EDGEWORTH EXPANSIONS OF SOME STATISTICS INCLUDING THE LB-STATISTIC AND V-STATISTIC

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As an estimator of an estimable parameter, Toda and Yamato (2001) introduced a linear combination Y_n of U-statistics which includes V-statistic and LB-statistic. We give the Edgeworth expansions of the standardized Y_n statistic with remainder $o(n^{-1})$ in case that the kernel is not degenerate. We also give the Edgeworth expansions of the studentized Y_n statistic using a jackknife variance estimator with remainder $o(n^{-1/2})$.

Key words and phrases: Edgeworth expansion, linear combination of U-statistics, V-statistic.

1. Introduction

Let $\theta(F)$ be an estimable parameter of an unknown distribution F and $g(x_1, \dots, x_k)$ be its symmetric kernel of degree $k(\geq 2)$. In this paper, we assume that the kernel g is not degenerate. Let $X_1, \dots, X_n$ be a random sample of size n from the distribution F . As an estimator of $\theta(F)$, a linear combination Y_n of U-statistics is introduced by Toda and Yamato (2001) as follows: Let $w(r_1, \dots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$, where k is the degree of the kernel g and is fixed. We assume that at least one of $w(r_1, \dots, r_j; k)$'s is positive. For $j = 1, \dots, k$, let $g_{(j)}(x_1, \dots, x_j)$ be the kernel given by

$$(1.1) \quad g_{(j)}(x_1, \dots, x_j) = \frac{1}{d(k, j)} \sum_{r_1 + \dots + r_j = k}^+ w(r_1, \dots, r_j; k) g(\underbrace{x_1, \dots, x_{r_1}}_{r_1}, \dots, \underbrace{x_j, \dots, x_{r_j}}_{r_j}),$$

where the summation $\sum_{r_1 + \dots + r_j = k}^+$ is taken over all positive integers $r_1, \dots, r_j$ satisfying $r_1 + \dots + r_j = k$ with j and k fixed and $d(k, j) = \sum_{r_1 + \dots + r_j = k}^+ w(r_1, \dots, r_j; k)$ for $j = 1, 2, \dots, k$. Let $U_n^{(j)}$ be the U-statistic associated with this kernel $g_{(j)}(x_1, \dots, x_j; k)$ for $j = 1, \dots, k$. The kernel $g_{(j)}(x_1, \dots, x_j; k)$ is symmetric because of the symmetry of $w(r_1, \dots, r_j; k)$. If $d(k, j)$ is equal to zero for some j , then the associated $w(r_1, \dots, r_j; k)$'s are equal to zero. In this case, we let the

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corresponding statistic $U_n^{(j)}$ be zero. The statistic Y_n is given by

$$(1.2) \quad Y_n = \frac{1}{D(n, k)} \sum_{j=1}^k d(k, j) \binom{n}{j} U_n^{(j)},$$

where $D(n, k) = \sum_{j=1}^k d(k, j) \binom{n}{j}$. Since w 's are nonnegative and at least one of them is positive, $D(n, k)$ is positive. Note that $U_n^{(k)} = U_n$ for $w(1, \dots, 1; k) > 0$, because $g_{(k)} = g$.

For example, let w be the function given by $w(1, 1, \dots, 1; k) = 1$ and $w(r_1, \dots, r_j; k) = 0$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k-1$ and $r_1 + \dots + r_j = k$. Then the corresponding statistic Y_n is equal to U-statistic U_n , which is given by

$$(1.3) \quad U_n = \binom{n}{j}^{-1} \sum_{1 \leq j_1 < \dots < j_k \leq n} g(X_{j_1}, \dots, X_{j_k}),$$

where $\sum_{1 \leq j_1 < \dots < j_k \leq n}$ denotes the summation over all integers $j_1, \dots, j_k$ satisfying $1 \leq j_1 < \dots < j_k \leq n$.

Let w be the function given by $w(r_1, \dots, r_j; k) = 1$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$. Then the corresponding statistic Y_n is equal to the LB-statistic B_n given by

$$(1.4) \quad B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1 + \dots + r_n = k} g(\underbrace{X_1, \dots, X_1}_{r_1}, \dots, \underbrace{X_n, \dots, X_n}_{r_n}),$$

where $\sum_{r_1 + \dots + r_n = k}$ denotes the summation over all non-negative integers $r_1, \dots, r_n$ satisfying $r_1 + \dots + r_n = k$. (The term LB-statistic is the abbreviation for the limit of the Bayes estimate and was introduced in Nomachi and Yamato (2001). For the limit of the Bayes estimate of θ , see Yamato (1977a, b).)

Let w be the function given by $w(r_1, \dots, r_j; k) = k!/(r_1! \dots r_j!)$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$. Then the corresponding statistic Y_n is equal to the V-statistic V_n given by

$$(1.5) \quad V_n = \frac{1}{n^k} \sum_{j_1=1}^n \dots \sum_{j_k=1}^n g(X_{j_1}, \dots, X_{j_k}).$$

(See Toda and Yamato (2001).)

Let w be the function given by $w(r_1, \dots, r_j; k) = k!/(r_1! \dots r_j!)$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$. Then, for the case of $k = 3$, the corresponding statistic Y_n for the third central moment of the distribution F is given by

$$S_n = \frac{n}{n^2 + 1} \sum_{i=1}^n (X_i - \bar{X})^3,$$

where $\bar{X}$ is the sample mean of $X_1, \dots, X_n$ (see Nomachi *et al.* (2002)).

In Section 2, we give an approximation of Y_n in order to derive its Edgeworth expansions in the following sections. For U-statistics, the Edgeworth expansions of its standardized statistics are studied by Callaert *et al.* (1980), Bickel *et al.* (1986), and Maesono (1987). Lai and Wang (1993) introduces asymptotic U-statistic and gives its Edgeworth expansion. In Section 3, with remainder $o(n^{-1})$, we derive the Edgeworth expansion of the standardized Y_n statistic using the result of Lai and Wang (1993). We also derive it with remainder $o(n^{-1/2})$.

For the practical inference of parameter θ , it is useful to get the Edgeworth expansion of the studentized Y-statistic. About a linear combination of U-statistics, L_n , of Sen (1977), Maesono (1996) gives its Edgeworth expansion for the studentized L-statistic with remainder $o(n^{-1})$, using a jackknife variance estimator. While two statistics Y_n and L_n are both linear combination of U-statistics, Y_n is different from L_n in the mean that the weight function w determines Y_n as an estimator of θ . In Section 4, we derive the Edgeworth expansion of the studentized Y-statistic using a jackknife variance estimator, with remainder $o(n^{-1/2})$. We will discuss the Edgeworth expansion with remainder $o(n^{-1})$ in a future paper. In Section 5, we give examples of Edgeworth expansions of the Y-statistic.

