ON THE ESTIMATION OF A RESTRICTED LOCATION PARAMETER FOR SYMMETRIC DISTRIBUTIONS

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For estimating the median θ of a spherically symmetric univariate distribution under squared error loss, when θ is known to be restricted to an interval $[-m, m]$, m known, we derive sufficient conditions for estimators δ to dominate the maximum likelihood estimator δ_{mle} . Namely: (i) we identify a large class of models where for sufficiently small m , all Bayesian estimators with respect to symmetric about 0 priors supported on $[-m, m]$ dominate δ_{mle} , and (ii) we provide for Bayesian estimators δ_π sufficient dominance conditions of the form $m \leq c_\pi$, which are applicable to various models and priors π . In terms of the models, applications include Cauchy and Student distributions, densities which are logconvex on (θ, ∞) including scale mixtures of Laplace distributions, and logconcave on (θ, ∞) densities with logconvex on (θ, ∞) first derivatives such as normal, logistic, Laplace and hyperbolic secant, among others. In terms of priors π which lead to dominating δ_π 's in (ii), applications include the uniform density, as well as symmetric densities about 0, which are also absolutely continuous, nondecreasing and logconcave on $(0, m)$.

Key words and phrases: Bayes estimator, Cauchy and Student models, dominance, logconcave densities, logconvex densities, maximum likelihood estimator, restricted parameter space, scale mixture of Laplace densities, squared error loss, symmetric location families.

1. Introduction

Consider the problem of estimating under squared-error loss, based on an observable X , the median θ of a spherically symmetric univariate model where θ is known to be restricted to an interval $[a, b]$. Without loss of generality, we set $a = -m$, $b = m$, with $m > 0$. Interesting questions pertain to the frequentist performance of Bayesian estimators, such as the determination or description of Bayesian estimators that improve upon the benchmark (but inadmissible; e.g., Charras and van Eeden (1991)) maximum likelihood estimator (δ_{mle}). With respect to the last question, findings have been obtained, for normal models and among others, by Casella and Strawderman (1981); Gatsonis *et al.* (1987); and Marchand and Perron (2001); and for more general models by Moors (1981, 1985) and Marchand and Perron (2005, 2007).

The results of Marchand and Perron (2001), which are developed for mul-

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tivariate versions with $X \sim N_p(\theta, \sigma^2 I_p)$, $\|\theta\| \leq m$, and known σ^2 , will be of particular relevance to us. Namely, their results for $p = 1$ imply that:

- (i) all Bayes estimators with respect to a symmetric about 0 prior dominate δ_{mle} as soon as $m \leq c_0 \sigma$ with $c_0 \approx 0,4837$;
- (ii) the Bayes estimator δ_{BU} with respect to a boundary uniform prior on $\{-m, m\}$ dominates δ_{mle} as soon as $m \leq \sigma$;
- (iii) the Bayes estimator δ_U with respect to a fully uniform on $[-m, m]$ prior dominates δ_{mle} as soon as $m \leq c_U \sigma$ with $c_U \approx 0,5230$;
- (iv) a Bayes estimator δ_π with respect to an absolutely continuous, symmetric and logconcave density π on $[-m, m]$, with $\pi(\cdot)$ also nondecreasing on $[0, m]$, dominates δ_{mle} for $m \leq c_\pi \sigma$ with $c_\pi \geq c_U$ (see their Remark 4, part 1).

Extensions of (ii) to a large class of symmetric models were also obtained by Marchand and Perron (2005, 2007).

The main original contributions of this paper are explicit extensions of (i), (iii), and (iv) to various other univariate, unimodal, and symmetric models. Although the findings of Marchand and Perron (2001) do suggest that the existence of such dominance results is perhaps not surprising, it remains particularly interesting that a unified development for logconcave densities with a first derivative which is logconvex on $(0, 2m)$, such as the one given here (e.g., Corollaries 3, 4; Example 4), is possible and leads to reasonable simple dominance conditions. Moreover, analogous dominance results to (i) are established, outside this framework; first to Student distributions (see Example 3) providing a stronger general Bayesian result than the one obtained by Marchand and Perron (2005) which is limited to the estimator δ_{BU} ; and (ii) secondly to logconvex on (θ, ∞) densities (Theorem 5) which will include scale mixtures of Laplace densities (Example 5). Various other illustrations are given. These include : examples of dominating Bayesian estimators, numerical evaluations (e.g., Table 1) of the cutoff points (e.g., c_π as in $m \leq c_\pi$) for dominance, applications to truncated linear and truncated linear minimax estimators (Examples 1, 2), and a multivariate Student dominance result (Theorem 4).

