MULTIPLE DECISION PROBLEM ON SIGNS OF MEAN VECTOR IN THREE-VARIATE NORMAL CASE

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Testing procedures can be considered decision procedures, and there exists the problem of how to deal with observed values of test statistics which fall in a neighborhood of the critical values. On the other hand, in the framework of the multiple decision problem, even if an incorrect decision is made, it may still be close to the correct decision, and may not exhibit the degree of error that would result from making decisions completely contrary to the truth. While testing procedures are conservative only in respect of rejection of the null hypothesis, multiple decision procedures are more conservative. In this paper we discuss the multiple decision problem on the signs of the components of the three-variate normal mean vector based on Takeuchi (1973), and study the behavior of multiple decision procedures by simulation.

Key words and phrases: Likelihood ratio test, multiple decision procedure, testing procedure, three-variate normal distribution.

1. Introduction

Testing procedures may be considered decision procedures, and as such, there exists the problem of dealing with observed values of test statistics that fall in a neighborhood of the critical values (i.e., the boundary between the acceptance region and the rejection region). If the observed values of test statistics fall into the rejection region and a testing procedure decides the rejection of going against the truth with probability less than a significance level due to the influence of a few outliers contained in the data, then a critical error arises. The testing procedure becomes a strong tool when observers have some degree of conviction about the hypotheses, but they need to consider the risks of making an error. By regarding the hypotheses as sets of parameters and the problem of testing a hypothesis as that of estimating an indicator of the null hypothesis, we can evaluate the risks via appropriate losses (see e.g. Maihara and Akahira (2003, 2004), Hwang et al. (1992)).

On the other hand, in the framework of the multiple decision problem, even if one makes a decision contrary to the truth, the decision may be close to the correct one, and may not exhibit the degree of error that would result from making decisions completely contrary to the truth. While testing procedures are conservative only in respect of rejection of the null hypothesis, multiple decision procedures are more conservative.

Takeuchi (1973) discussed the multiple decision problem as it applies to the signs of the components of a mean vector in the univariate and two-variate
normal case (see also Maihara (2005)). Related results may be found in Cohen and Sackrowitz (2005). In this paper, we firstly reconsider the formulation and theorems described in Takeuchi (1973), and then discuss the multiple decision problem in the context of the signs of the components of a three-variate normal mean vector. We also investigate the behavior of multiple decision procedures by means of simulations.

2. Formulation

In this section, we describe the formulation and the theorems given by Takeuchi (1973) in a mathematically stricter form.

Let \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) and \(\Theta\) be a sample space and a parameter space, respectively, where \(\mathcal{X}\) is an open set of \(\mathbb{R}^k\) and \(\Theta\) is an open set of \(\mathbb{R}^p\). Suppose that \(X_1, \ldots, X_n\) are independent and identically distributed (i.i.d.) \(k\)-dimensional random vectors according to a distribution belonging to a family \(P = \{P_\theta \mid \theta \in \Theta\}\) of probability distributions defined on the family \(\mathcal{B}(\mathcal{X})\) of Borel sets. We denote the \(k \times n\) random matrix by \(X := (X_1, \ldots, X_n)\).

**Definition 2.1.** Let \(\Omega_1, \ldots, \Omega_m\) be subsets of \(\Theta\) such that neither \(\Omega_i \subseteq \Omega_j\) nor \(\Omega_i \supseteq \Omega_j\) for \(i \neq j\), i.e.,

\[
\Omega_i \nsubseteq \Omega_j \quad (i \neq j),
\]

and such that \(\{\Omega_1, \ldots, \Omega_m\}\) is a covering of \(\Theta\) with no proper subcovering, that is,

\[
\Theta = \bigcup_{i=1}^{m} \Omega_i \supseteq \bigcup_{j \in J} \Omega_j \quad \text{for } J \subset \{1, \ldots, m\}.
\]

Let \(S\) be a family of subsets of \(\Theta\) containing the subsets \(\Omega_1, \ldots, \Omega_m\) and their union, that is,

\[
S := \left\{ \bigcup_{i \in I} \Omega_i \mid I \in \mathcal{M} \right\},
\]

where \(\mathcal{M} \subseteq 2^{\{1, \ldots, m\}} := \{A \mid A \subseteq \{1, \ldots, m\}\}, \{1\}, \ldots, \{m\} \in \mathcal{M},\) and \(\emptyset \notin \mathcal{M}\). Then, \(\Omega_1, \ldots, \Omega_m\) are called primary elements of \(S\) and the family \(\{\Omega_1, \ldots, \Omega_m\}\) of all primary elements will be denoted by \(S_e\). Furthermore, a mapping \(\varphi\) from \(\mathcal{X}^n\) to \(S\) will be called a multiple decision procedure:

\[
\varphi : \mathcal{X}^n \longrightarrow S \subseteq 2^\Theta = \{\Omega \mid \Omega \subseteq \Theta\},
\]

\[
\psi \quad \psi
\]

\[
x \longmapsto \varphi(x) \subseteq \Theta.
\]

**Remark 2.1.** For each \(\Omega \in S\), there is at least one primary element \(\Omega_e \in S_e\) such that \(\Omega_e \subseteq \Omega\), and no proper subset of any primary element \(\Omega_e \in S_e\) exists in
S. Furthermore, $S_e \subseteq S$ and $m \leq \#S = \#M \leq 2^m - 1$. If $S = 2^\Theta$, the multiple decision procedure is reduced to region estimation, and if $S = \{\Omega, \Theta \setminus \Omega\} (\Omega \subset \Theta)$, it is also reduced to testing the hypothesis.

**Definition 2.2.** If, for all $\Omega \in S$, there is a constant $\alpha_\Omega$ $(0 < \alpha_\Omega < 1/2)$ such that

\[ P_\theta \{ \theta \in \varphi(X) \} \geq 1 - \alpha_\Omega \quad \text{for} \quad \theta \in \Omega, \]  

(2.2)

then the multiple decision procedure $\varphi$ is called a confidence procedure with confidence system $\{1 - \alpha_\Omega \mid \Omega \in S\}$.

Under the above setup, the statistical decision problem described by $(\mathcal{X}, \Theta, \mathcal{P}, S, \varphi, \{1 - \alpha_\Omega\})$ will be called a multiple decision problem.

**Remark 2.2.** The inequality (2.2) means that the probability of making a decision for which the true parameter $\theta$ belongs is not less than the constant $1 - \alpha_\Omega$ when $\theta \in \Omega (\subseteq \Theta)$.

**Definition 2.3.** If $M = 2^{\{1, \ldots, m\}}$ in (2.1), that is, the unions of all the combinations of primary elements $\Omega_1, \ldots, \Omega_m$ belong to $S$, $S$ is said to be unrestricted. If $S$ is not unrestricted, $S$ is said to be restricted.

A confidence decision procedure, i.e. a multiple decision procedure, can be constructed as follows.

**Theorem 2.1.** Suppose that $S$ is unrestricted. Denote by $A_{\Omega_e}$ an acceptance region of testing a hypothesis $H_{\Omega_e}: \theta \in \Omega_e$ with a significance level $\alpha_{\Omega_e}$ for $\Omega_e \in S_e$. Let $\mathcal{X}^n = \bigcup_{\Omega_e \in S_e} A_{\Omega_e}$. Then the mapping

\[ \varphi(x) := \bigcup_{\Omega_e : \Omega_e \in S_e, x \in A_{\Omega_e}} \Omega_e \]  

(2.3)

from $\mathcal{X}^n$ to $S$ is a confidence procedure with the confidence system $\{1 - \alpha_\Omega \mid \Omega \in S\}$, where

\[ \alpha_\Omega := \sup_{\Omega_e : \Omega_e \in S_e, \Omega_e \subseteq \Omega} \alpha_{\Omega_e}. \]  

(2.4)

Especially, if the primary elements are mutually disjointed, every confidence procedure can be expressed in the form (2.3).

**Proof.** Firstly, since $\mathcal{X}^n = \bigcup A_{\Omega_e}$, it follows that at least one hypothesis $H_{\Omega_{e_0}}: \theta \in \Omega_{e_0}$ is accepted, and so $\varphi(x)$ defined in (2.3) is not empty for all $x \in \mathcal{X}^n$. Since $S$ is unrestricted, $S$ contains all the unions of primary elements $\Omega_e \in S_e$, and so $\varphi(x) \in S$ for all $x \in \mathcal{X}^n$. Every $\Omega$ in $S$ can be expressed as a
union of several primary elements $\Omega_e \in S_e$ and every $\theta$ in $\Omega$ belongs to some $\Omega_{e1}$ in $S_e$. If $\theta \in \Omega_{e1}$, then for each $x \in A_{\Omega_{e1}}$ the following holds

$$\theta \in \Omega_{e1} \subseteq \bigcup_{\Omega_e : \Omega_e \in S_e} \Omega_e = \varphi(x),$$

that is,

$$A_{\Omega_{e1}} \subseteq \{x \mid \theta \in \varphi(x)\} \quad \text{for} \quad \theta \in \Omega_{e1}. \quad (2.5)$$

Since the test associated with each acceptance region is of level $\alpha$, it follows that

$$P_{X \theta} \{A_{\Omega_{e1}}\} \geq 1 - \alpha_{\Omega_{e1}} \quad \text{for} \quad \theta \in \Omega_{e1}. \quad (2.6)$$

From (2.5) and (2.6) we have

$$\theta \in \varphi(x) \implies \{x \mid \theta \in \varphi(x)\} \subseteq \{x \mid \theta \in \varphi(x)\} \quad \text{for} \quad \theta \in \Omega_{e1}. \quad (2.7)$$

From (2.4) we obtain $1 - \alpha_{\Omega_{e1}} \geq 1 - \alpha$ for all $\Omega_{e1} \in S_e$ satisfying $\Omega_{e1} \subseteq \Omega$, hence from (2.7), we have

$$P_{X \theta} \{\theta \in \varphi(X)\} \geq 1 - \alpha \quad \text{for} \quad \theta \in \Omega_{e1}. \quad (2.8)$$

Since (2.8) holds for all $\Omega_{e1} \in S_e$ satisfying $\Omega_{e1} \subseteq \Omega$, it follows that

$$P_{\theta} \{\theta \in \varphi(X)\} \geq 1 - \alpha \quad \text{for} \quad \theta \in \Omega. \quad (2.9)$$

This means that $\varphi(x)$ is a confidence procedure with confidence system $\{1 - \alpha_\Omega \mid \Omega \in S\}$.