2. Approximation of Y-statistic

For the kernel $g_{(j)}(x_1, \dots, x_j)$ ($j = 1, \dots, k$), we put

$$\theta_j = Eg_{(j)}(X_1, \dots, X_j),$$

and

$$\psi_{(j),c}(x_1, \dots, x_c) = E[g_{(j)}(X_1, \dots, X_j) \mid X_1 = x_1, \dots, X_c = x_c], \quad c = 1, \dots, j.$$

For $j = 1, \dots, k$ and $c = 2, 3, \dots, k$, we put

$$\begin{aligned} g_{(j)}^{(1)}(x_1) &= \psi_{(j),1}(x_1) - \theta_j, \\ g_{(j)}^{(c)}(x_1, \dots, x_c) &= \psi_{(j),c}(x_1, \dots, x_c) - \sum_{i=1}^{c-1} \sum_{1 \leq l_1 < \dots < l_i \leq c} g_{(j)}^{(i)}(x_{l_1}, \dots, x_{l_i}) - \theta_j. \end{aligned}$$

Let $H_{(j),n}^{(c)}$ be the U-statistic associated with the kernel $g_{(j)}^{(c)}$, that is,

$$(2.1) \quad H_{(j),n}^{(c)} = \binom{n}{c}^{-1} \sum_{1 \leq l_1 < \dots < l_c \leq n} g_{(j)}^{(c)}(X_{l_1}, \dots, X_{l_c}).$$

Then, the U-statistic $U_n^{(j)}$ associated with the kernel $g_{(j)}$ can be written as follows,

$$U_n^{(j)} = \theta_j + \sum_{c=1}^j \binom{j}{c} H_{(j),n}^{(c)}.$$

This form is known as H-decomposition in the context of U-statistics, because it is due to Hoeffding (see, for example, Lee (1990), p. 26). We note that if

$d(k, k) = w(1, \dots, 1; k) > 0$ then we have $g_{(k)}(x_1, \dots, x_k) = g(x_1, \dots, x_k)$ and so $\theta_k = \theta$. Therefore, we shall delete the sub (k) . For example, we shall use g and θ instead of $g_{(k)}$ and θ_k , respectively. We shall also abbreviate ψ_c and $H_n^{(c)}$ instead of $\psi_{(k),c}$ and $H_{(k),n}^{(c)}$, respectively for $c = 1, 2, \dots, k$.

In the following, we assume $d(k, k) > 0$. Then, there exists a constant $\delta_k (\geq 0)$ such that

$$(2.2) \quad \frac{d(k, k)}{D(n, k)} \binom{n}{k} = 1 - \frac{\delta_k}{n} + O\left(\frac{1}{n^2}\right),$$

and

$$(2.3) \quad \frac{d(k, k-1)}{D(n, k)} \binom{n}{k-1} = \frac{\delta_k}{n} + O\left(\frac{1}{n^2}\right).$$

Therefore, it follows that

$$\sum_{j=1}^{k-2} \frac{d(k, j)}{D(n, k)} \binom{n}{j} = O\left(\frac{1}{n^2}\right).$$

For the U-statistic U_n , $d(k, k) \binom{n}{k} / D(n, k) = 1$ and $\delta_k = 0$. For the V-statistic V_n and the S-statistic S_n , $\delta_k = k(k-1)/2$. For the LB-statistic B_n , $\delta_k = k(k-1)$ (see Nomachi *et al.* (2002)).

PROPOSITION 2.1. *We suppose that $d(k, k) > 0$ and*

$$(2.4) \quad E \left[|g(X_{j_1}, \dots, X_{j_k})|^2 \right] < \infty \text{ for } 1 \leq j_1 \leq \dots \leq j_k \leq k.$$

Then we have

$$(2.5) \quad \begin{aligned} \sqrt{n}(Y_n - \theta) = & k \left(1 - \frac{\delta_k}{n} \right) \sqrt{n} H_n^{(1)} + \binom{k}{2} \sqrt{n} H_n^{(2)} + \binom{k}{3} \sqrt{n} H_n^{(3)} \\ & + \frac{(k-1)\delta_k}{\sqrt{n}} H_{(k-1),n}^{(1)} + \frac{\delta_k}{\sqrt{n}} (\theta_{k-1} - \theta) + R_n, \end{aligned}$$

where $E|R_n|^2 = O(n^{-3})$.

PROOF. We put

$$(2.6) \quad \sqrt{n}(Y_n - \theta) = \eta_1 + \eta_2 + \eta_3,$$

where

$$\begin{aligned} \eta_1 &= \frac{d(k, k)}{D(n, k)} \binom{n}{k} \sqrt{n} (U_n - \theta), \\ \eta_2 &= \frac{d(k, k-1)}{D(n, k)} \binom{n}{k-1} \sqrt{n} (U_n^{(k-1)} - \theta), \end{aligned}$$

and

$$\eta_3 = \sum_{j=1}^{k-2} \frac{d(k, j)}{D(n, k)} \binom{n}{j} \sqrt{n} (U_n^{(j)} - \theta).$$

By Minkowski's inequality and the relation immediately after (2.3), we have

$$\begin{aligned} (E | \eta_3 |^2)^{1/2} &\leq \sum_{j=1}^{k-2} \frac{d(k, j)}{D(n, k)} \binom{n}{j} n^{1/2} \left[(E | U_n^{(j)} - \theta_j |^2)^{1/2} + | \theta_j - \theta | \right] \\ &= O(n^{-3/2}) \end{aligned}$$

(see e.g. Lee (1990), p. 120). Hence we have

$$(2.7) \quad E | \eta_3 |^2 = O(n^{-3}).$$

Using the relation $E | H_{(k-1),n}^{(c)} |^2 = O(n^{-c})$, $c = 1, \dots, k-1$ (see Serfling (1980), p. 185) and (2.3), we have

$$\begin{aligned} (2.8) \quad \eta_2 &= \left(\frac{\delta_k}{n} + O\left(\frac{1}{n^2}\right) \right) \left[\sqrt{n} \left[\sum_{c=1}^{k-1} \binom{k-1}{c} H_{(k-1),n}^{(c)} \right] + \sqrt{n}(\theta_{k-1} - \theta) \right] \\ &= \delta_k \left[\frac{k-1}{\sqrt{n}} H_{(k-1),n}^{(1)} + \frac{1}{\sqrt{n}}(\theta_{k-1} - \theta) \right] + R_{2,n}, \end{aligned}$$

where $E | R_{2,n} |^2 = O(n^{-3})$. From (2.1) and (2.2), we have

$$\eta_1 = \left(1 - \frac{\delta_k}{n} + O\left(\frac{1}{n^2}\right) \right) \sqrt{n} \left[\sum_{c=1}^k \binom{k}{c} H_n^{(c)} \right].$$