As in Marchand and Perron (2001), the technical arguments relied upon to arrive at these findings begin with conditional risk decompositions of the type $|X| \leq m$, $|X| > m$, and $|X| = r$, $r > 0$. Perron (2003) exploited these types of decompositions to obtain analogous dominance results for estimating a Binomial proportion p when $|p - \frac{1}{2}|$ is constrained above, while Marchand and Perron (2005, 2007) made use of the conditioning on $|X| = r$ to obtain their previous extensions of (ii).

Along with the recent reviews of estimation problems in restricted parameter spaces given by Marchand and Strawderman (2004) or van Eeden (2006), a renewed interest in restricted parameter space inference has occurred with problems arising in particle physics as reported upon by Mandelkern (2002). An additional motivation for the type of problem considered here comes from a connection with inference problems in the presence of additional information, such as the problem of estimating a mean θ_1 based on observables Y_1, Y_2 with $E(Y_i) = \theta_i$; $i = 1, 2$,

subject to the additional information that $|\theta_1 - \theta_2| \leq c$ for some known constant c (e.g., van Eeden and Zidek (2004), Marchand and Strawderman (2004)).

The paper is organized as follows. Section 2 provides some basic technical aspects of the problem, as well as useful definitions and properties established early on in order to facilitate the presentation of the dominance results in Sections 3 and 4. Sections 3 and 4 contain various dominance results and illustrations. More specifically, Section 3 gives for various models, conditions on (m, δ) for which symmetric about 0 estimators δ dominate the mle. Section 4 makes connections between priors π and associated δ_π 's arriving at dominance conditions on π for Bayesian estimators δ_π . For several applications, such as Corollary 3, the results and the technical arguments leading to these, and beginning in Section 2, apply to logconcave densities on (θ, ∞) with a first derivative which is logconvex on $(\theta, \theta + 2m)$ (examples include Normal, Laplace, Logistic, Hyperbolic secant). However, as mentioned previously, not all the dominance results require such assumptions, as illustrated by the univariate Cauchy and Student distributions (Example 3).

2. Definitions and technical lemmas

Our work here relates to the estimation of the median θ , under squared error loss $(d - \theta)^2$, for symmetric and unimodal location families with densities $f(\cdot | \theta)$ on $\Re$ (with respect to Lebesgue measure) of the form:

$$(2.1) \quad f(x | \theta) = e^{-h(x-\theta)}, \quad \theta \in \Theta(m) = [-m, m], \quad (m \text{ fixed}).$$

The corresponding cdf's will be denoted $F_{\theta, h}$. In (2.1), h is a given continuous, even function, and increasing on $(0, \infty)$. As our model and loss are invariant under sign changes, we will consider equivariant estimators only (i.e., odd functions); and corresponding risk functions will be symmetric functions on $\Theta(m)$. We will be concerned with the specification of such estimators that dominate δ_{mle} , with $\delta_{\text{mle}}(x) = (x \wedge m)$ for $x > 0$.

DEFINITION 1. For $m > 0$, set

$$\begin{aligned} H^{**}(m) &= \{h : h \text{ increasing and convex on } (0, 2m)\}; \\ H^*(m) &= \{h : h \in H^{**}(m) \text{ and } h' \text{ is concave on } (0, 2m)\}. \end{aligned}$$

The above provides useful classifications for densities in (2.1). For instance, the omnipresent categorization $h \in H^*(m) \cap H^{**}(\infty)$ in this paper simply refers to densities in (2.1) that are logconcave on (θ, ∞) , with a first derivative which is logconvex on $(\theta, \theta + 2m)$ ¹. We also require the following quantities, defined for any h .

$$\rho_h(\theta, x) = \tanh\left(\frac{h(\theta + x) - h(\theta - x)}{2}\right); \quad \theta, x \in \Re;$$

¹ Observe that the underlying symmetry implies as well that these same densities are necessarily logconcave on $(-\infty, \theta)$, with a first derivative which is logconcave on $(\theta - 2m, \theta)$.

