \frac{\pi}{4} \textbf{R}_{\text{pm}}(\Gamma_1) \textbf{P} \frac{\pi}{4},$$  \hspace{0.5cm} (3)

where $\textbf{P}$ is the Mueller matrix of the linear polarizer, $\textbf{R}_{\text{pm}}$ and $\textbf{R}_{\text{QWP}}$ are the Mueller matrices of the phase modulator and the quarter-wave plate, $\textbf{S}$ is the Mueller matrix of the sample. The numerical data in parenthesis of each Muller matrix represent the azimuth $a$ and the polar angle $\alpha$ of each optical element.

According to the relationship between the Stokes parameter and the Mueller matrix of this system, the intensity of the emerging beam is expressed by

$$I = I_{DC} + I_0 \left[ (\cos \Delta \sin \Gamma_1 \cos \Gamma_2 + \sin \Delta \cos 2\phi \cos \Gamma_1) \cos \alpha + \sin \Delta \sin 2\phi \sin \Gamma_1 \sin \Gamma_2 \right].$$  \hspace{0.5cm} (4)

Birefringent parameters $\Delta$ and $\phi$ of the sample can be obtained from Eqs. (5) to (7),

$$\Delta = \tan^{-1} \frac{\sqrt{a_1^2 + a_2^2}}{a_3}, \quad 0 \leq \Delta \leq \frac{\pi}{2}.$$  \hspace{0.5cm} (8)

$$\phi = \frac{1}{2} \tan^{-1} \frac{a_1}{a_2}, \quad 0 \leq \phi \leq \frac{\pi}{4}.$$  \hspace{0.5cm} (9)

Substituting Eq. (2) and Eqs. (5) - (7) into Eq. (4) leads to

$$I = I_{DC} + I_0 \left[ a_1 \sin (\theta_1 \sin \omega_1 t) \cos (\theta_2 \sin \omega_2 t) + a_2 \cos (\theta_1 \sin \omega_1 t) + a_3 \sin (\theta_1 \sin \omega_1 t) \sin (\theta_2 \sin \omega_2 t) \right].$$  \hspace{0.5cm} (10)

where the terms $\cos(\theta_1 \sin \omega_1 t)$ and $\sin(\theta_1 \sin \omega_1 t)$ can be expanded, using the Bessel function $J_n(0)$ of the first kind, with $n=0, 1, 2, \ldots$, to Fourier series formula,

$$\cos(\theta_1 \sin \omega_1 t) = 2 \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) \sin[(2n-1)\omega_1 t].$$  \hspace{0.5cm} (11)

$$\sin(\theta_1 \sin \omega_1 t) = 2 \sum_{n=1}^{\infty} J_{2n}(\theta_1) \sin(2n \omega_1 t).$$  \hspace{0.5cm} (12)

Substituting Eqs. (11) and (12) into Eq. (10) leads to

$$I = I_{DC} + I_0 \left[ 2a_1 \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) \sin [(2n-1)\omega_1 t] \cos (\theta_2 \sin \omega_2 t) + a_2 \sum_{n=1}^{\infty} J_{2n}(\theta_1) \cos (2n \omega_1 t) \cos (\theta_2 \sin \omega_2 t) - 4a_3 \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) \sin [(2n-1)\omega_1 t] \sin (\theta_2 \sin \omega_2 t) \right].$$  \hspace{0.5cm} (13)

Here the coefficients of harmonic components will be obtained quickly by using lock-in amplifiers (LIAs), consequently, birefringent parameters will be calculated by using Eqs. (8) and (9). In Eq. (13), notice that the harmonic components included in the modulated intensities can be reduced to some extent by choosing certain magnitude of modulation amplitude $\theta_1$. In the following, we determine the frequency ratio of the EO crystal drivers in the cases of different magnitude of $\theta_1$.