Since $E | H_n^{(c)} |^2 = O(n^{-c})$, $c = 1, \dots, k$, we have

$$(2.9) \quad \eta_1 = k \left(1 - \frac{\delta_k}{n} \right) \sqrt{n} H_n^{(1)} + \binom{k}{2} \sqrt{n} H_n^{(2)} + \binom{k}{3} \sqrt{n} H_n^{(3)} + R_{3,n},$$

where $E | R_{3,n} |^2 = O(n^{-3})$. Applying (2.7), (2.8) and (2.9) to (2.6), we get (2.5). $\square$

We put

$$Y_n^* = k \left(1 - \frac{\delta_k}{n} \right) \sqrt{n} H_n^{(1)} + \binom{k}{2} \sqrt{n} H_n^{(2)} + \binom{k}{3} \sqrt{n} H_n^{(3)} + \frac{(k-1)\delta_k}{\sqrt{n}} H_{(k-1),n}^{(1)}$$

and

$$\mu_k = \delta_k(\theta_{k-1} - \theta).$$

For any random variables W and Δ , it holds that

$$(2.10) \quad \sup_x | P(W + \Delta \leq x) - P(W \leq x) | \leq 4(E | W \Delta | + E | \Delta |)$$

(see Shorack (2000), p. 261). Applying (2.10) to (2.5), we get the following.

PROPOSITION 2.2. *Under the conditions $d(k, k) > 0$ and $E[|g(X_{j_1}, \dots, X_{j_k})|^2] < \infty$ for $1 \leq j_1 \leq \dots \leq j_k \leq k$, we have*

$$\sup_{-\infty < x < \infty} \left| P\left(\frac{\sqrt{n}}{\sigma}(Y_n - \theta) \leq x\right) - P\left(\frac{1}{\sigma}\left(Y_n^* + \frac{\mu_k}{\sqrt{n}}\right) \leq x\right) \right| = O(n^{-3/2}),$$

where $\sigma = k\sigma_1$ and $\sigma_1^2 = \text{Var}[\psi_1(X)] > 0$.

We can write

$$\sqrt{n}H_n^{(2)} = \frac{2}{n^{3/2}} \sum_{1 \leq i < j \leq n} g^{(2)}(X_i, X_j) + \frac{1}{\sqrt{n}} H_n^{(2)}$$

and have $E[|H_n^{(2)}|/\sqrt{n}]^2 = O(n^{-3})$. We can also write

$$\sqrt{n}H_n^{(3)} = \frac{6}{n^{5/2}} \sum_{1 \leq i < j < l \leq n} g^{(3)}(X_i, X_j, X_l) + \frac{3n-2}{n^{3/2}} H_n^{(3)}$$

and have $E[|(3n-2)H_n^{(3)}|/n^{3/2}]^2 = O(n^{-4})$. We put

$$(2.11) \quad \begin{aligned} Y_n^{**} = & k \left(1 - \frac{\delta_k}{n}\right) \frac{1}{n^{1/2}} \sum_{i=1}^n g^{(1)}(X_i) \\ & + (k-1)\delta_k \frac{1}{n^{3/2}} \sum_{i=1}^n g_{(k-1)}^{(1)}(X_i) \\ & + k(k-1) \frac{1}{n^{3/2}} \sum_{1 \leq i < j \leq n} g^{(2)}(X_i, X_j) \\ & + k(k-1)(k-2) \frac{1}{n^{5/2}} \sum_{1 \leq i < j < l \leq n} g^{(3)}(X_i, X_j, X_l). \end{aligned}$$

Applying (2.10) to the above, we get the following.

COROLLARY 2.3. *We have*

$$\sqrt{n}(Y_n - \theta) = Y_n^{**} + \frac{\mu_k}{\sqrt{n}} + R'_n, \quad E|R'_n|^2 = O(n^{-3})$$

and

$$(2.12) \quad \sup_{-\infty < x < \infty} \left| P\left(\frac{\sqrt{n}}{\sigma}(Y_n - \theta) \leq x\right) - P\left(\frac{1}{\sigma}\left(Y_n^{**} + \frac{\mu_k}{\sqrt{n}}\right) \leq x\right) \right| = O(n^{-3/2}).$$

3. Expansion

We put as follows:

$$\begin{aligned}
\alpha(x) &= kg^{(1)}(x) = k[\psi_1(x) - \theta], \\
\alpha'(x) &= \delta_k[(k-1)g_{(k-1)}^{(1)}(x) - kg^{(1)}(x)] \\
&= \delta_k[\psi_2(x, x) + (k-2)E[\psi_3(x, X_2, X_2)] - (k-1)\theta_{k-1} - kg^{(1)}(x)], \\
\beta(x, y) &= k(k-1)g^{(2)}(x, y) = k(k-1)[\psi_2(x, y) - \psi_1(x) - \psi_1(y) + \theta], \\
\gamma(x, y, z) &= k(k-1)(k-2)g^{(3)}(x, y, z) \\
&= k(k-1)(k-2)[\psi_3(x, y, z) - \psi_2(x, y) - \psi_2(x, z) - \psi_2(y, z) \\
&\quad + \psi_1(x) + \psi_1(y) + \psi_1(z) - \theta].
\end{aligned}$$

Then we can write Y_n^{**} as

$$\begin{aligned}
(3.1) \quad Y_n^{**} &= \sum_{i=1}^n \left[\frac{\alpha(X_i)}{n^{1/2}} + \frac{\alpha'(X_i)}{n^{3/2}} \right] + \sum_{1 \leq i < j \leq n} \frac{\beta(X_i, X_j)}{n^{3/2}} \\
&\quad + \sum_{1 \leq i < j < l \leq n} \frac{\gamma(X_i, X_j, X_l)}{n^{5/2}}.
\end{aligned}$$

The Edgeworth expansion of the statistic given by this form has been derived by Lai and Wang (1993). We will derive the Edgeworth expansion of Y_n by using their result. Their conditions (A2) and (A3) are satisfied by the properties of H-decomposition. The condition (A1) is not related to our situation. Conditions (C) or (D) are described as follows by our notations for the one-dimensional random variable:

CONDITION (C). $E|g^{(2)}|^r < \infty$ for some $r > 2$ and there exists K Borel functions $f_j : R \rightarrow R$ such that $K(r-2) > 8(4r-5)$, $Ef_j^2(X_1) < \infty$ ($j = 1, \dots, K$), and the covariance matrix of $(W_1, \dots, W_K)$ is positive definite, where $W_j = (Lf_j)(X_1)$ and $(Lf)(y) = E[g^{(2)}(y, X_2)f(X_2)]$.

The original condition contains the term $I_{[E|g^{(3)}(X_1, X_2, X_3)| > 0]}$, which equals 1 since $g^{(3)}(X_1, X_2, X_3)$ is not likely to be zero under the assumption that the kernel g is not degenerate, that is, $\sigma_1^2 > 0$. It also contains $E|g^{(2)}|^r < \infty$ for some $r > 2$. This condition is satisfied with $r = 4$ under our condition $E[|\psi_3(X_1, X_2, X_3)|^4] < \infty$ of Theorem 3.2, which is assumed for the condition (A4) of Lai and Wang (1993).