Conversely, suppose that $\varphi(x)$ is a confidence procedure with confidence system $\{1 - \alpha_\Omega \mid \Omega \in S\}$ and that the elements of $S_e$ are mutually disjointed. Let $A_{\Omega_e} := \{x \mid \Omega_e \subseteq \varphi(x)\}$ for $\Omega_e \in S_e$. Now, for any $\theta \in \Omega_e$, $\Omega_e \subseteq \varphi(x)$ implies $\theta \in \varphi(x)$, which yields

$$\{x \mid \Omega_e \subseteq \varphi(x)\} \subseteq \{x \mid \theta \in \varphi(x)\} \quad \text{for} \quad \theta \in \Omega_e. \quad (2.10)$$

On the other hand, if $\theta \in \varphi(x)$ for every $\theta \in \Omega_e$, then $(\theta \in)\Omega_e \cap \varphi(x) \neq \emptyset$, and so $\Omega_e \subseteq \varphi(x)$, which implies

$$\{x \mid \theta \in \varphi(x)\} \subseteq \{x \mid \Omega_e \subseteq \varphi(x)\} \quad \text{for} \quad \theta \in \Omega_e. \quad (2.11)$$

From (2.9) and (2.10), we have

$$\{x \mid \Omega_e \subseteq \varphi(x)\} = \{x \mid \theta \in \varphi(x)\} \quad \text{for} \quad \theta \in \Omega_e.$$
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that is, \( A_{\Omega_e} \) is an acceptance region of testing the hypothesis \( H_{\Omega_e} : \theta \in \Omega_e \) with significance level \( \alpha_{\Omega_e} \). Now, if \( \vartheta \in \varphi(x) \), there exists \( \Omega_{e1} \in S_e \) such that \( \vartheta \in \Omega_{e1} \) and \( \Omega_{e1} \subseteq \varphi(x) \), and so \( x \in A_{\Omega_{e1}} \), which yields

\[
\vartheta \in \Omega_{e1} \subseteq \bigcup_{\Omega_e : \Omega_e \in S_e} \Omega_e,
\]

hence

\[
(2.12) \quad \varphi(x) \subseteq \bigcup_{\Omega_e : \Omega_e \in S_e} \Omega_e.
\]

On the other hand, if \( \vartheta \) belongs to the right side of (2.12), there exists \( \Omega_{e2} \in S_e \) such that \( \vartheta \in \Omega_{e2} \) and \( x \in A_{\Omega_{e2}} = \{ x \mid \Omega_{e2} \subseteq \varphi(x) \} \), and so \( \Omega_{e2} \subseteq \varphi(x) \). This implies

\[
(2.13) \quad \bigcup_{\Omega_e : \Omega_e \in S_e} \Omega_e \subseteq \varphi(x).
\]

From (2.12) and (2.13), it follows that the two sides of (2.12) are in fact identical.

**Definition 2.4.** Suppose the assumptions of Theorem 2.1 hold, and let \( \varphi \) be a confidence procedure represented in the form (2.3). Let

\[
(2.14) \quad \beta_{\Omega}(\theta) := 1 - P_{\theta}\{ \Omega \subseteq \varphi(X) \} = P_{\theta}\left\{ \bigcup_{\Omega_e : \Omega_e \in S_e, \Omega_e \subseteq \Omega} \{ X \in R_{\Omega_e} \} \right\}
\]

\[
= P_{\theta}^{X}\left\{ \bigcup_{\Omega_e : \Omega_e \in S_e, \Omega_e \subseteq \Omega} R_{\Omega_e} \right\}
\]

for all \( \Omega \in S \setminus \{ \Theta \} \) and all \( \theta \in \Theta \setminus \Omega \), where \( R_{\Omega_e} \) is the rejection region for testing the hypothesis \( H_{\Omega_e} \), that is, the complement of the acceptance region \( A_{\Omega_e} \). Then, \( \beta_{\Omega}(\theta) \) is called the power (function) of the confidence procedure \( \varphi \).

**Remark 2.3.** Since it is an error that the decision which contains \( \Omega \) as a subset is made when the true value of the parameter \( \theta \) is not contained in \( \Omega \), it is desirable that the probability of not making such a decision, i.e. the power, is greater. Furthermore, we observe that the power \( \beta_{\Omega}(\theta) \) can be evaluated in terms of the power of testing the hypotheses \( H_{\Omega_e} \).

When \( S \) is unrestricted and a confidence procedure (multiple decision procedure) is constructed using Theorem 2.1, the confidence procedure becomes so complicated that unnecessary and detailed decisions may have to be made. By restricting the decision space \( S \), simpler confidence procedures can be constructed.
Theorem 2.2. Let $\Theta \in S$. Let $\varphi_0$ be a confidence procedure corresponding to $S_0$ such that $S \subseteq S_0$ with confidence system $\{1 - \alpha_{\Omega_0} \mid \Omega_0 \in S_0\}$. Then a decision procedure $\varphi$ satisfying

$$(2.15) \quad \varphi(x) \in S, \quad \varphi(x) \supseteq \varphi_0(x)$$

is a confidence procedure corresponding to $S$ with confidence system $\{1 - \alpha_{\Omega} \mid \Omega \in S\}$. Furthermore, suppose that $S$ is closed under non-empty intersection operations, that is, $\Omega \cap \Omega' \neq \emptyset$ implies that $\Omega \cap \Omega' \in S$ for $\Omega \in S$ and $\Omega' \in S$. Then, there is the minimum confidence procedure satisfying (2.15) with confidence system $\{1 - \alpha_{\Omega} \mid \Omega \in S\}$.

Proof. The existence of $\varphi$ satisfying (2.15) is guaranteed by taking $\varphi(x) \equiv \Theta$ for all $x \in \mathcal{X}^n$, because $\Theta \in S$. Since $\varphi_0$ is a confidence procedure with confidence system $\{1 - \alpha_{\Omega_0} \mid \Omega_0 \in S_0\}$, it follows that for any $\Omega \in S(\subseteq S_0)$$$
P_\theta\{\theta \in \varphi_0(X)\} \geq 1 - \alpha_{\Omega} \quad \text{for} \quad \theta \in \Omega.$$

Since $\varphi(x) \supseteq \varphi_0(x)$, it holds that $$P_\theta\{\theta \in \varphi(X)\} \geq P_\theta\{\theta \in \varphi_0(X)\} \geq 1 - \alpha_{\Omega} \quad \text{for} \quad \theta \in \Omega,$$
hence, $\varphi$ is a confidence procedure from $\mathcal{X}^n$ to $S$ with confidence system $\{1 - \alpha_{\Omega} \mid \Omega \in S\}$.

To prove the latter half, let

$$(2.16) \quad \varphi^*(x) := \bigcap_{\Omega: \Omega \in S \subseteq \varphi_0(x)} \Omega.$$

Since $S$ is closed under the non-empty product operations, it follows that $\varphi^*(x) \in S$. Now, for any $\Omega' \in S$ satisfying $\varphi_0(x) \subseteq \Omega'$, $\vartheta \in \varphi_0(x)$ implies that $\vartheta \in \Omega'$, and so

$$\vartheta \in \bigcap_{\Omega: \Omega \in S \subseteq \varphi_0(x)} \Omega = \varphi^*(x).$$

Hence $\varphi_0(x) \subseteq \varphi^*(x)$, and so $\varphi^*$ satisfies (2.15). Thus, from the proof of the first half of this theorem, $\varphi^*$ is a confidence procedure with confidence system $\{1 - \alpha_{\Omega} \mid \Omega \in S\}$. The minimality of $\varphi^*$ is obvious from its construction (2.16).

For any confidence procedure $\varphi$ satisfying (2.15), it holds that

$$\varphi^*(x) = \bigcap_{\Omega: \Omega \in S \subseteq \varphi_0(x)} \varphi(x)$$

for all $x \in \mathcal{X}^n$. Therefore $\varphi^*$ is the minimum confidence procedure satisfying (2.15). \(\square\)

Remark 2.4. The condition $\Theta \in S$ is equivalent to the condition that the family $\mathcal{M}$ of sets of subscripts in (2.1) satisfies $\{1, \ldots, m\} \in \mathcal{M}$. 

Furthermore, confidence procedures can be constructed from the confidence regions, especially from confidence intervals, as can be seen from the following corollary.

**Corollary 2.1.** Suppose that $\Theta \in \mathcal{S}$. Let $\varphi_0$ be a confidence region with confidence coefficient $1 - \alpha$. Then a decision procedure $\varphi$ satisfying (2.15) is a confidence procedure with confidence system $\{1 - \alpha\}$. Furthermore, suppose that $\mathcal{S}$ is closed under non-empty intersections. Then there exists the minimum confidence procedure satisfying (2.15) with confidence system $\{1 - \alpha\}$.

**Proof.** Since $\varphi_0$ is a confidence region corresponding to the confidence coefficient $1 - \alpha$, we can define $\mathcal{S}_0$ by $\mathcal{S}_0 := 2^\Theta \setminus \{\emptyset\}$, $P_{\theta}\{\theta \in \varphi_0(X)\} \geq 1 - \alpha$ for any $\Omega \in \mathcal{S}_0$ and any $\theta \in \Omega (\subseteq \Theta)$. That is, $\varphi_0$ is a confidence procedure for $\mathcal{S}_0$ with confidence system $\{1 - \alpha\}$. Since $\mathcal{S} \subseteq (2^\Theta =) \mathcal{S}_0$, the proof is completed by applying Theorem 2.2. $\Box$

**Example 2.1 (Takeuchi (1973)).** Let $X_1, \ldots, X_n$ be i.i.d. random variables according to the normal distribution $N(\mu, \sigma^2)$, where the mean $\mu$ and the variance $\sigma^2$ are unknown. Let $\theta := (\mu, \sigma) \in \Theta := \{(\mu, \sigma) | -\infty < \mu < \infty, 0 < \sigma < \infty\} = \mathbb{R} \times \mathbb{R}_+$. Then we consider multiple decision problems on the sign of the mean $\mu$ by applying Theorem 2.1. The parameter space $\Theta$ is divided into the following three primary elements:

- $\Omega_1 := \{ (\mu, \sigma) | -\infty < \mu < 0, 0 < \sigma < \infty \}$,
- $\Omega_2 := \{ (\mu, \sigma) | \mu = 0, 0 < \sigma < \infty \}$,
- $\Omega_3 := \{ (\mu, \sigma) | 0 < \mu < \infty, 0 < \sigma < \infty \}$.