2.2.1 In the case of $J_0(\theta_1)=J_0(\theta_2)=0$

If the modulation amplitude is $\theta_1=\theta_2=2.4048$ rad, both $J_0(\theta_1)$ and $J_0(\theta_2)$ become zero. Then, Eq. (13) will be represented by

$$I = I_{DC} + 2I_0 \left[ a_1 \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) \cos (2n \omega_1 t) \right] + a_2 \sum_{n=1}^{\infty} \sum_{m=-1}^{\infty} J_{2n}(\theta_1) J_{2m}(\theta_2) \sin [(2n-1)\omega_1 t + 2m \omega_2 t] \right] + a_3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_{2n-1}(\theta_1) J_{2m-1}(\theta_2) \cos [(2n-1)\omega_1 t -(2m-1)\omega_2 t] \right] + a_4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_{2n}(\theta_1) J_{2m}(\theta_2) \cos [(2n-1)\omega_1 t +(2m-1)\omega_2 t].$$  \hspace{0.5cm} (14)

Table 1 shows the harmonic components be picked by the LIA. To get the parameters $a_1$, $a_2$, $a_3$, $a_4$, from signals of
LIAs, $\omega_1$ and $\omega_2$ in Eq. (14) will also be carefully selected to satisfy the following conditions, otherwise, several coefficients signals include in a harmonic component,
\[
\begin{align*}
|2n\omega_2| &\neq |2(n-1)\omega_1 + 2m\omega_2| \\
&\neq |2(n-1)\omega_1 - 2m\omega_2| \\
&\neq |2(n-1)\omega_1 - (2m-1)\omega_2| \\
&\neq |2(n-1)\omega_1 + (2m-1)\omega_2|
\end{align*}
\]

Therefore, it is also an important stage to select the best smaller signal that the LIA detects and its combinations. Simultaneously, it is also necessary to determine the frequency ratio of two phase modulators to satisfy the condition showed in Eq. (15).

Table 1: Lists of harmonic components

<table>
<thead>
<tr>
<th>No.</th>
<th>Frequency</th>
<th>Harmonic component</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2n\omega_1$</td>
<td>$2l_1a_1J_{2l_1}(\theta_1)$</td>
</tr>
<tr>
<td>2</td>
<td>$(2n-1)\omega_1+2m\omega_2$</td>
<td>$2a_1a_2J_{2m+1}(\theta_1)J_{2n+1}(\theta_1)$</td>
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<tr>
<td>3</td>
<td>$(2n-1)\omega_1-2m\omega_2$</td>
<td>$2a_1a_2J_{2m+1}(\theta_1)J_{2n+1}(\theta_1)$</td>
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<tr>
<td>4</td>
<td>$(2n-1)\omega_1-(2m-1)\omega_2$</td>
<td>$-2a_1a_2J_{2m+1}(\theta_1)J_{2n+1}(\theta_1)$</td>
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<tr>
<td>5</td>
<td>$(2n-1)\omega_1+(2m-1)\omega_2$</td>
<td>$2a_1a_2J_{2m+1}(\theta_1)J_{2n+1}(\theta_1)$</td>
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Table 2: Magnitude of Bessel functions and its combinations

<table>
<thead>
<tr>
<th>$J_{2m+1}(\theta_1)$</th>
<th>$J_1$</th>
<th>$J_3$</th>
<th>$J_5$</th>
<th>$J_7$</th>
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<tbody>
<tr>
<td>$J_{2m+1}(\theta_1)$</td>
<td>0.51915</td>
<td>0.19900</td>
<td>0.01639</td>
<td>0.0060</td>
</tr>
<tr>
<td>$J_1$</td>
<td>0.51915</td>
<td>0.26952</td>
<td>0.10331</td>
<td>0.00851</td>
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<td>$J_2$</td>
<td>0.43175</td>
<td>0.22414</td>
<td>0.08592</td>
<td>0.00708</td>
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<tr>
<td>$J_3$</td>
<td>0.19900</td>
<td>0.10331</td>
<td>0.03960</td>
<td>0.00326</td>
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<tr>
<td>$J_4$</td>
<td>0.06474</td>
<td>0.03361</td>
<td>0.01288</td>
<td>0.00106</td>
</tr>
<tr>
<td>$J_5$</td>
<td>0.01639</td>
<td>0.00851</td>
<td>0.00326</td>
<td>0.00027</td>
</tr>
<tr>
<td>$J_6$</td>
<td>0.00340</td>
<td>0.00177</td>
<td>0.00068</td>
<td>0.00006</td>
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<tr>
<td>$J_7$</td>
<td>0.00060</td>
<td>0.00031</td>
<td>0.00012</td>
<td>0.00001</td>
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<tr>
<td>$J_8$</td>
<td>0.00009</td>
<td>0.00005</td>
<td>0.00002</td>
<td>0.00000</td>
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Table 3: List of frequency ratio $\omega_1 : \omega_2$ ($\theta=2.4048$ rad)