CONDITION (D). There exist constants c_j and Borel functions $f_j : R \rightarrow R$ such that $Ef_j(X) = 0$, $E|f_j(X)|^r < \infty$ for some $r \geq 5$ and $g^{(2)} = \sum_{j=1}^K c_j f_j(X_1)f_j(X_2)$ a.s.; moreover, for some $0 < \varepsilon < \min\{1, 2(1 - 11r^{-1}/3)\}$,

$$\limsup_{|t| \rightarrow \infty} \sup_{|s_1| + \dots + |s_K| \leq |t|^{-\varepsilon}} \left| E \exp \left(it \left\{ \alpha(X) + \sum_{j=1}^K s_j f_j(X) \right\} \right) \right| < 1.$$

We shall use the same notations as Lai and Wang (1993) for constants σ , a_3 , a_4 , a' , b , c , κ_3 , κ_4 . In our situation these constants are given by as follows:

$$\begin{aligned}
\sigma^2 &= k^2 \sigma_1^2, \quad (\sigma_1^2 = \text{Var}[\psi_1(X)]), \\
a_3 &= k^3 E[(g^{(1)}(X))^3] = k^3 E[(\psi_1(X) - \theta)^3], \\
a_4 &= k^4 E[(g^{(1)}(X))^4] = k^4 E[(\psi_1(X) - \theta)^4], \\
a' &= k \delta_k \{ (k-1) E[g^{(1)}(X) g_{(k-1)}^{(1)}(X)] - k E[g^{(1)}(X)]^2 \} \\
&= k \delta_k \{ (k-1) E[g^{(1)}(X) g_{(k-1)}^{(1)}(X)] - k \sigma_1^2 \} \\
&= k \delta_k \{ E[\psi_1(X_1) \psi_2(X_1, X_1)] + (k-2) E[\psi_1(X_1) \psi_3(X_1, X_2, X_2)] \\
&\quad - (k-1) \theta_{k-1} \theta - k \sigma_1^2 \}, \\
b &= k^3 (k-1) E\{g^{(1)}(X_1) g^{(1)}(X_2) g^{(2)}(X_1, X_2)\} \\
&= k^3 (k-1) \{ E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_2)] - 2 \sigma_1^2 \theta - \theta^3 \}, \\
c &= k^4 (k-1)(k-2) E\{g^{(1)}(X_1) g^{(1)}(X_2) g^{(1)}(X_3) g^{(3)}(X_1, X_2, X_3)\} \\
&= k^4 (k-1)(k-2) \{ E[\psi_1(X_1) \psi_1(X_2) \psi_1(X_3) \psi_3(X_1, X_2, X_3)] \\
&\quad - 3 \theta E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_2)] + 3 \sigma_1^2 \theta^2 + 2 \theta^4 \}, \\
\kappa_3 &= a_3 + 3b \\
&= k^3 \{ E[(g^{(1)}(X))^3] + 3(k-1) E[g^{(1)}(X_1) g^{(1)}(X_2) g^{(2)}(X_1, X_2)] \}, \\
&= k^3 \{ E[(\psi_1(X))^3] + E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_2)] \\
&\quad - 3(2k-1) \sigma_1^2 \theta - (3k-2) \theta^3 \}, \\
\kappa_4 &= a_4 - 3\sigma^4 + 4c + 12k^4 (k-1) E[[g^{(1)}(X_1)]^2 g^{(1)}(X_2) g^{(2)}(X_1, X_2)] \\
&\quad + 12k^4 (k-1)^2 E[g^{(1)}(X_1) g^{(1)}(X_2) g^{(2)}(X_1, X_3) g^{(2)}(X_2, X_3)] \\
&= a_4 - 3\sigma^4 + 4c + 12k^4 (k-1) \{ E[[\psi_1(X_1)]^2 \psi_1(X_2) \psi_2(X_1, X_2)] \\
&\quad + (k-1) E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_3) \psi_2(X_2, X_3)] \\
&\quad - 2k\theta E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_2)] - \theta E[(\psi_1(X) - \theta)^3] \\
&\quad + k\theta^4 + (k-1)\theta^2 \sigma_1^2 - k\sigma_1^4 \}.
\end{aligned}$$

The constants σ , σ_1 , a' , κ_3 , κ_4 among the aboves, which are necessary for the Edgeworth expansion of Y-statistic, are calculated by using δ_k and the following quantities based on the kernel g :

$$\begin{aligned}
&\theta (= E[g(X_1, \dots, X_k)]), \quad \theta_{k-1} (= E[g(X_1, X_1, X_2, \dots, X_{k-1})]), \quad E[\psi_1^2(X)], \\
&E[\psi_1^3(X)], \quad E[\psi_1^4(X)], \quad E[\psi_1(X_1) \psi_2(X_1, X_1)], \quad E[\psi_1(X_1) \psi_3(X_1, X_2, X_2)], \\
&E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_2)], \quad E[\psi_1(X_1) \psi_1(X_2) \psi_1(X_3) \psi_3(X_1, X_2, X_3)], \\
&E\{[\psi_1(X_1)]^2 \psi_1(X_2) \psi_2(X_1, X_2)\}, \quad E[\psi_1(X_1) \psi_1(X_2) \psi_2(X_1, X_3) \psi_2(X_2, X_3)].
\end{aligned}$$

Let ϕ and Φ be the density function and the distribution function of the standard normal distribution $N(0, 1)$, respectively. We put the functions P_1 , P_2 , P_{21} , P_{22} and G_n as follows:

$$P_1(z) = \frac{\kappa_3}{6\sigma^3} (z^2 - 1) = \frac{\kappa_3}{6k^3 \sigma_1^3} (z^2 - 1),$$

$$\begin{aligned}
P_2(z) &= P_{21}(z) + P_{22}(z), \\
P_{21}(z) &= \left[a' + \frac{1}{4}k^2(k-1)^2[\sigma_2^2 - 2\sigma_1^2] \right] \frac{z}{\sigma^2}, \\
P_{22}(z) &= \frac{\kappa_4}{24\sigma^4}(z^3 - 3z) + \frac{\kappa_3^2}{72\sigma^6}(z^5 - 10z^3 + 15z), \\
G_n(z) &= \Phi(z) - \frac{1}{n^{1/2}}\phi(z)P_1(z) - \frac{1}{n}\phi(z)P_2(z),
\end{aligned}$$

where $\sigma_2^2 = \text{Var}[\psi_2(X_1, X_2)] > 0$. Under certain conditions given by Lai and Wang (1993), we have Edgeworth expansion of Y_n^{**} such that

$$\sup_{-\infty < z < \infty} \left| P\left(\frac{Y_n^{**}}{\sigma} \leq z\right) - G_n(z) \right| = o\left(\frac{1}{n}\right).$$