The elements of the decision space $\mathcal{S}$ which can be derived from the multiple decision procedures are the following 7 ($= 2^3 - 1$) subsets of $\Theta$:

- $\Omega_1, \Omega_2, \Omega_3$, $\Omega_1 \oplus \Omega_2 = \{ (\mu, \sigma) | -\infty < \mu \leq 0, 0 < \sigma < \infty \}$,
- $\Omega_2 \oplus \Omega_3 = \{ (\mu, \sigma) | 0 \leq \mu < \infty, 0 < \sigma < \infty \}$,
- $\Omega_3 \oplus \Omega_1 = \{ (\mu, \sigma) | \mu \neq 0, 0 < \sigma < \infty \}$, $\Omega_1 \oplus \Omega_2 \oplus \Omega_3 = \Theta$.

For the three hypotheses $H_1 : \mu < 0$, $H_2 : \mu = 0$, $H_3 : \mu > 0$ corresponding to the primary elements, the uniformly most powerful unbiased (UMPU) tests of level $\alpha$ exist, and the corresponding acceptance regions are given by

- $\mathcal{A}_1 := \{ x \in \mathbb{R}^n | T(x) \leq t_\alpha \}$,
- $\mathcal{A}_2 := \{ x \in \mathbb{R}^n | |T(x)| \leq t_{\alpha/2} \}$,
- $\mathcal{A}_3 := \{ x \in \mathbb{R}^n | T(x) \geq -t_\alpha \}$,

respectively, where

$$T = T(X) := \frac{\sqrt{n} \bar{X}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2/(n - 1)}}, \quad \bar{X} := \frac{1}{n} \sum_{i=1}^n X_i,$$
Figure 1. The acceptance regions $A_1, A_2, A_3$ for the hypotheses $H_1, H_2, H_3$ ($n = 20, \alpha = 0.05$ (left figure), $\alpha = 0.50$ (right figure)).

and $t_\alpha$ is the upper 100$\alpha$ percentage point of the $t$ distribution with $n - 1$ degrees of freedom.

Then it follows from Theorem 2.1 that a confidence procedure (multiple decision procedure), with confidence system \{$1 - \alpha$\}, based on the $t$ test statistic $T$ is given by

$$\varphi^T(x) := \bigcup_{x \in A_i} \Omega_i$$

$$= \begin{cases} 
\{(\mu, \sigma) \mid -\infty < \mu < 0, 0 < \sigma < \infty\} & \text{for } T(x) < -t_{\alpha/2}, \\
\{(\mu, \sigma) \mid -\infty < \mu \leq 0, 0 < \sigma < \infty\} & \text{for } -t_{\alpha/2} \leq T(x) < -t_{\alpha}, \\
\{(\mu, \sigma) \mid -\infty < \mu < \infty, 0 < \sigma < \infty\} & \text{for } -t_{\alpha} \leq T(x) \leq t_{\alpha}, \\
\{(\mu, \sigma) \mid 0 \leq \mu < \infty, 0 < \sigma < \infty\} & \text{for } t_{\alpha} < T(x) \leq t_{\alpha/2}, \\
\{(\mu, \sigma) \mid 0 < \mu < \infty, 0 < \sigma < \infty\} & \text{for } t_{\alpha/2} < T(x). 
\end{cases}$$

For example, when $t_{\alpha} < T \leq t_{\alpha/2}$, i.e. $x \in A_2 \cap A_3 \setminus A_1$ (see Fig. 1), only $H_2 : \mu = 0$ and $H_3 : \mu > 0$ are accepted and the decision $\mu \geq 0$ is taken. Now, if $|T| \leq t_{\alpha/2}$, then the decision $\mu = 0$ is made in the individual test for the simple hypothesis $H_2$ against the composite alternative $K_2 : \mu \neq 0$, but this decision is not produced by the multiple decision procedure $\varphi^T$. This follows from the fact that the acceptance region $A_2$ of the simple hypothesis $H_2$ is covered with the acceptance regions $A_1$ and $A_3$ on both sides of $H_2$, i.e. the regions corresponding to hypotheses $H_1$ and $H_3$. Thus, $H_2$ is not accepted alone. When the observed values are concentrated in a neighborhood of the origin, that is, when the true value of the mean $\mu$ exists in a neighborhood of the origin, it may become a critical error that the observer decides incorrectly that $\mu$ is positive or negative. In this case, the multiple decision procedure suspends a decision on the sign of $\mu$ to avoid the critical error from a conservative point of view. Moreover, it does not occur that only $H_2$ is rejected and the decision $\mu = 0$ is taken, because the rejection region $R_2 = A_2^c$ of testing the hypothesis $H_2$ is also covered with $A_1$ and $A_3$. Furthermore, if $T \geq -t_{\alpha}$, then the decision $\mu > 0$ is made in the individual test for the composite hypothesis $H_3$ against the composite alternative $K_3 : \mu \leq 0$, but the multiple decision procedure $\varphi^T$ does not derive this decision unless $T > t_{\alpha/2}(> -t_{\alpha})$. To derive the decision that $\mu$ is away from the origin, the multiple decision procedure needs more definite behavior of the test statistic $T$ away from the origin than the individual testing procedure. These phenomena also occur in the three-variate normal case, to be discussed in the next section.
The behavior of the multiple decision procedure is shown later in this paper using simulations (see Tables 1 to 6).

We now consider the power of the confidence procedure (the multiple decision procedure) \( \varphi^T \). Although the power is defined for \( \theta \in \Theta \setminus \Omega \), it follows from (2.14) that

\[
(2.17) \quad \beta_\Omega(\theta) = P_\theta^X \left\{ \bigcup_{1 \leq i < 3} \mathcal{R}_i \right\} \geq P_\theta^X \{ \mathcal{R}_j \} \quad \text{for} \quad \theta \notin \Omega,
\]

since \( \theta \notin \Omega \) means that \( \theta \notin \Omega_j \) for subsets \( \Omega_j \) (1 \( \leq j \leq 3 \)) of \( \Omega \). Hence, the lower bounds for the power of the confidence procedure \( \varphi^T \) are given by the power for testing the hypothesis \( H_j : \theta \in \Omega_j \). In particular, since \( \varphi^T \) is based on the three acceptance regions of three UMPU tests with the same level \( \alpha \), it follows that

\[
(2.18) \quad P_\theta^X \{ \mathcal{R}_j \} \geq \alpha \quad \text{for} \quad \theta \notin \Omega,
\]

and from (2.17) and (2.18) we obtain

\[
\beta_\Omega(\theta) \geq \alpha \quad \text{for} \quad \theta \notin \Omega.
\]

3. **Multiple decision problem on the signs of three means**

Next, we discuss the multiple decision problem on the signs of the components of the mean vector in the three-variate normal case. A similar method to that applied for the two-variate normal case in Takeuchi (1973) and Maihara (2005) is used. Its behavior is then investigated using a simulation.

Let \( X_1, \ldots, X_n \) be i.i.d. random vectors according to the three-variate normal distribution \( N_3(\mu, I_3) \), where \( \mu := \mu(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 \) and \( I_3 \) is the three-dimensional unit matrix.\(^1\) By applying Theorem 2.1, a confidence procedure (multiple decision procedure) can be constructed from Likelihood Ratio Tests (LRTs) for the hypotheses

\[
H_1 : \mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \quad H_2 : \mu_1 = 0, \mu_2 = 0, \mu_3 > 0, \quad H_3 : \mu_1 = 0, \mu_2 = 0, \mu_3 < 0,
\]

\[
H_4 : \mu_1 > 0, \mu_2 > 0, \mu_3 = 0, \quad H_5 : \mu_1 = 0, \mu_2 < 0, \mu_3 = 0, \quad H_6 : \mu_1 > 0, \mu_2 = 0, \mu_3 = 0,
\]

\[
H_7 : \mu_1 < 0, \mu_2 = 0, \mu_3 > 0, \quad H_8 : \mu_1 = 0, \mu_2 > 0, \mu_3 < 0, \quad H_9 : \mu_1 = 0, \mu_2 < 0, \mu_3 < 0,
\]

\[
H_{10} : \mu_1 = 0, \mu_2 < 0, \mu_3 > 0, \quad H_{11} : \mu_1 = 0, \mu_2 < 0, \mu_3 < 0, \quad H_{12} : \mu_1 > 0, \mu_2 = 0, \mu_3 > 0,
\]

\[
H_{13} : \mu_1 > 0, \mu_2 = 0, \mu_3 < 0, \quad H_{14} : \mu_1 < 0, \mu_2 = 0, \mu_3 > 0, \quad H_{15} : \mu_1 < 0, \mu_2 = 0, \mu_3 < 0,
\]

\[
H_{16} : \mu_1 > 0, \mu_2 > 0, \mu_3 = 0, \quad H_{17} : \mu_1 > 0, \mu_2 < 0, \mu_3 = 0, \quad H_{18} : \mu_1 < 0, \mu_2 > 0, \mu_3 = 0,
\]

\[
H_{19} : \mu_1 < 0, \mu_2 < 0, \mu_3 > 0, \quad H_{20} : \mu_1 > 0, \mu_2 > 0, \mu_3 < 0, \quad H_{21} : \mu_1 > 0, \mu_2 < 0, \mu_3 < 0,
\]

\[
H_{22} : \mu_1 > 0, \mu_2 < 0, \mu_3 > 0, \quad H_{23} : \mu_1 > 0, \mu_2 < 0, \mu_3 < 0, \quad H_{24} : \mu_1 < 0, \mu_2 > 0, \mu_3 > 0,
\]

\[
H_{25} : \mu_1 < 0, \mu_2 > 0, \mu_3 < 0, \quad H_{26} : \mu_1 < 0, \mu_2 < 0, \mu_3 > 0, \quad H_{27} : \mu_1 < 0, \mu_2 < 0, \mu_3 < 0.
\]