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>1</th>
<th>2</th>
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Table 1 shows that when $\theta$ is 2.4048 rad, the magnitudes of $J_{2m+1}(\theta)$ are all much less than 1, and the higher the order is, the less the magnitude is, which results in smaller signals that the LIA detects. Therefore, it is also an important stage to select the best $n$ and $m$ in formula (15) in order to be able to get larger signals.

From the standpoint of sensitivity of the LIA, here, the terms including Bessel functions with magnitudes less than 0.001 is disregarded in this theoretical analysis. Table 2 shows the magnitudes of several orders of Bessel functions and their combinations.

From Table 2, the combination of $J_{2n}$, $J_{2n+1}$, and $J_{2n}$ $(n=m=1)$ are considered as the best choice for practical measurement with LIAs. Simultaneously, it is also necessary to determine the frequency ratio of two phase modulators to satisfy the condition showed in Eq. (15).

Table 3 shows the possible ratio of two frequencies $\omega_1$ and $\omega_2$. The round symbol indicates an acceptable ratio, and the cells filled with gray shows unsuitable ratio. The disadvantage of using the LN crystal is whether the driver can support high voltage for the EO crystal at high frequency or not. Here, we continuously analyze when $J_{2n}(0)$ do not equal to zero.

2.2.2 In the case of $J_{2n}(\theta) \neq 0$

Since the zeroth order of Bessel function is not zero, Eq. (13) is expanded as,

\[
I = I_{dc} + I_{ac}J_{2n}(\theta) + 2I_{ac}\sum_{m=1}^{\infty} \left| \frac{\sum_{m=1}^{\infty} J_{2m+1}(\theta_1) \cos (2m+1)\omega_1 t}{\sum_{m=1}^{\infty} J_{2m+1}(\theta_1) \cos (2m+1)\omega_1 t} \right|
\]

Similar to the analysis carried in section 2.1.1, it is necessary to decide both the order of Bessel functions and frequency ratio. Table 4 shows the harmonic components to be obtained by LIAs.

Table 4: Lists of Harmonic components

<table>
<thead>
<tr>
<th>No.</th>
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<tbody>
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<td>1</td>
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<td>$2n\omega_1$</td>
<td>$2a_1a_2J_{2n+1}(\theta_1)$</td>
</tr>
<tr>
<td>5</td>
<td>$(2n-1)\omega_1-(2m-1)\omega_2$</td>
<td>$-2a_1a_2J_{2n+1}(\theta_1)J_{2n+1}(\theta_1)$</td>
</tr>
<tr>
<td>6</td>
<td>$(2n-1)\omega_1+(2m-1)\omega_2$</td>
<td>$2a_1a_2J_{2n+1}(\theta_1)J_{2n+1}(\theta_1)$</td>
</tr>
</tbody>
</table>

In this case, the six harmonics should be different mutually, $|2n\omega_1| \neq |2n-1\omega_1| \neq |2n-2\omega_1| \neq |2n-3\omega_1| \neq |2n-4\omega_1|$ (17)