Therefore we have

$$(3.2) \quad \sup_{-\infty < z < \infty} \left| P\left(\frac{1}{\sigma}\left(Y_n^{**} + \frac{\mu_k}{\sqrt{n}}\right) \leq z\right) - G_n\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) \right| = o\left(\frac{1}{n}\right).$$

Now we shall compute

$$\begin{aligned}
(3.3) \quad G_n\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) &= \Phi\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) - \frac{1}{\sqrt{n}}\phi\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right)P_1\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) \\
&\quad - \frac{1}{n}\phi\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right)P_2\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right).
\end{aligned}$$

LEMMA 3.1.

$$(3.4) \quad \sup_{-\infty < z < \infty} \left| G_n\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) - G_n^*(z) \right| = O(n^{-3/2}),$$

where

$$(3.5) \quad G_n^*(z) = \Phi(z) - \frac{1}{\sqrt{n}}\phi(z)P_1^*(z) - \frac{1}{n}\phi(z)P_2^*(z),$$

$$P_1^*(z) = \frac{\mu_k}{\sigma} + P_1(z) = \frac{\mu_k}{\sigma} + \frac{\kappa_3}{6\sigma^3}(z^2 - 1)$$

and

$$P_2^*(z) = P_{21}^*(z) + P_{22}(z)$$

$$\begin{aligned}
P_{21}^*(z) &= \frac{\mu_k^2}{2\sigma^2}z + \frac{\mu_k\kappa_3}{6\sigma^4}(z^3 - 3z) + P_{21}(z) \\
&= \left[\frac{\mu_k^2}{2} + a' + \frac{1}{4}k^2(k-1)^2[\sigma_2^2 - 2\sigma_1^2] \right] \frac{1}{\sigma^2}z + \frac{\mu_k\kappa_3}{6\sigma^4}(z^3 - 3z).
\end{aligned}$$

That is,

$$\begin{aligned}
P_2^*(z) &= \left[\frac{\mu_k^2}{2} + a' + \frac{1}{4}k^2(k-1)^2[\sigma_2^2 - 2\sigma_1^2] \right] \frac{1}{\sigma^2}z \\
&\quad + \left[\frac{\mu_k\kappa_3}{6} + \frac{\kappa_4}{24} \right] \frac{1}{\sigma^4}(z^3 - 3z) + \frac{\kappa_3^2}{72\sigma^6}(z^5 - 10z^3 + 15z).
\end{aligned}$$

PROOF. By applying the Taylor expansion and noting $\sup_z |z|^j \phi(z) < \infty$ ($j = 1, 2, \dots$), we have

$$(3.6) \quad \sup_z \left| \Phi \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) - \left\{ \Phi(z) - \frac{\mu_k}{\sigma\sqrt{n}} \phi(z) - \frac{\mu_k^2}{2\sigma^2 n} z \phi(z) \right\} \right| = O(n^{-3/2})$$

and

$$\sup_z \left| \phi \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) - \left\{ \phi(z) + \frac{\mu_k}{\sigma\sqrt{n}} z \phi(z) + \frac{\mu_k^2}{2\sigma^2 n} (z^2 - 1) \phi(z) \right\} \right| = O(n^{-3/2}).$$

Therefore by the finiteness of $\sup_z |z|^j \phi(z)$ ($j = 1, 2, \dots$), we have

$$(3.7) \quad \sup_z \left| \phi \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) P_1 \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) - \frac{\kappa_3}{6\sigma^3} \left\{ (z^2 - 1) \phi(z) + \frac{\mu_k}{\sigma\sqrt{n}} (z^3 - 3z) \phi(z) \right\} \right| = O(n^{-1})$$

and

$$(3.8) \quad \sup_z \left| \phi \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) P_2 \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) - \phi(z) P_2(z) \right| = O(n^{-1/2}).$$

Hence applying (3.6), (3.7) and (3.8) to the right-hand side of (3.3), we get (3.4). $\square$

Thus by (2.12), (3.2) and (3.4) we get the following.

THEOREM 3.2. *We assume that $d(k, k) > 0$, $\sigma_1^2 > 0$,*

$$\begin{aligned} E [|g(X_{j_1}, \dots, X_{j_k})|^2] &< \infty \text{ for } 1 \leq j_1 \leq \dots \leq j_k \leq k, \\ E [|\psi_1(X_1)|^4] &< \infty, \quad E [|\psi_2(X_1, X_2)|^4] < \infty, \quad E [|\psi_2(X_1, X_1)|^3] < \infty, \end{aligned}$$

and

$$E [|\psi_3(X_1, X_2, X_2)|^2] < \infty, \quad E [|\psi_3(X_1, X_2, X_3)|^4] < \infty.$$

Suppose that

$$(3.9) \quad \limsup_{|t| \rightarrow \infty} |E e^{it\psi_1(X_1)}| < 1$$

and either Condition (C) or (D) holds. Then we have

$$(3.10) \quad \sup_{-\infty < z < \infty} \left| P \left(\frac{\sqrt{n}}{k\sigma_1} (Y_n - \theta) \leq z \right) - G_n^*(z) \right| = o(n^{-1}),$$

where $G_n^*(z)$ is given by (3.5).

Let us put

$$G_{U,n}^*(z) = \Phi(z) - \left[\frac{1}{\sqrt{n}} \phi(z) P_1(z) + \frac{1}{n} \phi(z) P_{22}(z) \right]$$

which is the Edgeworth expansion of $\sqrt{n}(U_n - \theta)/(k\sigma_1)$ with remainder $o(n^{-1})$ (see Maesono (1987)). Then by (3.5) we can write

$$(3.11) \quad G_n^*(z) = G_{U,n}^*(z) - \left[\frac{\mu_k}{\sqrt{n}\sigma} \phi(z) + \frac{1}{n} \phi(z) P_{21}^*(z) \right].$$

The second term of the right-hand side of this equation shows the difference between the Edgeworth expansions of the Y-statistic and U-statistic with remainder $o(n^{-1})$. As seen in (3.11), the effect of the weight function w appears only in terms of μ_k and a' , where P_{21}^* includes μ_k and a' . For the V-statistic V_n and S-statistic S_n , $\delta_k = k(k-1)/2$. Hence, for V_n and S_n ,

$$\mu_k = \frac{k(k-1)}{2}(\theta_{k-1} - \theta), \quad a' = \frac{k^2(k-1)}{2} \left[(k-1)E[g^{(1)}(X)g_{(k-1)}^{(1)}(X)] - k\sigma_1^2 \right].$$

The Edgeworth expansion of V_n is also derived by Takahashi (1988) using a different method. For the LB-statistic B_n , $\delta_k = k(k-1)$, and

$$\mu_k = k(k-1)(\theta_{k-1} - \theta), \quad a' = k^2(k-1) \left[(k-1)E[g^{(1)}(X)g_{(k-1)}^{(1)}(X)] - k\sigma_1^2 \right].$$