By analogy with the two-variate normal case, these 27 hypotheses which divide the parameter space \( \Theta = \mathbb{R}^3 \) may be grouped into four groups: \( \{ H_1 \} \),

---

\(^1\) Generally, the problem is to judge whether \( \mu_i \leq \mu_i^{(0)} \) for \( i = 1, 2, 3 \), but it can be assumed without loss of generality that \( \mu_1^{(0)} = \mu_2^{(0)} = \mu_3^{(0)} = 0 \). Also, when the covariance matrix is known, the unit matrix \( I_3 \) can be taken as the covariance matrix, again without loss of generality.
{H_2, \ldots, H_7}, \{H_8, \ldots, H_{19}\}, \text{ and } \{H_{20}, \ldots, H_{27}\}. \text{ Let } S \text{ be a family of subsets of } \Theta = \mathbb{R}^3, \text{ and suppose that } S \text{ contains all unions of the primary elements corresponding to these 27 hypotheses. Thus, } S \text{ is unrestricted. It is sufficient to construct the acceptance regions for only } H_1, H_2, H_8 \text{ and } H_{20}. \text{ The acceptance region of the LRT for } H_1 : \mu_1 = \mu_2 = \mu_3 = 0 \text{ with level } \alpha \text{ is}

\begin{equation}
A_1 := \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 + (\bar{x}_3)^2 \leq c_1^2/n\},
\end{equation}

where \(x_i = t(x_{i1}, x_{i2}, x_{i3}) \ (i = 1, \ldots, n)\), \(\bar{x} = t(\bar{x}_1, \bar{x}_2, \bar{x}_3) := (1/n) \sum_{i=1}^n x_i\), \text{ and } c_1 = c_1(\alpha) \text{ is the upper } 100\alpha \text{ percentage point of the chi-square distribution with three degrees of freedom. The radius } c_1 \text{ of the acceptance region } A_1 \text{ satisfies}

\begin{equation}
1 - \alpha = P_0\{X \in A_1\} = P_0^X\{A_1\} = \int\int\int_{x^2+y^2+z^2 \leq c_1^2} d\Phi(x) d\Phi(y) d\Phi(z)
= \int_0^{c_1} r^2 \phi(r) dr \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = 2\Phi(c_1) - 2c_1 \phi(c_1) - 1,
\end{equation}

where \(\phi \) and \(\Phi \) are the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of the standard normal distribution \(N(0, 1)\) respectively. When \(\alpha = 0.05\), \(c_1 \approx 2.795\). The acceptance region of the LRT for \(H_2 : \mu_1 = \mu_2 = 0, \mu_3 > 0\) is

\begin{equation}
A_2 := \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 \leq c_2^2/n, \bar{x}_3 > 0\}
\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 + (\bar{x}_3)^2 \leq c_2^2/n, \bar{x}_3 \leq 0\},
\end{equation}

where \(c_2 = c_2(\alpha)\) is the unique positive constant satisfying

\begin{equation}
\inf_{\mu_3 > 0} P_{0,0,\mu_3}^X\{A_2\} = 1 - \alpha.
\end{equation}

Since the probability \(P_{0,0,\mu_3}^X\{A_2\}\) is strictly monotone increasing in \(\mu_3 \ (> 0)\), the infimum in the left side of (3.5), that is, the least favorable configuration is given as \(\mu_3 \to 0^+\). Further, the order of the limit and integration is exchangeable in the calculation of the limit \(\lim_{\mu_3 \to 0^+} P_{0,0,\mu_3}^X\{A_2\}\) (see Maihara (2005) for details). Thus, it follows that

\begin{equation}
\inf_{\mu_3 > 0} P_{0,0,\mu_3}^X\{A_2\} = \lim_{\mu_3 \to 0^+} P_{0,0,\mu_3}^X\{A_2\} = \Phi(c_2) - \left(c_2 + \sqrt{\pi/2}\phi(c_2)\right).
\end{equation}

Since \(\Phi(c) - (c + \sqrt{\pi/2})\phi(c)\) is strictly monotone increasing in \(c \ (> 0)\), from (3.5) and (3.6), it is sufficient to obtain the unique positive constant \(c_2 = c_2(\alpha)\) satisfying

\begin{equation}
\Phi(c_2) - (c_2 + \sqrt{\pi/2})\phi(c_2) = 1 - \alpha.
\end{equation}
When $\alpha = 0.05$, $c_2 \approx 2.654$. The acceptance region of the LRT for $H_8 : \mu_1 = 0$, $\mu_2 > 0$, $\mu_3 > 0$ is

\[(3.8) \quad A_8 := \{(x_1, \ldots, x_n) \mid -c_8 \leq \sqrt{n} \bar{x}_1 \leq c_8, \bar{x}_2 > 0, \bar{x}_3 > 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_3)^2 \leq c_8^2/n, \bar{x}_2 > 0, \bar{x}_3 \leq 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 \leq c_8^2/n, \bar{x}_2 \leq 0, \bar{x}_3 > 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 + (\bar{x}_3)^2 \leq c_8^2/n, \bar{x}_2 \leq 0, \bar{x}_3 \leq 0\}, \]

where $c_8 = c_8(\alpha)$ is the unique positive constant satisfying

\[(3.9) \quad \inf_{\mu_2 > 0, \mu_3 > 0} P_{\alpha, \mu_2, \mu_3}^X \{A_8\} = 1 - \alpha. \]

Since the infimum of $P_{\alpha, \mu_2, \mu_3}^X \{A_8\}$ is attained as $\mu_2 \to 0+$, $\mu_3 \to 0+$ and the order of the limit and integration is exchangeable as above, it follows that

\[(3.10) \quad \inf_{\mu_2 > 0, \mu_3 > 0} P_{\alpha, \mu_2, \mu_3}^X \{A_8\} = \lim_{\mu_2 \to 0+, \mu_3 \to 0+} P_{\alpha, \mu_2, \mu_3}^X \{A_8\} = P_0^X \{A_8\} \]

\[= \frac{1}{4} \{2\Phi(c_8) - 1\} + \frac{1}{2} \int \int x^2 + y^2 \leq c_8^2 \Phi(x) d\Phi(y) \]

\[+ \frac{1}{4} \int \int \int x^2 + y^2 + z^2 \leq c_8^2 \Phi(x) d\Phi(y) d\Phi(z) \]

\[= \Phi(c_8) - (1/2) (c_8 + 2\sqrt{\pi/2}) \phi(c_8). \]

Since $\Phi(c) - (1/2) (c + 2\sqrt{\pi/2}) \phi(c)$ is strictly monotone increasing in $c (> 0)$, from (3.9) and (3.10), it is sufficient to obtain the unique positive constant $c_8 = c_8(\alpha)$ satisfying

\[(3.11) \quad \Phi(c_8) - (1/2) (c_8 + 2\sqrt{\pi/2}) \phi(c_8) = 1 - \alpha. \]

When $\alpha = 0.05$, $c_8 \approx 2.501$. The acceptance region of the LRT for $H_{20} : \mu \in R_+^3$ is

\[(3.12) \quad A_{20} := \{(x_1, \ldots, x_n) \mid \bar{x} \in R_+^3\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid x_1 > 0, x_2 > 0, -c_{20} \leq \sqrt{n} \bar{x}_3 \leq 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid x_1 > 0, -c_{20} \leq \sqrt{n} \bar{x}_2 \leq 0, \bar{x}_3 > 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid -c_{20} \leq \sqrt{n} \bar{x}_1 \leq 0, \bar{x}_2 > 0, \bar{x}_3 > 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_2)^2 + (\bar{x}_3)^2 \leq c_{20}^2/n, \bar{x}_1 > 0, \bar{x}_2 \leq 0, \bar{x}_3 \leq 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_3)^2 \leq c_{20}^2/n, \bar{x}_2 \leq 0, \bar{x}_3 \leq 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 \leq c_{20}^2/n, \bar{x}_1 \leq 0, \bar{x}_2 \leq 0, \bar{x}_3 > 0\} \]

\[\oplus \{(x_1, \ldots, x_n) \mid (\bar{x}_1)^2 + (\bar{x}_2)^2 + (\bar{x}_3)^2 \leq c_{20}^2/n, \bar{x}_1 \leq 0, \bar{x}_2 \leq 0, \bar{x}_3 \leq 0\}. \]
where \( c_{20} = c_{20}(\alpha) \) is the unique positive constant satisfying

\[
\inf_{\mu \in \mathbb{R}^3_+} P^X_{\mu} \{ A_{20} \} = 1 - \alpha.
\]

Since the infimum of \( P^X_{\mu} \{ A_{20} \} \) is attained as \( \mu_1 \to 0^+, \mu_2 \to 0^+, \mu_3 \to 0^+ \) and the order between the limit and integration is exchangeable as above, it follows that

\[
\inf_{\mu \in \mathbb{R}^3_+} P^X_{\mu} \{ A_{20} \} = \lim_{\mu \to 0^+} P^X_{\mu} \{ A_{20} \} = P^X_0 \{ A_{20} \} = \Phi(c_{20}) - \frac{1}{4} (c_{20} + 3\sqrt{\pi/2}) \phi(c_{20}).
\]

Since \( \Phi(c) - \frac{1}{4} (c + 3\sqrt{\pi/2}) \phi(c) \) is strictly monotone increasing in \( c (> 0) \), it is sufficient to obtain the unique positive constant \( c_{20} = c_{20}(\alpha) \) satisfying

\[
(3.13) \quad \Phi(c_{20}) - \frac{1}{4} (c_{20} + 3\sqrt{\pi/2}) \phi(c_{20}) = 1 - \alpha.
\]