Here it is also necessary to decide the modulation amplitude $\theta$. The basic consideration for selection of $\theta$ is that magnitudes of Bessel function of each order or its combination are not too small to be detected. According to our sorting, the combination of $J_{2n}$, $J_{2n+1}$, and $J_{2n+1}$ and $\theta=\theta_2=1.6057$ rad are considered as the best choice for practical measurement with LIAs. This means that $n$ and $m$ are also selected to be 1. Table 5 shows the magnitude of Bessel functions and their combinations.
Similarly, it is also necessary to determine the frequency ratio of the two phase modulators to satisfy the conditions showed in Eq. (16). Table 6 shows the possible ratio of two frequencies \( \omega_1 \) and \( \omega_2 \). The round symbol indicates acceptable ratios, and the cells filled with gray shows unsuitable ratios, respectively.

### 2.3 Numerical analysis

In section 2.2, the intensities are derived in two cases, respectively. In this section, the signals to be obtained through LIAs will be discussed with an example in detail. In this example, the ratio of two frequencies is chosen with \( \omega_1 : \omega_2 = 8 : 1 \). Figure 1 shows the waveform of intensity signals for modulation amplitude \( \theta=2.4048 \) and 1.6057rad, respectively.

The modulated intensity signals detected by an optical detector pass through LIAs, and then the harmonic components will be selected in accordance with the parameters to be used in Eqs. (8) and (9). In section 2.2, we described that these harmonic components to be detected will also be decided according to the modulation amplitude.

When the modulation amplitude is \( \theta=2.4048 \) rad, the following harmonic components will be selected,

\[
2I_0a_2J_2(\theta)\sin 16t, \quad 2I_0a_2J_2(\theta)\cos 16t, \quad 2I_0a_2J_2(\theta)\sin 9t
\]

and the signals sensed by LIAs will be,

\[
b_1 = \sqrt{2I_0a_1J_1(\theta)}J_1(\theta), \quad b_2 = \sqrt{2I_0a_2J_2(\theta)}, \quad b_3 = \sqrt{2I_0a_3J_3(\theta)}
\]

The square root of 2 is required in Eqs. (21) – (23) because the output of the lock-in amplifier is a root-mean-square voltage.

Since \( J_1(2.4048) = 0.51915 \) and \( J_3(2.4048) = 0.43175 \), the parameters \( a_1, a_2, \) and \( a_3 \) calculated from detected \( b_1, b_2, \) and \( b_3 \) will be respectively,

\[
a_1 = \frac{b_1}{\sqrt{2I_0J_1(2.4048)J_1(2.4048)}} = 3.155 \frac{b_1}{I_0}, \quad a_2 = \frac{b_2}{\sqrt{2I_0J_2(2.4048)J_2(2.4048)}} = 1.638 \frac{b_2}{I_0},
\]

\[
a_3 = \frac{b_3}{\sqrt{2I_0J_3(2.4048)J_3(2.4048)}} = 2.624 \frac{b_3}{I_0}
\]

By substituting Eqs. (24) - (26) into Eqs. (8) and (9), the birefringent parameters of samples can be obtained.

Similarly, when the modulation amplitude is \( \theta=1.6057 \) rad, the following harmonic components will be selected by Lock-in amplifiers,

\[
2I_0a_2J_2(\theta)\sin 16t, \quad 2I_0a_2J_2(\theta)\cos 16t, \quad 2I_0a_2J_2(\theta)\sin 9t
\]

and the sensed signals will be,
\[ b_1 = \sqrt{2I_0a_1J_j(\theta)J_1(\theta)}, \quad (30) \]
\[ b_2 = \sqrt{2I_0a_2J_j(\theta)}, \quad (31) \]
\[ b_3 = \sqrt{2I_0a_3J_1^2(\theta)}. \quad (32) \]

The parameters \( a_1, a_2, \) and \( a_3 \) will also be calculated from \( b_1, b_2, \) and \( b_3, \) respectively,

\[ a_1 = \frac{b_1}{\sqrt{2I_0J_j(1.6057)J_1(1.6057)}} = 2.741 \frac{b_1}{I_0}, \quad (33) \]
\[ a_2 = \frac{b_2}{\sqrt{2I_0J_2(1.6057)}} = 2.737 \frac{b_2}{I_0}, \quad (34) \]
\[ a_3 = \frac{b_3}{\sqrt{2I_0J_1(1.6957)}} = 2.173 \frac{b_3}{I_0}. \quad (35) \]

Here, \( J_1(1.6057) = 0.57046 \) and \( J_2(1.6057) = 0.25893, \) respectively, as shown in Table 5.