Using a similar method to the above, we can easily derive the second-order expansion. Taking the similar method to Proposition 2.1, we have the following. We suppose (2.4). Then we have

$$(3.12) \quad \sqrt{n}(Y_n - \theta) = \sqrt{n}(U_n - \theta) + \frac{\mu_k}{\sqrt{n}} + R_n'',$$

where $E|R_n''|^2 = O(n^{-3/2})$. By (2.10), we have

$$(3.13) \quad \sup_{-\infty < z < \infty} \left| P\left(\frac{\sqrt{n}}{\sigma}(Y_n - \theta) \leq z\right) - P\left(\frac{\sqrt{n}}{\sigma}(U_n - \theta) + \frac{\mu_k}{\sigma\sqrt{n}} \leq z\right) \right| = O(n^{-3/4}).$$

Under certain conditions which are given in the following Proposition 3.3, we have

$$(3.14) \quad \sup_{-\infty < z < \infty} \left| P\left(\frac{\sqrt{n}}{\sigma}(U_n - \theta) \leq z\right) - G'_{U,n}(z) \right| = o(n^{-1/2}),$$

where

$$G'_{U,n}(z) = \Phi(z) - \frac{1}{\sqrt{n}}\phi(z)P_1(z),$$

(see Maesono and Yamato (1994), Theorem 2.2). Therefore we have

$$(3.15) \quad \sup_z \left| P\left(\frac{\sqrt{n}}{\sigma}(U_n - \theta) + \frac{\mu_k}{\sigma\sqrt{n}} \leq z\right) - G'_{U,n}\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) \right| = o(n^{-1/2}).$$

Applying (3.6) and (3.7) to $G'_{U,n}(z - \mu_k/(\sigma\sqrt{n}))$,

$$(3.16) \quad \sup_z \left| G'_{U,n} \left(z - \frac{\mu_k}{\sigma\sqrt{n}} \right) - \left[\Phi(z) - \frac{1}{\sqrt{n}} \left[\frac{\mu_k}{\sigma} + \frac{\kappa_3}{6\sigma^3}(z^2 - 1) \right] \phi(z) \right] \right| = O(n^{-1}).$$

From (3.13), (3.14), (3.15) and (3.16), we have the second-order Edgeworth expansion.

PROPOSITION 3.3. *We assume that $d(k, k) > 0$, $\sigma_1 > 0$, $E|g(X_1, \dots, X_k)|^3 < \infty$, and $E[|g(X_{j_1}, \dots, X_{j_k})|^2] < \infty$ for $1 \leq j_1 \leq \dots \leq j_k \leq k$. Suppose that $\limsup_{|t| \rightarrow \infty} |Ee^{it\psi_1(X_1)}| < 1$. Then*

$$\sup_{-\infty < z < \infty} \left| P \left(\frac{\sqrt{n}}{k\sigma_1} (Y_n - \theta) \leq z \right) - G''_n(z) \right| = o(n^{-1/2}),$$

where

$$(3.17) \quad G''_n(z) = \Phi(z) - \frac{1}{\sqrt{n}} \left[\frac{\mu_k}{\sigma} + \frac{\kappa_3}{6\sigma^3}(z^2 - 1) \right] \phi(z).$$

We can write

$$(3.18) \quad G''_n(z) = G'_{U,n}(z) - \frac{\mu_k}{\sqrt{n}\sigma} \phi(z).$$

That is, the difference between the Edgeworth expansions of the standardized Y-statistic and U-statistic is $[\mu_k/(\sqrt{n}\sigma)]\phi(z)$, with remainder $o(n^{-1/2})$.

4. Studentized Y-statistic

Using the results of Maesono (1995, 1996) we shall evaluate the Edgeworth expansion of the studentized Y-statistic with a remainder term $o(n^{-1/2})$. For studentization of the Y-statistic, we consider the jackknife variance estimator $\hat{\sigma}_n^2$ of $\sqrt{n}Y_n$ given by

$$(4.1) \quad \hat{\sigma}_n^2 = (n-1) \sum_{i=1}^n (Y_n^{(i)} - Y_n)^2,$$

where for $i = 1, \dots, n$, $Y_n^{(i)}$ is the Y-statistic computed from a sample of size $n-1$ with X_i left out. We put

$$(4.2) \quad f^*(x) = \frac{1}{2} \left(\left[g^{(1)}(x) \right]^2 - \sigma_1^2 \right) + (k-1) E \left[g^{(1)}(X_2) g^{(2)}(x, X_2) \right].$$

Thus by Maesono (1996, p. 205) we have the asymptotic evaluation of $\hat{\sigma}_n^{-1}$ as follows:

LEMMA 4.1. *We suppose that $E|g(X_1, \dots, X_k)|^{4+\varepsilon} < \infty$ and $E|g_{(k-1)}(X_1, \dots, X_{k-1})|^{4+\varepsilon} < \infty$, for $\varepsilon > 0$, $\sigma_1^2 > 0$, and $E|g_{(j)}(X_1, \dots, X_j)|^2 < \infty$ for $j = 1, 2, \dots, k-2$. Then we have*

$$(4.3) \quad k\sigma_1\hat{\sigma}_n^{-1} = \sigma\hat{\sigma}_n^{-1} = 1 - \frac{1}{n\sigma_1^2} \sum_{i=1}^n f^*(X_i) + R_{1,n},$$

where $E|R_{1,n}|^2 = O(n^{-2})$.

We note that relation (4.3) also holds for the jackknife estimator based on the U-statistic U_n with the same order with respect to the error term; from Lemma 4 of Maesono (1995). Then, by Theorem 3 of Maesono (1995), we have the following.

LEMMA 4.2. *We suppose that $E|g(X_1, \dots, X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$, $\sigma_1^2 > 0$, and $\limsup_{|t| \rightarrow \infty} |E[\exp\{itg^{(1)}(X)\}]| < 1$. Then*

$$(4.4) \quad \sup_{-\infty < z < \infty} |P(\sqrt{n}\hat{\sigma}_n^{-1}(U_n - \theta) \leq z) - H_n(z)| = o(n^{-1/2}),$$

where

$$H_n(z) = \Phi(z) + \frac{1}{\sqrt{n}} \frac{1}{6\sigma^3} [a_3 z^2 + \kappa_3(z^2 + 1)] \phi(z).$$

From (4.3) we have

$$(4.5) \quad \frac{1}{\sqrt{n}} \mu_k \hat{\sigma}_n^{-1} = \frac{\mu_k}{\sigma\sqrt{n}} + R_{2,n},$$

where $E|R_{2,n}|^2 = O(n^{-2})$. Therefore by (3.12) and (4.5) we have

$$(4.6) \quad \sqrt{n}\hat{\sigma}_n^{-1}(Y_n - \theta) = \sqrt{n}\hat{\sigma}_n^{-1}(U_n - \theta) + \frac{\mu_k}{\sigma\sqrt{n}} + R_{3,n},$$

where $E|R_{3,n}|^2 = O(n^{-3/2})$. We note that

$$H_n\left(z - \frac{\mu_k}{\sigma\sqrt{n}}\right) = H_n^*(z) + O\left(\frac{1}{n}\right),$$

where

$$(4.7) \quad \begin{aligned} H_n^*(z) &= H_n(z) - \frac{\mu_k}{\sqrt{n}\sigma} \phi(z) \\ &= \Phi(z) + \frac{1}{\sqrt{n}} \left(-\frac{\mu_k}{\sigma} + \frac{1}{6\sigma^3} [a_3 z^2 + \kappa_3(z^2 + 1)] \right) \phi(z). \end{aligned}$$

Thus by the similar method to the last section, from (4.4), (4.6) and (4.7) we have the following.