$$\begin{aligned}
\bar{\rho}_h(m, x) &= \sup\{\rho_h(\theta, x) : \theta \in [0, m]\}; & m, x > 0; \\
\alpha_h(m, \theta) &= E_\theta[\rho_h(\theta, |X|) \mid |X| > m]; & 0 \leq \theta \leq m; \\
\bar{\alpha}_h(m) &= \sup\{\alpha_h(m, \theta); \theta \in [0, m]\}; & m > 0; \\
\beta_h(m, \theta) &= E_\theta\left[\frac{\theta\rho_h(\theta, X)}{X} \mid |X| \leq m\right]; & 0 \leq \theta \leq m; \\
\bar{\beta}_h(m) &= \sup\{\beta_h(m, \theta); \theta \in [0, m]\}; & m > 0; \\
G_h(m) &= mh'(m), & \text{for } h \text{ differentiable at } m; m > 0; \\
G_h^{-1}(a) &= \inf\{m > 0 : mh'(m) > a\}; & a > 0.
\end{aligned}$$

We continue with various useful properties related to the quantities above and how their behaviour depends on h and m .

LEMMA 1. *For a fixed nondecreasing h , we have*

- (a) $\rho_h(\cdot, x)$ is nondecreasing on $[0, m]$ for all $x > 0$ whenever $h \in H^{**}(m)$;
- (b) $\rho_h(\theta, \cdot)$ is nondecreasing on $(0, \infty)$ for all $\theta \in (0, m)$ if and only if $h \in H^{**}(\infty)$;
- (c) $E_\theta[\text{sgn}(X) \mid |X| = r] = \rho_h(\theta, r)$; $r \geq 0$;
- (d) for any fixed $\theta \geq 0$, $\lim_{x \rightarrow 0^+} \frac{\rho_h(\theta, x)}{x} = h'(\theta)$; (h differentiable at θ);
- (e) for any fixed $\theta \in [0, m]$, $\frac{\rho_h(\theta, x)}{x}$ is decreasing in x on $(0, m)$ whenever $h \in H^*(m)$;
- (f) whenever $h \in H^*(m)$, we have $\bar{\rho}_h(m, x) \leq xh'(m)$ for all $x \in (0, m)$.

PROOF. Parts (a, b, c) are given² by Marchand and Perron (2007, lemmas 1 and 2), while part (d) follows by a direct evaluation. Part (f) is a direct consequence of (d, e). Finally, the proof of part (e) is postponed to Appendix.

LEMMA 2. *For any fixed h ,*

- (a) the function $\alpha_h(m, \cdot)$ increases on $[0, m]$ with $\bar{\alpha}_h(m) = \alpha_h(m, m) = \frac{1-2F_{0,h}(-2m)}{1+2F_{0,h}(-2m)}$, $\lim_{m \rightarrow 0^+} \bar{\alpha}_h(m) = 0$, $\lim_{m \rightarrow \infty} \bar{\alpha}_h(m) = 1$, and $\bar{\alpha}_h^{-1}(1/2) = \frac{1}{2}F_{0,h}^{-1}(\frac{5}{6})$;
- (b) $\bar{\beta}_h(m) \leq G_h(m)$ for any fixed (m, h) such that $h \in H^*(m)$.

PROOF. For part (a), a direct evaluation gives us for $\theta \in [0, m]$:

$$\begin{aligned}
\alpha_h(m, \theta) &= E_\theta\left[\tanh\left(\frac{h(\theta + |X|) - h(\theta - |X|)}{2}\right) \mid |X| > m\right] \\
&= \frac{\int_m^\infty (e^{-h(\theta-x)} - e^{-h(\theta+x)})dx}{\int_m^\infty (e^{-h(\theta-x)} + e^{-h(\theta+x)})dx} \\
&= \frac{F_{0,h}(m + \theta) - F_{0,h}(m - \theta)}{2 - F_{0,h}(m + \theta) - F_{0,h}(m - \theta)} \\
&= 1 - 2 \frac{1}{1 + \frac{1 - F_{0,h}(m - \theta)}{1 - F_{0,h}(m + \theta)}}.
\end{aligned}$$

² They actually give a stronger result than (a) which implies the one given here.

From the above, the increasing property of $\alpha_h(m, \cdot)$ follows at once, with $\bar{\alpha}_h(m) = \alpha_h(m, m)$ as given. The other given properties of (a) follow in a straightforward manner.