### 2.4 Error analysis

Theoretical analysis was, so far, conducted under the ideal conditions with no errors of optical elements. Practically the errors of alignment or optical element itself result in systematic error of measurement. Here we determine intensities deduced by errors in retardation of phase modulators and quarter wave plate. Denoting the natural retardance error of phase modulators as \( e_1, e_2, \) the error of azimuth angle as \( e_3, \) and the retardance error of quarter wave plate \( e_4, \) Eq. (4) can be rewritten as:

\[
I = I_{dc} + 2I_0 \sin \Delta \sin(2\phi) \left[ e_1 \cos \Delta \sin(2\phi) \cos(2\phi) - e_2 \sin \Delta \sin(2\phi) \right] \\
- \cos e_3 \sin \Delta \sin(2\phi) \left[ 1 - \cos(\Gamma_2 + e_4) \right] \sin(2e_3) \\
+ \sin e_1 \left[ \cos \Delta \cos(2\phi) + \cos e_3 \sin \Delta \cos(2\phi) \right] \sin(2\phi) \sin(2e_3) \\
+ I_0 \left[ \sin e_1 \left[ 1 - \cos \Delta \sin(2\phi) \right] + e_3 \sin \Delta \sin(2\phi) \right] \cos(2\phi) \\
+ \sin e_1 \sin \Delta \cos(2\phi) \left[ 1 - \cos(\Gamma_1 + e_4) \right] \cos(2e_3) \\
- I_0 \left[ \sin e_1 \left[ 1 - \cos \Delta \sin(2\phi) \right] - e_3 \sin \Delta \sin(2\phi) \right] \cos(2\phi).
\]

Rearranging Eq. (36) by sorting the terms of birefringent parameters leads to

\[
I = I_{dc} + 2I_0 \sin \Delta \sin(2\phi) \left[ e_1 \cos \Delta \sin(2\phi) \cos(2\phi) - e_2 \sin \Delta \sin(2\phi) \right] \\
- e_3 \sin \Delta \sin(2\phi) \left[ 1 - \cos(\Gamma_2 + e_4) \right] \sin(2e_3) \\
+ \sin e_1 \left[ \cos \Delta \cos(2\phi) + \cos e_3 \sin \Delta \cos(2\phi) \right] \sin(2\phi) \cos(2e_3) \\
+ I_0 \left[ \sin e_1 \left[ 1 - \cos \Delta \sin(2\phi) \right] + e_3 \sin \Delta \sin(2\phi) \right] \sin(2\phi) \\
+ \sin e_1 \sin \Delta \cos(2\phi) \left[ 1 - \cos(\Gamma_1 + e_4) \right] \cos(2e_3) \\
- I_0 \left[ \sin e_1 \left[ 1 - \cos \Delta \sin(2\phi) \right] - e_3 \sin \Delta \sin(2\phi) \right] \sin(2\phi).
\]

With no sample installed, the intensity becomes

\[
I = I_{dc} + I_0 \cos \Delta \left[ \frac{1}{2} e_1 \sin \Gamma_1 - e_2 \sin \Gamma_2 + e_3 \cos \Gamma_1 + e_4 \sin \Gamma_1 \right. \\
+ \sin \Gamma_2 \cos \Gamma_1 + (e_1 - e_2) \cos \Gamma_1 \sin \Gamma_2 \] \\
+ I_0 \left[ \frac{1}{2} e_1 \sin \Gamma_2 - e_2 \sin \Gamma_2 + e_3 \cos \Gamma_2 \right. \\
+ \sin \Gamma_2 \cos \Gamma_1 - (e_1 - e_2) \cos \Gamma_1 \sin \Gamma_2 \]
\]