PROPOSITION 4.3. *We suppose that $E|g(X_1, \dots, X_k)|^{4+\varepsilon} < \infty$ and $E|g_{(k-1)}(X_1, \dots, X_{k-1})|^{4+\varepsilon} < \infty$, for $\varepsilon > 0$, and $E|g_{(j)}(X_1, \dots, X_j)|^2 < \infty$ for $j = 1, 2, \dots, k-2$. We also suppose that $\sigma_1^2 > 0$, and*

$$\limsup_{|t| \rightarrow \infty} \left| E \left[\exp\{itg^{(1)}(X)\} \right] \right| < 1.$$

Then

$$(4.8) \quad \sup_{-\infty < z < \infty} |P(\sqrt{n}\hat{\sigma}_n^{-1}(Y_n - \theta) \leq z) - H_n^*(z)| = o(n^{-1/2}).$$

As seen in (4.7), the difference between the Edgeworth expansions of the studentized Y-statistic and U-statistic using a jackknife variance estimator is $[\mu_k/(\sqrt{n}\sigma)]\phi(z)$ with remainder $o(n^{-1/2})$. With the remainder $o(n^{-1/2})$, this difference is as same as the difference between the Edgeworth expansions of the standardized Y-statistic and U-statistic which is given by (3.18).

We used the the jackknife variance estimator for studentization of Y-statistic and obtained its Edgeworth expansion with a remainder of term $o(n^{-1/2})$. We shall discuss the Edgeworth expansion by using other estimators and/or with a remainder of term $o(n^{-1})$ in a future paper.

5. Examples

From (3.11), the difference between the Edgeworth expansions of standardized Y-statistic and U-statistic with remainder $o(n^{-1})$ can be written as

$$(5.1) \quad G_n^*(z) - G_{U,n}^*(z) = - \left\{ \frac{1}{\sqrt{n}\sigma} A_1 \phi(z) + \frac{1}{n\sigma^2} A_2 z \phi(z) + \frac{1}{6n\sigma^4} A_3 (z^3 - 3z) \phi(z) \right\},$$

where

$$A_1 = \mu_k, \quad A_2 = \frac{\mu_k^2}{2} + a' + \frac{1}{4}k^2(k-1)^2[\sigma_2^2 - 2\sigma_1^2], \quad A_3 = \mu_k \kappa_3.$$

We show two examples of the expansions $G_n^*(z)$, by giving $G_{U,n}^*(z)$ and A_1, A_2, A_3 . Incidentally, we also give the expansions $G_n'', G_{U,n}^*, H_n$ and H_n^* .

(1) At first we consider the third central moment $\theta = \int (x - \mu)^3 dF(x)$, where μ is the mean of F . Its kernel $g(x_1, x_2, x_3)$ is given by

$$\frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \frac{1}{2}(x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2) + 2x_1 x_2 x_3.$$

We assume that the distribution F has a density, it is symmetric about zero and its mean is zero. We also assume $E|X|^{12+\varepsilon} < \infty$ ($0 < \varepsilon < 1$) for X having the distribution F and denote its moments about the origin by m'_j ($j = 2, 4, 6, 8, 10, 12$). We can write $g^{(2)}(X_1, X_2) = (x_1 + x_2)(m'_2 - x_1 x_2)/2 = \sum_{j=1}^5 c_j f_j(X_1) f_j(X_2)$, where

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{2}, \quad c_3 = \frac{1}{2}(1 - m'_2), \quad c_4 = \frac{1}{2}m'_2, \quad c_5 = -\frac{1}{2}m'_2,$$

and

$$f_1(x) = x^2, \quad f_2(x) = x^2 + x, \quad f_3(x) = x, \quad f_4(x) = x + 1, \quad f_5(x) = 1.$$

We note that

$$|Ee^{it[\alpha(X) + \sum_{j=1}^5 s_j f_j(X)]}| \leq |I| + |Ee^{it\alpha(X)}|,$$

where

$$I = Ee^{it\alpha(X)} \{e^{it\sum_{j=1}^5 s_j f_j(X)} - 1\}.$$

Then, it follows that under the condition $|s_1| + \dots + |s_5| \leq |t|^{-\delta}$ ($0 < \forall \delta < 1$),

$$|I| \leq E|e^{it\sum_{j=1}^5 s_j f_j(X)} - 1| \leq |t|^{-\delta} (EX^2 + E|X| + 1),$$

which tends to zero as $|t| \rightarrow \infty$. Since X has the density, $Ee^{it\alpha(X)} = Ee^{it(X^3 - 3X)} \rightarrow 0$ as $t \rightarrow \infty$. Thus the condition (D) is satisfied.

Now, we have

$$\begin{aligned} \sigma_1^2 &= \frac{1}{9}(m'_6 - 6m'_4m'_2 + 9m'^3_2), \\ \sigma_2^2 &= \frac{2}{9}m'_6 - \frac{5}{6}m'_4m'_2 + \frac{3}{2}m'^3_2, \\ \sigma^2 &= m'_6 - 6m'_4m'_2 + 9m'^3_2, \\ a_3 &= 0, \\ a_4 &= m'_{12} - 12m'_{10}m'_2 + 54m'_8m'^2_2 - 108m'_6m'^3_2 + 81m'_4m'^4_2, \\ a' &= -\delta_3(m'_6 - 6m'_4m'_2 + 9m'^3_2), \\ b &= 0, \\ c &= 12(m'_4 - 3m'^2_2)^3, \\ \kappa_3 &= 0, \\ \kappa_4 &= m'_{12} - 12m'_{10}m'_2 - 36m'_8m'_4 + 162m'_8m'^2_2 - 3m'^2_6 + 360m'_6m'_4m'_2 \\ &\quad - 1134m'_6m'^3_2 + 156m'^3_4 - 2268m'^2_4m'^2_2 + 5913m'_4m'^4_2 - 3483m'^6_2, \\ \mu_k &= 0. \end{aligned}$$

The coefficients necessary for the Edgeworth expansion are calculated by using δ_3 and the moments $m'_2, m'_4, m'_6, m'_8, m'_{10}, m'_{12}$. In this case, we have $P_1(z) = 0$, $P_{22}(z) = \kappa_4(z^3 - 3z)/(24\sigma^4)$ and

$$G_{U,n}^*(z) = \Phi(z) - \frac{1}{n}\phi(z)P_{22}(z).$$

The coefficients of (5.1) are

$$A_1 = A_3 = 0$$

and

$$A_2 = \begin{cases} -3m'_6 + \frac{45}{2}m'_4m'_2 - \frac{63}{2}m'^3_2 & (\text{V, S - statistic}), \\ -6m'_6 + \frac{81}{2}m'_4m'_2 - \frac{117}{2}m'^3_2 & (\text{LB - statistic}). \end{cases}$$

On the other hand, we have $G_n''(z) = G'_{U,n}(z) = \Phi(z)$ and $H_n^*(z) = H_n(z) = \Phi(z)$. Hence, we must consider the Edgeworth expansion of higher order to see the effect of Studentizing. We shall consider this problem in a future paper.