When \( \alpha = 0.05 \), \( c_{20} \approx 2.331 \). By rotating these three acceptance regions \( A_2, A_8, A_{20} \) w.r.t. the origin. Three-dimensional Euclidean space \( \mathbb{R}^3 \), the acceptance regions \( A_3, \ldots, A_7, A_9, \ldots, A_{19}, A_{21}, \ldots, A_{27} \) of the LRTs for the other 23 hypotheses \( H_3, \ldots, H_7, H_9, \ldots, H_{19}, H_{21}, \ldots, H_{27} \) can be obtained respectively (see Figs. 2 and 4). A multiple decision procedure (confidence procedure) \( \varphi \) can be constructed from these 27 acceptance regions by applying Theorem 2.1. These 27 acceptance regions divide the space \( \mathbb{R}^3 \) using four spheres, nine circular cylinders, and twelve planes. The decisions of the multiple decision procedure \( \varphi \) constructed according to Theorem 2.1 depend on which of these parts the sample mean vector \( \bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \) lies. Thus the multiple decision procedure \( \varphi \) can be identified with the solid figure in Fig. 4. Unlike the univariate and two-variate cases, it is difficult to know how many parts the space \( \mathbb{R}^3 \) is divided into by these 27 acceptance regions, that is, how many decisions \( \varphi \) may derive. In other words, it is difficult to characterize the range of \( \varphi \), i.e.

\[
\varphi(\mathcal{X}^n) := \{ \Omega \in \mathcal{S} \mid \exists x \in \mathcal{X}^n \text{ s.t. } \Omega = \varphi(x) \} = \{ \varphi(x) \mid x \in \mathcal{X}^n \}.
\]

By constructing an evenly spaced grid of points in the space \( \mathbb{R}^3 \) and computing the decisions which \( \varphi \) produces for each point, we find that the number

![Figure 2. The acceptance regions \( A_2, A_8, A_{20} \) of the LRTs for the hypotheses \( H_2, H_8, H_{20} \).](image)
of decisions derived by \( \varphi \) will be 2110. That is, the computation shows that 
\[ \#\varphi(X^n) = 2110. \]
For example, when \( \sqrt{n}\bar{x}_i > c_8 \) (\( i = 1, 2, 3 \)), the only accepted hypothesis is 
\( H_{20} : \mu_1 > 0, \mu_2 > 0, \mu_3 > 0, \) and so the multiple decision \( \varphi \) makes the most definite and most desirable decision that all components of the mean vector \( \mu \) are positive. When 
\[ n^{1/2} \bar{x}_i = n\{\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2\} \leq c_{20}^2, \]
all 27 hypotheses are accepted simultaneously, and so \( \varphi \) derives the vaguest decision "\( \mu \in \mathbb{R}^3 \)" which means that deciding the signs of components of the mean vector \( \mu \) has been suspended and improvement of the subject (e.g., medicine, product, and so on) of the experiment is required. Now, since the acceptance regions \( A_1, \ldots, A_{19} \) for the nineteen hypotheses \( H_1, \ldots, H_{19} \) are covered by the unions of other acceptance regions, these nineteen are not accepted individually. Hence the multiple decision procedure constructed from 27 acceptance regions does not derive the decisions corresponding to these nineteen hypotheses. Only the eight hypotheses \( H_{20}, \ldots, H_{27} \) may be accepted individually. These eight hypotheses correspond to the most definite and most desirable decisions such as "\( \mu_1 > 0, \mu_2 > 0, \mu_3 > 0 \)".

Here, via a simulation, it is possible to see how some of the 27 hypotheses are accepted simultaneously and which decisions are derived by \( \varphi \). Using point symmetry w.r.t. the origin \( \mu = 0 \), the parameters are chosen with \( 0 \leq \mu_3 \leq \mu_2 \leq \mu_1 \) (see Table 1). The method of simulation is as follows:

i) Generating a random number \( \bar{x} \) from \( N_3(\mu, (1/(rn))I_3) \), where \( n \) and \( r \) denote the size of sample and the number of repetitions, respectively. The author has used \( n = 20 \) and \( r = 10000 \) for this experiment.

ii) Using the formulas for the 27 acceptance regions \( A_1, \ldots, A_{27} \) based on (3.2), (3.4), (3.8), (3.12), and observing the value of
\[ \delta := (\delta_1, \ldots, \delta_{27}), \]
where
\[ \delta_i = \delta_i(\bar{x}) := \begin{cases} 1 & \text{if } \bar{x} \text{ falls in } A_i, \\ 0 & \text{otherwise} \end{cases} \]
\( (i = 1, \ldots, 27) \).

iii) Calculating the decision derived from the multiple decision procedure
\[ \varphi(x) = \bigcup_{1 \leq i \leq 27} \Omega_i \] 
when \( \delta_i(x) = 1 \).

As can be observed in Table 1, the decisions are vague and expressed by unions of several subsets when the parameter \( \mu \) is near the origin. They become more definite as \( \mu \) is moved away from the origin.

3.1. Simplification of multiple decision procedures I by restricting the decision space \( S \)
Let \( S_0 \) be a family of subsets of the parameter space \( \Theta = \mathbb{R}^3 \), and suppose that \( S_0 \) contains all the unions of primary elements corresponding to these 27 hypotheses in (3.1), that is, \( S_0 \) is unrestricted. According to Theorem 2.1, let \( \varphi_0 \) be the multiple decision procedure corresponding to the unrestricted decision.

---

2 The tables in this paper are simplified. The detailed ones are given in Maihara (2005).
Table 1. Simulation results on the behavior of multiple decision procedures based on the usual LRTs \((n = 20, \alpha = 0.05)\).

<table>
<thead>
<tr>
<th>(\mu_1)</th>
<th>(\mu_2)</th>
<th>(\mu_3)</th>
<th>accepted hypotheses</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>0.1</td>
<td>1, \ldots, 27</td>
<td>(\mu \in \mathbb{R}^3)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td>1, \ldots, 26</td>
<td>(\mu_1 \geq 0 \lor \mu_2 \geq 0 \lor \mu_3 \geq 0)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0</td>
<td>1, \ldots, 25</td>
<td>(\mu_1 \geq 0 \lor \mu_2 \geq 0 \lor \mu_3 = 0)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.1</td>
<td>1, \ldots, 9, 12, 13, 16, 17, 18, 20, \ldots, 25</td>
<td>(\mu_1 &gt; 0 \lor \mu_2 &gt; 0 \lor \mu_1 = \mu_2 = 0) \lor \mu_2 = \mu_3 = 0 \lor \mu_3 = \mu_1 = 0)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.3</td>
<td>2, 4, 6, 8, 9, 12, 13, 16, 17, 18, 20, \ldots, 25</td>
<td>(\mu_1 &gt; 0 \lor \mu_2 &gt; 0 \lor (\mu_3 &gt; 0 \land \mu_1 = \mu_2 = 0))</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0.1</td>
<td>1, \ldots, 27</td>
<td>(\mu \in \mathbb{R}^3)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>0.1</td>
<td>1, \ldots, 25</td>
<td>(\mu_1 \geq 0 \lor \mu_2 \geq 0 \lor \mu_3 = 0)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0.1</td>
<td>4, 6, 8, 9, 12, 13, 16, 17, 18, 20, \ldots, 25</td>
<td>(\mu_1 &gt; 0 \lor \mu_2 &gt; 0)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.1</td>
<td>4, 6, 8, 9, 12, 13, 16, 17, 18, 20, \ldots, 24</td>
<td>(\mu_1 &gt; 0 \lor (\mu_2 &gt; 0 \land \mu_3 \geq 0) \lor (\mu_1 = 0 \land \mu_2 &gt; 0))</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>8, 12, 16, 20, 21, 22, 24</td>
<td>(\mu_1 &gt; 0 \lor \mu_2 &gt; 0 \lor \mu_3 &gt; 0 \lor (\mu_3 &gt; 0 \land \mu_1 &gt; 0))</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0</td>
<td>0</td>
<td>1, 6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>(\mu_1 &gt; 0 \lor \mu = 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2</td>
<td>0</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>(\mu_1 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
<td>6, 12, 16, 20, 21, 22</td>
<td>(\mu_1 &gt; 0 \land (\mu_2 &gt; 0 \lor \mu_3 &gt; 0 \lor \mu_2 = \mu_3 = 0))</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>0.2</td>
<td>6, 12, 13, 16, 17, 20, 21, 22</td>
<td>(\mu_1 &gt; 0 \land (\mu_2 \geq 0 \lor \mu_3 \geq 0))</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>0.4</td>
<td>12, 16, 20, 21, 22</td>
<td>(\mu_1 &gt; 0 \land (\mu_2 &gt; 0 \lor \mu_3 &gt; 0))</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0</td>
<td>16, 20, 21</td>
<td>(\mu_1 &gt; 0 \land \mu_2 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>20</td>
<td>(\mu_1 &gt; 0 \land \mu_2 &gt; 0 \land \mu_3 &gt; 0)</td>
</tr>
</tbody>
</table>

Table 2. Simulation results on the behavior of the multiple decision procedures in the case where \(S\) is restricted \((n = 20, \alpha = 0.05)\).

<table>
<thead>
<tr>
<th>(\mu_1)</th>
<th>(\mu_2)</th>
<th>(\mu_3)</th>
<th>accepted hypotheses</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>1, \ldots, 27</td>
<td>(\mu \in \mathbb{R}^3)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>1, \ldots, 27</td>
<td>(\mu \in \mathbb{R}^3)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0</td>
<td>0</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>(\mu_1 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1</td>
<td>0</td>
<td>1, 6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>(\mu_1 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1</td>
<td>0.1</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>(\mu_1 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>0.5</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>(\mu_1 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.5</td>
<td>16, 20, 21</td>
<td>(\mu_1 &gt; 0 \land \mu_2 &gt; 0)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>20</td>
<td>(\mu_1 &gt; 0 \land \mu_2 &gt; 0 \land \mu_3 &gt; 0)</td>
</tr>
</tbody>
</table>
space $S_0$. As in the two-variate case in Takeuchi (1973) and Maihara (2005), a
decision space $S$ can be made restricted by removing the complicated subsets
such as

$$\{(\mu_1, \mu_2, \mu_3) \mid \mu_1 > 0 \vee \mu_2 > 0 \vee \mu_1 = \mu_2 = 0 \vee \mu_2 = \mu_3 = 0 \vee \mu_3 = \mu_1 = 0\}$$

from $S_0$, and a simplified multiple decision procedure $\varphi$ corresponding to $S$
can be constructed according to Theorem 2.2. Suppose that the elements of $S$
are the subsets corresponding to the following twenty hypotheses and those which
are (point) symmetric w.r.t. the origin $0$:

$$H'_1 : \mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \quad H'_2 : \mu_1 = 0, \mu_2 = 0, \mu_3 > 0,$$
$$H'_3 : \mu_1 = 0, \mu_2 = 0, \mu_3 \geq 0, \quad H'_4 : \mu_1 = 0, \mu_2 = 0, \mu_3 \in R,$$
$$H'_5 : \mu_1 = 0, \mu_2 > 0, \mu_3 > 0, \quad H'_6 : \mu_1 = 0, \mu_2 > 0, \mu_3 \geq 0,$$
$$H'_7 : \mu_1 = 0, \mu_2 > 0, \mu_3 \in R, \quad H'_8 : \mu_1 = 0, \mu_2 \geq 0, \mu_3 \geq 0,$$
$$H'_9 : \mu_1 = 0, \mu_2 \geq 0, \mu_3 \in R, \quad H'_{10} : \mu_1 = 0, \mu_2 \in R, \mu_3 \in R,$$
$$H'_{11} : \mu_1 > 0, \mu_2 > 0, \mu_3 > 0, \quad H'_{12} : \mu_1 > 0, \mu_2 > 0, \mu_3 \geq 0,$$
$$H'_{13} : \mu_1 > 0, \mu_2 > 0, \mu_3 \in R, \quad H'_{14} : \mu_1 > 0, \mu_2 \geq 0, \mu_3 \geq 0,$$
$$H'_{15} : \mu_1 > 0, \mu_2 \geq 0, \mu_3 \in R, \quad H'_{16} : \mu_1 > 0, \mu_2 \in R, \mu_3 \in R,$$
$$H'_{17} : \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \quad H'_{18} : \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \in R,$$
$$H'_{19} : \mu_1 \geq 0, \mu_2 \in R, \mu_3 \in R, \quad H'_{20} : \mu_1 \in R, \mu_2 \in R, \mu_3 \in R.$$ (3.14)

The primary elements of $S$ are the same as in (3.1), that is, 27 subsets
corresponding to the four hypotheses $H'_1, H'_2, H'_3, H'_{11}$ in (3.14) and the 23 which
are symmetric w.r.t. the origin. Then, $\#S = 6^3 = 216$ and, needless to say, it
is less than $\#S_0 = 2^{27} - 1$. The decisions corresponding to the elements of $S$
make judgments on the sign of at least one component of the three-dimensional
mean vector $\mu$, or make no judgment about any of the signs. For a sample
$x \in \mathcal{X}^n$, the image $\varphi(x)$ of $\varphi$ is given by the smallest subset $\Omega$ in $S$, which
satisfies $\varphi_0(x) \subseteq \varOmega = \varphi(x)$. Then, $\#\varphi(\mathcal{X}^n) = 5^3 = 125 < 2110$. For example,
when $x$ lies in the intersection of the acceptance regions $A_6, A_{12}, A_{16}, A_{20}, A_{21},$
$A_{22}$ of hypotheses $H_6, H_{12}, H_{16}, H_{20}, H_{21}, H_{22}, \varphi_0$ corresponding to $S_0$
makes the decision “$\mu_1 > 0 \land (\mu_2 > 0 \lor \mu_3 > 0 \lor \mu_2 = \mu_3 = 0)$” (see Table 1), while $\varphi$
corresponding to $S$ derives the decision $H'_{16} : “\mu_1 > 0, \mu_2 \in R, \mu_3 \in R”$ which
makes a judgment only about the sign of the first component $\mu_1$ (see Table 2).
When the decision space $S$ is restricted as above, the vague and complicated
decisions expressed by using the word “or” and the symbol “\lor” are not derived.

### 3.2. Simplification of multiple decision procedures II by combining
boundaries of acceptance regions

Although the confidence system $\{\alpha_i \mid i = 1, \ldots, 27\}$ in the discussion above
is considered as $\{\alpha_i \mid i = 1, \ldots, 27\} = \{\alpha\} := \{0.05\}$, it seems to be very difficult
to grasp the structure of multiple decision procedures visually and theoretically.
To combine the boundaries of the acceptance regions $A_1, \ldots, A_{27}$ and to unify
the radii of their spherical parts, we modify the values for \( \{\alpha_i \mid i = 1, \ldots, 27\} \). Let

\[
\begin{align*}
\alpha_1(c) &:= 1 - \{2\Phi(c) - 2c\phi(c) - 1\}, \\
\alpha_2(c) &:= 1 - \{\Phi(c) - (c + \sqrt{\pi/2})\phi(c)\}, \\
\alpha_8(c) &:= 1 - \{\Phi(c) - (1/2)(c + 2\sqrt{\pi/2})\phi(c)\}, \\
\alpha_{20}(c) &:= 1 - \{\Phi(c) - (1/4)(c + 3\sqrt{\pi/2})\phi(c)\},
\end{align*}
\]

according to equations (3.3), (3.7), (3.11), and (3.13). Then it follows that all four functions are strictly monotone decreasing and \( \alpha_1(c) > \alpha_2(c) > \alpha_8(c) > \alpha_{20}(c) \) \( (c > 0) \), which means that the same order relation also holds for their inverse functions as is observed in Fig. 3. Hence, \( A_1 \) has the biggest radius \( c_1 \) in four kinds of acceptance regions for the same level \( \alpha \). Thus, from the conservative viewpoint, the significance levels \( \alpha_2, \ldots, \alpha_7, \alpha_8, \ldots, \alpha_{19}, \alpha_{20}, \ldots, \alpha_{27} \) for testing hypotheses \( H_2, \ldots, H_7, H_8, \ldots, H_{19}, H_{20}, \ldots, H_{27} \) are reduced to \( \alpha_2 = \cdots = \alpha_7 = \alpha_2(c_1) \approx 0.03505, \alpha_8 = \cdots = \alpha_{19} = \alpha_8(c_1) \approx 0.02384, \alpha_{20} = \cdots = \alpha_{27} = \alpha_{20}(c_1) \approx 0.01573 \) respectively, and the radii \( c_2, \ldots, c_7, c_8, \ldots, c_{19}, c_{20}, \ldots, c_{27} \) of

Figure 3. The significance levels \( \alpha_1, \alpha_2, \alpha_8, \alpha_{20} \) and the radii \( c_1, c_2, c_8, c_{20} \) of acceptance regions \( A_1, A_2, A_8, A_{20} \) for the hypotheses \( H_1, H_2, H_8, H_{20} \).

Figure 4. The acceptance regions of LRTs with significance level \( \alpha = 0.05 \) for the hypotheses \( H_1, \ldots, H_{27} \) (left figure), those in the case where their boundaries are combined (right figure: A sphere is hidden by the six planes).
Table 3. Simulation results on the behavior of the multiple decision procedures in the case where the boundaries of $A_1, \ldots, A_{27}$ are put together ($n = 20, \alpha = 0.05$).

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>accepted hypotheses</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.2</td>
<td>1, \ldots, 27</td>
<td>$\mu \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.3</td>
<td>2, 4, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, \ldots, 26</td>
<td>$\mu_1 &gt; 0 \lor \mu_2 &gt; 0 \lor \mu_3 &gt; 0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3</td>
<td>0.2</td>
<td>1, \ldots, 27</td>
<td>$\mu \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0</td>
<td>4, 6, 8, 9, 12, 13, 16, 17, 18, 20, \ldots, 25</td>
<td>$\mu_1 &gt; 0 \lor \mu_2 &gt; 0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0.4</td>
<td>6, 8, 12, 13, 16, 17, 20, \ldots, 24</td>
<td>$\mu_1 &gt; 0 \lor (\mu_2 &gt; 0 \land \mu_3 &gt; 0)$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.4</td>
<td>8, 12, 16, 20, 21, 22, 24</td>
<td>$(\mu_1 &gt; 0 \land \mu_2 &gt; 0) \lor (\mu_2 &gt; 0 \land \mu_3 &gt; 0) \lor (\mu_3 &gt; 0 \land \mu_1 &gt; 0)$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5</td>
<td>0.3</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>$\mu_1 &gt; 0$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5</td>
<td>0.4</td>
<td>12, 16, 20, 21, 22</td>
<td>$\mu_1 &gt; 0 \land (\mu_2 &gt; 0 \lor \mu_3 &gt; 0)$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6</td>
<td>0</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>$\mu_1 &gt; 0$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0</td>
<td>16, 20, 21</td>
<td>$\mu_1 &gt; 0 \land \mu_2 &gt; 0$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>20</td>
<td>$\mu_1 &gt; 0 \land \mu_2 &gt; 0 \land \mu_3 &gt; 0$</td>
</tr>
</tbody>
</table>

The spherical parts of these acceptance regions are expanded to that of $c_1 \approx 2.795$ when $\alpha_1 = 0.05$ (see Fig. 3). Then, $\# \varphi (X''') = 103$ and the space $\mathbb{R}^3$ is divided by a sphere, three circular cylinders, and six planes tangent to the sphere (see Fig. 4). The various 103 decisions can be enumerated paying attention to only the eighth quadrant, by utilizing point symmetry about the origin. By combining the boundaries of acceptance regions, the structure of the multiple decision procedure constructed from the acceptance regions is simplified and decisions derived by the procedure also become simpler (see Table 3).

3.3. Simplification of multiple decision procedures III by changing primary elements

Other subsets of the parameter space $\Theta = \mathbb{R}^3$, which provide a covering of $\Theta$, can also be considered as primary elements of the decision space $S$. Suppose that the primary elements of $S$ are eight subsets corresponding to the hypotheses

\[
H''_{20} : \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \quad H''_{21} : \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \leq 0, \\
H''_{22} : \mu_1 \geq 0, \mu_2 \leq 0, \mu_3 \geq 0, \quad H''_{23} : \mu_1 \geq 0, \mu_2 \leq 0, \mu_3 \leq 0, \\
H''_{24} : \mu_1 \leq 0, \mu_2 \geq 0, \mu_3 \geq 0, \quad H''_{25} : \mu_1 \leq 0, \mu_2 \geq 0, \mu_3 \leq 0, \\
H''_{26} : \mu_1 \leq 0, \mu_2 \leq 0, \mu_3 \geq 0, \quad H''_{27} : \mu_1 \leq 0, \mu_2 \leq 0, \mu_3 \leq 0,
\]

respectively. Although these eight primary elements cover $\Theta = \mathbb{R}^3$, they do not give the partition of $\Theta$, unlike the 27 primary elements corresponding to the hypotheses in (3.1). The acceptance region of the LRT for the hypothesis $H''_{20} : \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$ with level $\alpha$ is the same as that for $A_{20}$ in (3.12) for $H_{20} : \mu_1 > 0, \mu_2 > 0, \mu_3 > 0$ in (3.1). The acceptance regions of the LRTs for
Table 4. Simulation results on the behavior of multiple decision procedures in the case where the primary elements of $S$ are $H_{20}^{''}, \ldots, H_{27}^{''}$ ($n = 20, \alpha = 0.05$).