For the case of \( \theta_1 = \theta_2 = 2.4048, \) Eq. (39) is rewritten as
\[ I = 2I_e \varepsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} J_{2n}(\theta_1) J_{2m}(\theta_2) \cos(2n\omega_1 + 2m\omega_2) \]

\[ + 2I_e \varepsilon \sum_{n=1}^{\infty} J_{2n}(\theta_1) J_{2m}(\theta_2) \cos(2n\omega_1 - 2m\omega_2) \]

\[ - 2I_e \varepsilon \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) J_{2m-1}(\theta_2) \cos[(2n-1)\omega_1 - (2m-1)\omega_2] \]

\[ + 2I_e \varepsilon \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) J_{2m}(\theta_2) \sin[(2n-1)\omega_1 + (2m+1)\omega_2] \]

\[ + 2I_e \varepsilon \sum_{n=1}^{\infty} J_{2n}(\theta_1) \cos 2n\omega_1 t \]

\[ - 4I_e \varepsilon \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) \sin(2n-1)\omega_1 t \]

\[ + 4I_e (\varepsilon_1 - \varepsilon_2) \sum_{n=1}^{\infty} J_{2n}(\theta_1) J_{2m}(\theta_2) \sin[2n\omega_1 + (2m+1)\omega_2] \]

\[ - 4I_e (\varepsilon_1 - \varepsilon_2) \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) J_{2m-1}(\theta_2) \sin[2(n-1)\omega_1 - (2m+1)\omega_2] \]

\[ + 2I_e \sum_{n=1}^{\infty} J_{2n-1}(\theta_1) J_{2m}(\theta_2) \sin[(2n-1)\omega_1 + 2m\omega_2] \]

\[ + 2I_e \sum_{n=1}^{\infty} J_{2n+1}(\theta_1) J_{2m}(\theta_2) \sin[(2n+1)\omega_1 - 2m\omega_2] \]

\[ \text{(40)} \]

For case of \[ \theta_1 = \theta_2 = 1.6057 \text{ rad}, \] Eq. (39) will be represented by:

\[ I = I_{JC} + I_{JC} J_{2n}(\theta_1) J_{2m}(\theta_2) + I_e \varepsilon \sum_{n=1}^{\infty} J_{2n}(\theta_1) \cos 2n\omega_1 t \]

\[ + 2I_e \varepsilon \sum_{n=1}^{\infty} J_{2n}(\theta_1) \cos 2m\omega_1 t \]

\[ \text{(41)} \]

3. Conclusion

A fast birefringence measurement system consisting of two Y-Z LiNbO3 EO crystals has been proposed and theoretically analyzed. The advantage of this technique is to obtain birefringent parameters from modulated polarization states only. This condition requires the two LN crystals to be driven at different frequencies. This work analyzed the effect of parameters such as modulation amplitude, the order of Bessel functions on decision of the frequency ratio. According to our theoretical analysis, it was found that the modulation amplitude has a large influence on the selection of the driving frequency. Considering the practical problems of driving voltage amplifier, the analysis was carried out for two cases, when the modulation amplitude was 2.4048 rad and 1.6057 rad, respectively. The influence of errors that may exist in optical alignment of this system was also analyzed. All theoretical analysis was carried out for the ratio of two driving frequencies \[ \omega_1 : \omega_2 = 8 : 1. \] When a different ratio is chosen according to Tables 3 and 5, the harmonic components to be sensed by LIAs will be different from the components expressed by Eqs. (18) - (20), and (27) - (29), as well as waveform shown in Fig. 1. The thinking way of deducing equations, however, will not be changed. Eqs (40) and (41) derived in section 2.4 also consist of any frequency ratios.

Our future work is to design an EO crystal driver in accordance with our theoretical analysis, and carry out real time birefringence measurements.

References