For (b), we have indeed from parts (d,e) of Lemma 1, and for all $\theta \in [0, m]$: $\beta_h(m, \theta) \leq \theta h'(\theta) \leq mh'(m) = G_h(m)$, since $h \in H^*(m)$.

3. General dominance results

Here, for an equivariant estimator δ , we provide conditions on (h, m, δ) for which δ dominates δ_{mle} on $\Theta(m)$, with risk analyzes for $\theta \in [0, m]$ sufficing, given the symmetry of the risk functions involved. As in Marchand and Perron (2001), we initially proceed by deriving improvements with respect to the conditional risks for: (i) $|X| > m$, (ii) $|X| \leq m$, and (iii) $|X| = r$, $r > 0$. Then, combining the conditions obtained for these conditional risks, we obtain various unconditional risk improvements. Notwithstanding the fact that a large portion of the development below does indeed represent an extension of the univariate normal case analyzed by Marchand and Perron (2001) to general models in (2.1), the technical arguments used match for the most part those given by Marchand and Perron (2001), with results of Section 2 called upon for various bounds on monotonicity properties. Hence, we have relegated some proofs to Appendix.

THEOREM 1. *For models in (2.1) with logconcave densities on (θ, ∞) , and δ a nondecreasing function on (m, ∞) , the condition*

$$(3.1) \quad m \left(\frac{1 - 6F_{0,h}(-2m)}{1 + 2F_{0,h}(-2m)} \right) \leq \delta(x) \leq m, \quad \text{for all } x > m,$$

with strict inequalities for some $x \in (m, \infty)$, is sufficient to have

$$(3.2) \quad E_\theta[|\theta - \delta_{\text{mle}}(X)|^2 - |\theta - \delta(X)|^2 \mid |X| > m] > 0, \quad \text{for all } \theta \in \Theta(m).$$

PROOF. See Appendix.

THEOREM 2. *For models in (2.1) with logconcave densities on (θ, ∞) , having a first derivative which is logconvex on $(\theta, \theta + 2m)$ (i.e., $m \leq \sup\{m \geq 0 : H^*(m) \ni h\}$), $\bar{\beta}_h(m) \leq 1$, and δ a nondecreasing function on $(0, m)$ such that $x(x - \delta(x))$ increases in x for $x \in (0, m)$, the condition*

$$(3.3) \quad (2\bar{\beta}_h(m) - 1)x \leq \delta(x) \leq x, \quad \text{for all } x \in (0, m),$$

with strict inequalities for some $x \in (0, m)$, is sufficient to have

$$(3.4) \quad E_\theta[|\theta - \delta_{\text{mle}}(X)|^2 - |\theta - \delta(X)|^2 \mid |X| \leq m] > 0, \quad \text{for all } \theta \in \Theta(m).$$

PROOF. See Appendix.

Combining the dominance conditions of Theorems 1 and 2 leads directly to the following.

COROLLARY 1. *For models in (2.1) with logconcave densities on (θ, ∞) , having a first derivative which is logconvex on $(\theta, \theta + 2m)$, $\bar{\beta}_h(m) \leq 1$, and δ a nondecreasing function on $(0, \infty)$ such that: (i) $\delta(x) < x \wedge m$ for $x > 0$, (ii) $x(x - \delta(x))$ increases in x on $(0, m)$, the conditions*

$$(3.5) \quad \begin{aligned} \frac{\delta(m)}{m} &> \frac{1 - 6F_{0,h}(-2m)}{1 + 2F_{0,h}(-2m)}, \quad \text{and} \\ \frac{\delta(x)}{x} &> 2\bar{\beta}_h(m) - 1 \quad \text{for all } x \in (0, m), \end{aligned}$$

are jointly sufficient for δ to dominate δ_{mle} on $\Theta(m)$.

Remark 1. For densities in (2.1) that are logconcave on (θ, ∞) , with a first derivative which is logconvex on (θ, ∞) (i.e., $h \in H^*(\infty)$), the conditions on δ of Corollary 1 yield many dominating estimators as long as $\bar{\beta}_h(m) \leq 1$, which in turn holds whenever $m \leq G_h^{-1}(1)$ (see Lemma 2). Furthermore, a wide class of solutions is obtained by selecting δ 's such that $\frac{\delta(x)}{x}$ decreases for $x \in (0, m)$ (which together with (i) implies (ii)), with $\frac{\delta(m)}{m} > (2\bar{\beta}_h(m) - 1) \vee \left(\frac{1 - 6F_{0,h}(-2m)}{1 + 2F_{0,h}(-2m)}\right)$.