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>accepted hypotheses</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>0.2</td>
<td>$H_{20}^{''}, \ldots, H_{27}^{''}$</td>
<td>$\mu \in R^3$</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>$H_{20}^{''}, \ldots, H_{26}^{''}$</td>
<td>$\mu_1 \geq 0 \lor \mu_2 \geq 0 \lor \mu_3 \geq 0$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>0.1</td>
<td>$H_{20}^{''}, \ldots, H_{27}^{''}$</td>
<td>$\mu \in R^3$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0</td>
<td>$H_{20}^{''}, \ldots, H_{25}^{''}$</td>
<td>$\mu_1 \geq 0 \lor \mu_2 \geq 0$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>$H_{20}^{''}, H_{21}^{''}, H_{22}^{''}, H_{24}^{''}$</td>
<td>$(\mu_1 \geq 0 \land \mu_2 \geq 0) \lor (\mu_2 \geq 0 \land \mu_3 \geq 0) \lor (\mu_3 \geq 0 \land \mu_1 \geq 0)$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0</td>
<td>$H_{20}^{''}, \ldots, H_{27}^{''}$</td>
<td>$\mu \in R^3$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>0</td>
<td>$H_{20}^{''}, \ldots, H_{25}^{''}$</td>
<td>$\mu_1 \geq 0 \lor \mu_2 \geq 0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>0.2</td>
<td>$H_{20}^{''}, \ldots, H_{24}^{''}$</td>
<td>$\mu_1 \geq 0 \lor (\mu_2 \geq 0 \land \mu_3 \geq 0)$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0.4</td>
<td>$H_{20}^{''}, H_{21}^{''}, H_{22}^{''}, H_{24}^{''}$</td>
<td>$(\mu_1 \geq 0 \land \mu_2 \geq 0) \lor (\mu_2 \geq 0 \land \mu_3 \geq 0) \lor (\mu_3 \geq 0 \land \mu_1 \geq 0)$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.3</td>
<td>$H_{20}^{''}, \ldots, H_{23}^{''}$</td>
<td>$\mu_1 \geq 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
<td>$H_{20}^{''}, H_{21}^{''}, H_{22}^{''}$</td>
<td>$\mu_1 \geq 0 \land (\mu_2 \geq 0 \land \mu_3 \geq 0)$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0</td>
<td>$H_{20}^{''}, H_{21}^{''}$</td>
<td>$\mu_1 \geq 0 \land \mu_2 \geq 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>$H_{20}^{''}$</td>
<td>$\mu_1 \geq 0 \land \mu_2 \geq 0 \land \mu_3 \geq 0$</td>
</tr>
</tbody>
</table>

the hypotheses $H_{21}^{''}, \ldots, H_{27}^{''}$ are also the same as $A_{21}, \ldots, A_{27}$ for $H_{21}, \ldots, H_{27}$ in (3.1), respectively. Then, $\# \varphi(X^n) = 103$ as in Subsection 3.2. All the radii of the eight acceptance regions are equal to $c_20 \approx 2.331$ which is smaller than the radii which are equal to $c_1 \approx 2.795$ in the case of Subsection 3.1 for $\alpha = 0.05$. Although definite decisions expressed using strict inequality signs “<” and “>” can be made by the simplifications of Subsections 3.1 and 3.2, the present method results in somewhat vague decisions expressed in terms of inequality signs “≤” and “≥” such as “$\mu_j$ is equal to or not less than 0” (see Table 4).

3.4. Simplification of multiple decision procedures IV by using rectangular acceptance regions

Although so far in this discussion multiple decision procedures have been based on acceptance regions of LRTs w.r.t. the three-variate normal distribution $N_3(\mu, I_3)$, rectangular parallelepipeds which are products of acceptance regions of tests w.r.t. the marginal distributions $N(\mu_1, 1)$, $N(\mu_2, 1)$, and $N(\mu_3, 1)$ can be considered as acceptance regions on which multiple decision procedures are based as in the two-variate case. For example, as an acceptance region of testing hypotheses $H_1 : \mu_1 = \mu_2 = \mu_3 = 0$ with level $\alpha$, the cube

$$A_1^R := \{(x_1, \ldots, x_n) \mid |\sqrt{n}x_1| \leq c_1^R, |\sqrt{n}x_2| \leq c_1^R, |\sqrt{n}x_3| \leq c_1^R\}$$

may be taken, where $c_1^R$ is the upper 50{$1 - (1 - \alpha)^{1/3}$} percentage point of $N(0, 1)$ such that

$$(3.15) \quad 1 - \alpha = P_0((X_1, \ldots, X_n) \in A_1^R) = \{2\Phi(c_1^R) - 1\}^3.$$
When \( \alpha = 0.05 \), \( c_1^2 \approx 2.388 \). The cube \( A^R_1 \) is the product of the three acceptance regions for testing the hypotheses \( H_1^{(1)} : \mu_1 = 0 \), \( H_1^{(2)} : \mu_2 = 0 \), and \( H_1^{(3)} : \mu_3 = 0 \). Similarly, as acceptance regions for testing the hypotheses \( H_2 : \mu_1 = \mu_2 = 0 \), \( \mu_3 > 0 \), \( H_8 : \mu_1 = 0 \), \( \mu_2 > 0 \), \( \mu_3 > 0 \), \( H_{20} : \mu_1 > 0 \), \( \mu_2 > 0 \), \( \mu_3 > 0 \) with level \( \alpha \),

\[
A^R_2 := \{(x_1, \ldots, x_n) | |\sqrt{n}\bar{x}_1| \leq c_2^R, |\sqrt{n}\bar{x}_2| \leq c_2^R, |\sqrt{n}\bar{x}_3| \geq -c_2^R\},
\]

\[
A^R_8 := \{(x_1, \ldots, x_n) | |\sqrt{n}\bar{x}_1| \leq c_8^R, |\sqrt{n}\bar{x}_2| \geq -c_8^R, |\sqrt{n}\bar{x}_3| \geq -c_8^R\},
\]

\[
A^R_{20} := \{(x_1, \ldots, x_n) | \sqrt{n}\bar{x}_1 \geq -c_{20}^R, \sqrt{n}\bar{x}_2 \geq -c_{20}^R, \sqrt{n}\bar{x}_3 \geq -c_{20}^R\}
\]
can be considered respectively. Here \( c_2^R, c_8^R, c_{20}^R \) are the unique positive constants satisfying

\[
(3.16) \quad 1 - \alpha = P_0\{(X_1, \ldots, X_n) \in A^R_2\} = \{2\Phi(c_2^R) - 1\}^2\Phi(c_2^R),
\]

\[
(3.17) \quad 1 - \alpha = P_0\{(X_1, \ldots, X_n) \in A^R_8\} = \{2\Phi(c_8^R) - 1\}\Phi(c_8^R)^2,
\]

\[
(3.18) \quad 1 - \alpha = P_0\{(X_1, \ldots, X_n) \in A^R_{20}\} = \Phi(c_{20}^R)^3,
\]

respectively. That is, \( c_2^R, c_8^R, c_{20}^R \) are the upper points of \( N(0, 1) \) of percentages

\[
100(4 - q_2^{-1/3} - q_2^{1/3})/6, 100(5 - q_8^{-1/3} - q_8^{1/3})/6, 100(1 - (1 - \alpha)^{1/3})\]

respectively, where

\[
q_2 := 26 - 27\alpha + 3\sqrt{3}\sqrt{25 - 52\alpha + 27\alpha^2},
\]

\[
q_8 := 55 - 54\alpha + 6\sqrt{3}\sqrt{28 - 55\alpha + 27\alpha^2},
\]

which are derived from Cardano’s formula. When \( \alpha = 0.05 \), \( c_2^R \approx 2.320 \), \( c_8^R \approx 2.235 \) and \( c_{20}^R \approx 2.121 \). Then, the acceptance regions \( A^R_1, \ldots, A^R_{27} \) divide threedimensional Euclidean space \( \mathbb{R}^3 \) using a cube and eighteen planes (see Fig. 5). The computation shows that the number of divisions of \( \mathbb{R}^3 \), which is equal to the number of decisions derived from the multiple decision procedure, is 513 in a similar way to the case based on LRTs (see Table 5).

As for the previous (LRT) case, by changing the confidence system \( \{\alpha_i \mid i = 1, \ldots, 27\} \) so as to combine the boundaries of the 27 acceptance regions

\[
\begin{align*}
\end{align*}
\]

Figure 5. The frames of acceptance regions \( A^R_1, \ldots, A^R_{27} \).
Table 5. Simulation results for the behavior of multiple decision procedures in the case where rectangular acceptance regions are used (as before, $n = 20$, $\alpha = 0.05$).