(2) We consider the kernel $g(x_1, x_2, \dots, x_k) = x_1 x_2 \cdots x_k$ ($k \geq 3$). This kernel yields the estimable parameter $\theta(F) = \mu^k$, where μ is the mean of the distribution F . We assume that the distribution F is symmetric about the mean μ (> 0) and $EX^{2k} < \infty$. We denote the central moments about the mean by m_j ($j = 2, 4$). Furthermore, we assume that F have a density and the characteristic function ψ of $F(x - \mu)$ is decreasing on $[0, \infty)$. We can write $g^{(2)}(x_1, x_2) = \mu^{k-2}(x_1 - \mu)(x_2 - \mu)$.

Hence, we have $E \exp(it[\alpha(X) + sf(X)]) = \psi(\mu^{(k-2)/2}[k\mu^{k/2} + s]t)$. We shall take a sufficiently large $|t|$ such that $|t|^{-\varepsilon} < \mu$ for any $\varepsilon > 0$. By decreasing values of ψ , for $|s| \leq |t|^{-\varepsilon}$ we have

$$\psi(\mu^{(k-2)/2}[k\mu^{k/2} + s]t) \leq \max \left\{ \psi(\mu^{(k-2)/2}[k\mu^{k/2} \pm |t|^{-\varepsilon}]t) \right\}.$$

Thus $\sup_{|s| \leq |t|^{-\varepsilon}} \psi(\mu^{(k-2)/2}[k\mu^{k/2} + s]t)$ converges to zero as $|t| \rightarrow \infty$, since F has the density. Hence, the condition (D) is satisfied.

Now, we have

$$\begin{aligned} \sigma_1^2 &= \mu^{2(k-1)}m_2, \\ \sigma_2^2 &= \mu^{2(k-2)}(m_2^2 + 2\mu^2m_2), \\ \sigma^2 &= k^2\mu^{2(k-1)}m_2, \\ a_3 &= 0, \\ a_4 &= k^4\mu^{4(k-1)}m_4, \\ a' &= \delta_k k(k-2)\mu^{2(k-2)}m_2^2, \\ b &= k^3(k-1)\mu^{3k-4}m_2^2, \\ c &= k^4(k-1)(k-2)\mu^{2(2k-3)}m_2^3, \\ \kappa_3 &= 3k^3(k-1)\mu^{3k-4}m_2^2, \\ \kappa_4 &= k^4\{\mu^{4(k-1)}m_4 + 4(k-1)(4k-5)\mu^{2(2k-3)}m_2^3 - 3\mu^{4(k-1)}m_2^2 \\ &\quad - 12(k-1)^2\mu^{2(2k-1)}m_2 - 36(k-1)\mu^{4k}\}, \\ \mu_k &= \delta_k\mu^{k-2}m_2. \end{aligned}$$

The coefficients necessary for the Edgeworth expansion are calculated by using δ_k and the moments μ , m_2 , m_4 . In this case,

$$\begin{aligned} P_1(z) &= \frac{(k-1)\sqrt{m_2}}{2\mu}(z^2 - 1), \\ P_{22}(z) &= \frac{\mu^2m_4 + 4(k-1)(4k-5)m_2^3 - 3\mu^2m_2^2 - 12(k-1)^2\mu^4m_2 - 36(k-1)\mu^6}{24\mu^2m_2^2} \\ &\quad \times (z^3 - 3z) + \frac{(k-1)^2m_2}{8\mu^2}(z^5 - 10z^3 + 15z) \end{aligned}$$

and

$$G_{U,n}^*(z) = \Phi(z) - \left[\frac{1}{\sqrt{n}} \phi(z) P_1(z) + \frac{1}{n} \phi(z) P_{22}(z) \right].$$

The coefficients of the right-hand side of (5.1) are as follows.

$$\begin{aligned} A_1 &= \begin{cases} \frac{k(k-1)\mu^{k-2}m_2}{2} & (\text{V, S - statistic}) \\ k(k-1)\mu^{k-2}m_2 & (\text{LB - statistic}). \end{cases} \\ A_2 &= \begin{cases} \frac{k^2(k-1)(7k-11)\mu^{2(k-2)}m_2^2}{8} & (\text{V, S - statistic}) \\ \frac{k^2(k-1)(7k-11)\mu^{2(k-2)}m_2^2}{4} & (\text{LB - statistic}). \end{cases} \\ A_3 &= \begin{cases} \frac{3k^4(k-1)^2\mu^{2(2k-3)}m_2^3}{2} & (\text{V, S - statistic}) \\ 3k^4(k-1)^2\mu^{2(2k-3)}m_2^3 & (\text{LB - statistic}). \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} G_n''(z) &= \Phi(z) - \frac{1}{\sqrt{n}} \left[\frac{\delta_k \sqrt{m_2}}{k\mu} + \frac{(k-1)\sqrt{m_2}}{2\mu} (z^2 - 1) \right] \phi(z), \\ G_n''(z) - G_{U,n}'(z) &= -\frac{(k-1)\sqrt{m_2}}{2\mu\sqrt{n}} \phi(z) \quad (\text{V, S - statistic}) \end{aligned}$$

and

$$G_n''(z) - G_{U,n}'(z) = -\frac{(k-1)\sqrt{m_2}}{\mu\sqrt{n}} \phi(z) \quad (\text{LB - statistic}).$$

While, under the condition $E|X|^{8+\varepsilon} < \infty$ for ε ,

$$\begin{aligned} H_n(z) &= \Phi(z) + \frac{(k-1)\sqrt{m_2}}{2\mu\sqrt{n}} (z^2 + 1) \phi(z), \\ H_n^*(z) - H_n(z) &= -\frac{(k-1)\sqrt{m_2}}{2\mu\sqrt{n}} \phi(z) \quad (\text{V, S - statistic}) \end{aligned}$$

and

$$H_n^*(z) - H_n(z) = -\frac{(k-1)\sqrt{m_2}}{\mu\sqrt{n}} \phi(z) \quad (\text{LB - statistic}).$$

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