Example 1. (Truncated linear and truncated linear minimax estimators.)

Interesting and simple alternatives to δ_{mle} are truncated linear estimators of the form $\delta_a(x) = (ax \wedge m) \operatorname{sgn}(x)$, for $a \in (0, 1)$. A particularly appealing choice, available for cases where the variance in (2.1) exists (say equal to V), is given by the choice $a_0 = \frac{m^2}{m^2 + V}$ corresponding to the truncation onto $[-m, m]$ of the linear minimax estimator for $\theta \in \Theta(m)$. Although $a_0 X$ dominates X on $\Theta(m)$, there is no guarantee that dominance will carry over to the truncated versions δ_{a_0} and δ_{mle} .

For the estimator δ_a with $a \in (0, 1)$, it is easy to see that conditions (i) and (ii) of Corollary 1 are satisfied. Hence for models in (2.1), Corollary 1 (or Remark 1) tells us, for (m, h) satisfying its conditions, that δ_a dominates δ_{mle} whenever:

$$(3.6) \quad (2\bar{\beta}_h(m) - 1) \vee \left(\frac{1 - 6F_{0,h}(-2m)}{1 + 2F_{0,h}(-2m)} \right) \leq a < 1.$$

For the truncated linear minimax estimator δ_{a_0} , the evaluation above is more delicate since a_0 depends on m . The following is specific to the normal case, but similar analyzes can be arrived at for other logconcave on (θ, ∞) densities.

Example 2. For a normal $N(\theta, \sigma^2)$ model with $|\theta| \leq m$, we show that δ_{a_0} dominates δ_{mle} whenever $m \leq c_1 \sigma \approx 0.60936\sigma$, with c_1 being the unique positive solution in c to the equation $\psi(c) = 0$, with $\psi(c) = \frac{c^2}{c^2 + 1} - \left(\frac{1 - 6\Phi(-2c)}{1 + 2\Phi(-2c)}\right)$; Φ being

the standard normal cdf. To establish this, set $\sigma = 1$ without loss of generality and apply (3.6) with $a = \frac{m^2}{m^2+1}$. We seek to determine the values of m such that:

(i) $\frac{m^2}{m^2+1} \geq 2\bar{\beta}_h(m) - 1$, and (ii) $\psi(m) \geq 0$. Make use of part (b) of Lemma 2 with $h(y) = (y^2)/2$ to infer that (i) holds as long as $\frac{m^2}{m^2+1} \geq 2m^2 - 1$; i.e., $m \leq (\frac{1}{2})^{1/4}$. With regards to (ii), observe that $\psi(0) > 0$, and that

$$\psi(m) \geq 0 \Leftrightarrow T(m) \geq \frac{1}{2}, \quad \text{with } T(m) = (4m^2 + 3)\Phi(-2m).$$

A direct evaluation yields $T'(m) = 2\Phi(-2m)(4m - (4m^2 + 3)\frac{\phi(2m)}{\Phi(-2m)})$; ϕ being the standard normal pdf. From this and by virtue of the inequality $\frac{\phi(x)}{\Phi(-x)} \geq x$ ($x \geq 0$), we infer that $T(m)$ decreases in m on $[0, \infty)$. Since $\lim_{m \rightarrow \infty} T(m) = 0$, we conclude that $\psi(m)$ changes signs once from $+$ to $-$ as m ranges on $(0, \infty)$. Finally, a direct evaluation tells us that $\psi((\frac{1}{2})^{1/4}) \leq 0$, yielding the result.

THEOREM 3 (Marchand and Perron (2005)). *For models in (2.1), the (symmetric) estimator δ dominates δ_{mle} on $\Theta(m)$ whenever, for all $x > 0$,*

$$(3.7) \quad 2\sup\{\theta\rho_h(\theta, x) : \theta \in [0, m]\} - (m \wedge x) < \delta(x) < (m \wedge x),$$

for all $x \in A_{h,m} = \{x > 0 : \sup\{\theta\rho_h(\theta, x) : \theta \in [0, m]\} < x\}$, and $\delta(x) = x$ otherwise.