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>accepted hypotheses</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>1, ..., 27</td>
<td>$\mu \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>1, ..., 23</td>
<td>$\mu_1 \geq 0 \lor \mu_2 = 0 \lor \mu_3 = 0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0</td>
<td>1, ..., 7, 12, 13, 16, 17, 20, ..., 23</td>
<td>$\mu_1 &gt; 0 \lor \mu_1 = \mu_2 = 0 \lor \mu_2 = \mu_3 = \mu_3 = 0$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>1, ..., 17, 19</td>
<td>$(\mu_1 &gt; 0 \land \mu_2 &gt; 0 \land \mu_3 &gt; 0)$ \lor $\mu_1 = 0 \lor \mu_2 = 0 \lor \mu_3 = 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
<td>6, 12, 13, 16, 17, 20, ..., 23</td>
<td>$\mu_1 &gt; 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>0</td>
<td>6, 16, 20, 21</td>
<td>$\mu_1 &gt; 0 \land (\mu_2 &gt; 0 \lor \mu_3 = 0)$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>0.1</td>
<td>6, 12, 13, 16, 17, 20, 21</td>
<td>$\mu_1 &gt; 0 \land (\mu_2 \geq 0 \lor \mu_3 = 0)$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
<td>16, 20, 21</td>
<td>$\mu_1 &gt; 0 \lor \mu_2 &gt; 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.5</td>
<td>20</td>
<td>$\mu_1 &gt; 0 \land \mu_2 &gt; 0 \land \mu_3 &gt; 0$</td>
</tr>
</tbody>
</table>

$\mathcal{A}^R_1, \ldots, \mathcal{A}^R_27$, the structure of multiple decision procedures (confidence procedures) constructed from these acceptance regions can be simplified. Since the boundary of $\mathcal{A}^R_1$ has the longest distance $c^R_1 \approx 2.388$ from the coordinate planes in the four types of acceptance regions, the significance levels $\alpha_2, \ldots, \alpha_7$, $\alpha_8, \ldots, \alpha_{19}$, $\alpha_{20}, \ldots, \alpha_{27}$ corresponding to hypotheses $H_2, \ldots, H_7$, $H_8, \ldots, H_{19}$, $H_{20}, \ldots, H_{27}$ are reduced to

$$
\alpha_2 = \cdots = \alpha_7 = 1 - \{2\Phi(c^R_1) - 1\}^2 \Phi(c^R_1) \approx 0.04181,
$$

$$
\alpha_8 = \cdots = \alpha_{19} = 1 - \{2\Phi(c^R_1) - 1\} \{\Phi(c^R_1)\}^2 \approx 0.03355,
$$

$$
\alpha_{20} = \cdots = \alpha_{27} = 1 - \{\Phi(c^R_1)\}^3 \approx 0.02521,
$$

respectively, according to equations (3.16), (3.17), and (3.18). Then, $\# \varphi(\mathcal{X}^n) = 27$ and $\mathbb{R}^3$ is divided by six planes (see Fig. 6 and Table 6). Hence basing multiple decision procedures on rectangular acceptance regions results in a considerable

Figure 6. The acceptance regions $\mathcal{A}^R_1, \ldots, \mathcal{A}^R_27$ (left figure), and its part of the first quadrant and decisions made in each part (right figure) after the confidence system has been changed.
MULTIPLE DECISION PROBLEM IN THREE-VARIATE NORMAL MEAN

Table 6. Simulation results for the behavior of multiple decision procedures in the case where the boundaries of $A_1^R, \ldots, A_{27}^R$ are combined ($n = 20, \alpha = 0.05$).

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>accepted hypotheses</th>
<th>decision</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1, \ldots, 27</td>
<td>$\mu \in \mathbb{R}^3$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>6, 12, 13, 16, 17, 20, \ldots, 23</td>
<td>$\mu_1 &gt; 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.5</td>
<td>16, 20, 21</td>
<td>$\mu_1 &gt; 0 \land \mu_2 &gt; 0$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>20</td>
<td>$\mu_1 &gt; 0 \land \mu_2 &gt; 0 \land \mu_3 &gt; 0$</td>
</tr>
</tbody>
</table>

simplification of structure as can be observed in Fig. 6.

For a general positive integer $k$, rectangular acceptance regions of testing hypotheses on the signs of the components of the mean vector $\mu$ of the $k$-variate normal distribution $N_k(\mu, I_k)$ are given by Takeuchi (1973).

3.5. Power functions of multiple decision procedures

In this section, we consider the power of multiple decision procedures (confidence procedures). The two-variate normal case is discussed by Takeuchi (1973). Since the power of multiple decision procedures constructed from testing procedures according to Theorem 2.1 can be evaluated in terms of the power of testing procedures against the hypotheses corresponding to the primary elements as in (2.14), it is sufficient to consider the power of multiple decision procedure against just the primary elements. Considering the special case of the power against the primary element

$\Omega_1 := \{(\mu_1, \mu_2, \mu_3) \mid \mu_1 = \mu_2 = \mu_3 = 0\} = \{0\},$

corresponding to the hypothesis $H_1$ in (3.14), we compare the multiple decision procedure based on LRTs with that based on acceptance regions of rectangular type. Since the squared norm of the standardized sample mean vector $n^t \bar{X} \bar{X} = (\sqrt{n} \bar{X}_1)^2 + (\sqrt{n} \bar{X}_2)^2 + (\sqrt{n} \bar{X}_3)^2$ is distributed according to the non-central chi-square distribution with three degrees of freedom and non-centrality parameter $n\|\mu\|^2$, it follows from (2.14) that the power of the multiple decision procedure based on LRTs against $\Omega_1$ is

\[
\beta_{\Omega_1}^L(\mu) := P_{\mu}(X \notin A_1) = 1 - P_{\mu}(t(n\bar{X})) \leq \chi^2_{3,\alpha} = Q_{\chi^2_{3,n\|\mu\|^2}}(\chi^2_{3,\alpha}),
\]

for $\mu \in \mathbb{R}^3 \setminus \Omega_1$, where $Q_{\chi^2_{3,n\|\mu\|^2}}(\cdot)$ is the upper probability of that distribution. Since the non-central chi-square distribution with positive non-centrality parameter is stochastically larger than the central one, it can be seen that

\[
\beta_{\Omega_1}^L(\mu) \geq Q_{\chi^2_{3,0}}(\chi^2_{3,\alpha}) = \alpha \quad \text{for } \mu \in \mathbb{R}^3 \setminus \Omega_1.
\]
On the other hand, the power of the multiple decision procedure based on rectangular acceptance regions against $\Omega_1$ is

$$
(3.20) \quad \beta_{\Omega_1}^R(\mu) := P_\mu\{X \not\in \mathcal{A}_1^R\} = 1 - P_\mu\{\sqrt{n}X_1 \leq c_1^R, |\sqrt{n}X_2| \leq c_1^R, |\sqrt{n}X_3| \leq c_1^R\} = 1 - \prod_{i=1}^{3}\{\Phi(\sqrt{n}\mu_i + c_1^R) - \Phi(\sqrt{n}\mu_i - c_1^R)\}
$$

for $\mu \in \mathbb{R}^3 \setminus \Omega_1$. Since the function $g(x) := \Phi(x + c) - \Phi(x - c)$ takes its maximum value $g(0) = 2\Phi(c) - 1$ when the constant $c$ is positive, it follows from (3.15) that

$$
\beta_{\Omega_1}^R(\mu) \geq 1 - \{2\Phi(c_1^R) - 1\}^3 = \alpha \quad \text{for} \quad \mu \in \mathbb{R}^3 \setminus \Omega_1.
$$

Using (3.19) and (3.20), we compared the power $\beta_{\Omega_1}^R$ with $\beta_{\Omega_1}^L$ when the mean vector $\mu$ is contained in the primary elements

$$
\begin{align*}
\Omega_2 &:= \{(\mu_1, \mu_2, \mu_3) \mid \mu_1 = \mu_2 = 0, \mu_3 > 0\}, \\
\Omega_8 &:= \{(\mu_1, \mu_2, \mu_3) \mid \mu_1 = 0, \mu_2 > 0, \mu_3 > 0\}, \\
\Omega_{20} &:= \{(\mu_1, \mu_2, \mu_3) \mid \mu_1 > 0, \mu_2 > 0, \mu_3 > 0\},
\end{align*}
$$

which correspond to the hypotheses $H_2$, $H_8$, $H_{20}$ in (3.1) respectively (see Fig. 7 and Table 7). From the symmetry of the 27 hypotheses, the powers against $\Omega_1$ can be calculated similarly for the other 23 primary elements.

As can be seen from Fig. 7 and Table 7, generally, the power $\beta_{\Omega_1}^L$ of multiple decision procedures based on LRTs is greater than the power $\beta_{\Omega_1}^R$ of those based on rectangular acceptance regions. Although the power $\beta_{\Omega_1}^R$ is a little greater than $\beta_{\Omega_1}^L$ when the mean vector $\mu$ is on the coordinate axis $\Omega_2 = \{\mu \mid \mu_1 = \mu_2 = 0, \mu_3 > 0\}$, $\beta_{\Omega_1}^L$ becomes greater than $\beta_{\Omega_1}^R$ as $\mu$ moves away from the coordinate axis. In particular, $\beta_{\Omega_1}^L$ is much greater than $\beta_{\Omega_1}^R$ when $\mu$ is on the diagonal line $\{\mu \mid \mu_1 = \mu_2 = \mu_3 > 0\}$. These results are due to the fact that the acceptance regions of the LRTs are narrower than those of the rectangular acceptance regions in the diagonal directions and conversely wider in the directions of the coordinate axes.

![Figure 7](image-url)

Figure 7. The powers $\beta_{\Omega_1}^L$ of the multiple decision procedure based on LRTs (solid line), $\beta_{\Omega_1}^R$ of that based on the rectangular acceptance regions (broken line), (a) when $\mu \in \Omega_2$, (b) when $\mu \in \{\mu \mid \mu_1 = 0, \mu_2 = \mu_3 > 0\}$ ($\subset \Omega_8$), (c) when $\mu \in \{\mu \mid \mu_1 = \mu_2 = \mu_3 > 0\}$ ($\subset \Omega_{20}$) ($n = 20$, $\alpha = 0.05$).
axes. For example, $c_1^R < c_1 < \sqrt{3} c_1^R$. Further, since most methods of simplifying the structure of multiple decision procedures (confidence procedures) are conservative, multiple decision procedures as a set are extended by simplification, so the power tends to decrease in general.

4. Concluding remarks

For the multiple decision problem of the signs of the components of the mean vector of a three-variate normal distribution with a known covariance matrix, we constructed multiple decision procedures from the testing procedures. Firstly, we reconsidered the formulation and the theorems described in Takeuchi (1973), and then by simulation examined the behavior of multiple decision procedures, finding that they are more conservative than the testing procedures.

One may consider several loss functions and risks within the framework of the multiple decision problem. Furthermore, problems of ordering and selecting the maximum from the means of several populations are closely related to applications in fields such as medical statistics (see e.g., Gupta and Panchapakesan (1979) and Hirotsu (2003)).
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