The above theorem is also used in Marchand and Perron (2007) to determine conditions for which the Bayes estimator with respect to a uniform 2-point prior on $\{-m, m\}$, given by $\delta_{BU}(x) = m\rho_h(m, x)$ for $x > 0$, dominates δ_{mle} on $\Theta(m)$. In particular, they show that if (m, h) are such that $h \in H^*(m)$ and $m \leq G_h^{-1}(1)$ (see Table 1), then (3.7) is satisfied for $\delta = \delta_{BU}$ as seen as a consequence of part (f) of Lemma 1 (they actually give somewhat weaker conditions along with illustrations). We will not use Theorem 3 in the same way. Instead, we will, as in the next result and the examples that follow, exploit conditions for which the left-hand side of (3.7) is bounded above by 0, which leaves a large window in (3.7) for δ to vary and still dominate δ_{mle} .

COROLLARY 2. *For models in (2.1), an equivariant estimator δ of θ , such that $\delta \neq \delta_{\text{mle}}$ and $0 \leq \delta(x) \leq (m \wedge x)$ for all $x > 0$, dominates δ_{mle} when either:*

- (i) $2m\bar{\rho}_h(m, x) \leq (m \wedge x)$ for all $x > 0$,
- (ii) or δ is nondecreasing on (m, ∞) , and (m, h) are such that $h \in H^*(m)$ with $m \leq G_h^{-1}(\frac{1}{2}) \wedge \frac{1}{2}F_{0,h}^{-1}(\frac{5}{6})$.

PROOF. The sufficiency of (i) is a direct consequence of (3.7). For (ii), we verify (3.7) for all $x \in (0, m)$ and (3.1) for all $x \geq m$. For $x \in (0, m)$, apply part (f) of Lemma 1 to infer that $2m\bar{\rho}_h(m, x) \leq 2mh'(m)x \leq x$ as soon as $m \leq G_h^{-1}(1/2)$. For $x \geq m$, it is straightforward to see that the mere nonnegativity of δ implies that (3.1) is satisfied whenever $\bar{\alpha}_h(m) \leq \frac{1}{2}$, that is $m \leq \frac{1}{2}F_{0,h}^{-1}(\frac{5}{6})$ as seen in Lemma 2.

Remark 2. For densities in (2.1) that are logconcave on (θ, ∞) , with a first derivative which is logconvex on (θ, ∞) (i.e., $h \in H^*(\infty)$), condition **(ii)** of Corollary 2 gives, for shrinkage estimators δ which increase on $[m, \infty)$, universal dominance with no further conditions on δ , but requiring only small enough m . Furthermore, it will be established below (i.e., Lemma 3, Corollary 3) that the conditions on g are shared by all Bayes estimators with respect to symmetric about 0 priors on $\Theta(m)$. This universal dominance result, which illustrates vividly the inadequacy of δ_{mle} for small parameter spaces (under squared error loss) extends the univariate normal result of Marchand and Perron (2001) (they also give a multivariate normal version). Observe that the estimator $\delta(X) = 0$, which has been further studied by Dou and van Eeden (2006) in the normal case, belongs to the class of dominating estimators. A selection of further examples of models for which $h \in H^*(\infty)$, which include members of the Exponential power family such as Laplace (or Double-Exponential) distributions, Hyperbolic Secant, and Logistic distributions (see Marchand and Perron (2007), for more details and examples); along with the corresponding upper bounds on m , are given in Table 1 below.

Example 3 (Student and Cauchy distributions). Consider families of distributions in (2.1) with densities

$$(3.8) \quad f(x | \theta) = \frac{1}{\text{Beta}(\nu/2, 1/2)\sigma\sqrt{\nu}} \left\{ 1 + \frac{1}{\nu} \left(\frac{x - \theta}{\sigma} \right)^2 \right\}^{-(\nu+1)/2},$$

corresponding to Student distributions with positive scale and shape parameters σ and ν . Even though the associated h functions belong to $H^{**}(\frac{\sigma\sqrt{\nu}}{2})$, they do not belong to $H^{**}(\infty)$ (see Marchand and Perron (2007)), and part **(ii)** of Corollary 2 cannot be applied. However, turning to the potential applicability of part **(i)** of Corollary 2, $\bar{\rho}_h$ admits a useful lower bound as established by Marchand and Perron (2005, Lemma 2 and equation A.4) which tells us that

$$m\bar{\rho}_h(m, x) \leq \frac{m^2 x}{m^2 + x^2 + \sigma^2 \nu} (1 + \nu), \quad \text{for all } x > 0,$$

with equality for all $x > 0$ if and only if $\nu = 1$ (i.e., Cauchy case). Simple analysis with the upper bound tells us that: **(a)** $2m\bar{\rho}_h(m, x) \leq x$ for $x \in (0, m)$ whenever $m \leq \sigma\sqrt{\frac{\nu}{1+2\nu}}$, and **(b)** $2m\bar{\rho}_h(m, x) \leq m$ for $x > m$ whenever $m \leq \sigma\sqrt{\frac{1}{\nu+2}}$. Therefore, unifying both conditions for $\nu \geq 1$, we infer that any shrinkage estimator δ (toward 0) with respect to δ_{mle} is a dominator of δ_{mle} under (3.8) as soon as $m \leq \sigma\sqrt{\frac{1}{\nu+2}}$. In the Cauchy case, the bound becomes $m \leq \sigma\sqrt{\frac{1}{3}}$. We conclude by mentioning that, for the **specific case** of the Bayes estimator δ_{BU} with respect to the boundary uniform prior on $\{-m, m\}$, Marchand and Perron (2005) establish the stronger condition $m \leq \sigma$ for dominance. But the new condition above is universal in the sense that it applies to **all** shrinkage estimators. And, more interestingly, these shrinkage estimators will include **all** Bayesian estimators with respect to symmetric priors (see Example 2 continued below).

Remark 3. Although this paper does not address multivariate extensions with a constraint of the form $\|\theta\| \leq m$, as in Marchand and Perron (2001, 2005), one wonders about such extensions. Several difficulties persist, namely with regards to Bayesian estimators, but we point out that the above Student case admits the following relatively straightforward extension, which we establish in Appendix.

THEOREM 4. *Let $X - \theta$ be distributed as a p -variate student with d degrees of freedom and $\|\theta\| \leq m$. Then, any equivariant shrinkage estimator δ ($\delta \neq \delta_{\text{mle}}$) with respect to δ_{mle} (i.e., $\delta(x) = g(\|x\|) \frac{x}{\|x\|}$, with $0 \leq g(r) \leq m \wedge r$ for all $r > 0$) dominates δ_{mle} as soon as $m \leq \sqrt{\frac{p^2}{d+2p} \wedge \frac{d}{3}}$; (i.e., $m \leq \sqrt{\frac{d}{3}}$ for $d < p$, and $m \leq \sqrt{\frac{p^2}{d+2p}}$ for $d \geq p$).*

4. Bayesian dominance results

We now turn our attention to Bayesian estimators δ_π , associated with symmetric about 0 priors π , and work toward conditions on π which will justify the applicability of the dominance results of Section 3. The next lemma summarizes the key relationships between π and δ_π which we will make use of.

LEMMA 3. *Let π be a symmetric about 0 prior density on $\Theta(m)$ (with respect to a symmetric σ -finite measure ν), and δ_π the corresponding Bayes estimator for squared error loss under model (2.1). Then,*

- (a) *if $h \in H^{**}(\infty)$, then the family of posterior densities $\pi(\cdot | x)$ possess increasing monotone likelihood ratio;*
- (b) *δ_π is nondecreasing whenever $h \in H^{**}(\infty)$;*
- (c) *δ_π admits the representation: $\delta_\pi(x) = E_x[U\rho_h(U, x)]x \geq 0$, where U has density (with respect to ν) proportional to $\{(e^{-h(x-u)} + e^{-h(x+u)})I_{(0,m]}(u) + e^{-h(x)}I_{\{0\}}(u)\}\pi(u)$;*
- (d) *if (m, h) are such that $h \in H^*(m)$, then $\delta_\pi(x) \leq x \wedge m$, for all $x > 0$, whenever (i) $E_x[Uh'(U)] \leq 1$, or (ii) $m \leq G_h^{-1}(1)$;*
- (e) *if π is a continuous, logconcave, and nondecreasing density on $(0, m)$, then both $x - \delta_\pi(x)$ and $x(x - \delta_\pi(x))$ increase in x , $x \geq 0$;*
- (f) *if π^* is another prior (also with respect to ν) such that $\frac{\pi^*(\theta)}{\pi(\theta)}$ increases in θ on $[0, m]$, and if (m, h) are such that $h \in H^{**}(m)$, then $\delta_{\pi^*}(x) \geq \delta_\pi(x)$ for all $x \geq 0$.*

PROOF. (a, b) Part (b) follows immediately from (a) since $\delta_\pi(x)$ is a posterior expectation. For (a), observe that, for any (x_1, x_2) such that $x_1 < x_2$,

$$\frac{\pi(\theta | x_2)}{\pi(\theta | x_1)} \propto e^{h(\theta-x_1)-h(\theta-x_2)},$$

which is indeed increasing in θ , $\theta \in \Theta(m)$, whenever h is convex; i.e, $h \in H^{**}(\infty)$.

(c) Start with the expression

$$\delta_\pi(x) = \frac{\int_{[-m,m]} te^{-h(x-t)}\pi(t)d\nu(t)}{\int_{[-m,m]} e^{-h(x-t)}\pi(t)d\nu(t)},$$

set $u = -t$ for $t \in [-m, 0]$, and exploit the symmetry of both h and π to obtain

$$\delta_\pi(x) = \frac{\int_{(0,m]} t\{e^{-h(x-t)} - e^{-h(x+t)}\}\pi(t)d\nu(t)}{\int_{(0,m]} \{e^{-h(x-t)} + e^{-h(x+t)}\}\pi(t)d\nu(t) + e^{-h(x)}\pi(0)\nu(\{0\})}.$$

Finally, the representation holds with the definition of ρ_h , and since $\frac{e^{-A}-e^{-B}}{e^{-A}+e^{-B}} = \tanh(\frac{A-B}{2})$, for all A, B .

(d) Clearly $\delta_\pi(x) \leq m$ for all $x \geq m$. Now, to have the property $\delta_\pi(x) \leq x$ for $x \in [0, m]$, condition (i) suffices as seen by making use of part (f) of Lemma 1. Condition (ii), which says that $mh'(m) \leq 1$, implies (i) given that h is convex on $(0, m)$ by assumption that $h \in H^*(m)$.

(e) The increasing property of $x - \delta_\pi(x)$ is established within Proposition 3.1 of Kubokawa (2005), and implies that $x(x - \delta_\pi(x))$ increases for $x \geq 0$ (since $x - \delta_\pi(x) \geq 0 - \delta_\pi(0) = 0$ given the symmetry of π and h).

(f) In part (c) above, set $g_{\pi,x}$ and $g_{\pi^*,x}$ as the densities of U associated with observation x and with priors π and π^* , respectively. Since the increasing property of the ratio $\frac{\pi^*(u)}{\pi(u)}$ transfers over to the ratio $\frac{g_{\pi^*,x}(u)}{g_{\pi,x}(u)}$; $u \in [0, m]$; we infer that $\delta_{\pi^*}(x) \geq \delta_\pi(x)$; $x \geq 0$; given that $u\rho_h(u, x)$ is, by part (a) of Lemma 1 and since $h \in H^{**}(m)$, increasing in u ; $u \in [0, m]$.

From the above, we now infer the following as a consequence of Corollary 2.

COROLLARY 3.

- (a) For models in (2.1), with logconcave densities on (θ, ∞) having a first derivative which is logconvex on $(\theta, \theta + 2m)$, **any** Bayes estimator δ_π with respect to a symmetric about 0 prior dominates δ_{mle} on $\Theta(m)$ as soon as $m \leq G_h^{-1}(\frac{1}{2}) \wedge \frac{1}{2}F_{0,h}^{-1}(\frac{5}{6})$.
- (b) For models in (2.1), and an arbitrary Bayes estimator δ_π with respect to a symmetric about 0 prior density π on $\Theta(m)$, condition (i) of Corollary 2 suffices for δ_π to dominate δ_{mle} .

PROOF. By virtue of Corollary 2, we only need to show that δ_π is a nondecreasing shrinkage estimator in (a); and a shrinkage estimator in (b). With the given assumption on (m, h) , the nondecreasing and shrinkage properties required in (a) follow from parts (a) and (d, ii) (respectively) of Lemma 3. Now, suppose condition (i) of Corollary 2 holds. Then, the shrinkage property required in (b) follows from part (c) of Lemma 3 as $U\rho_h(U, x) \leq m\bar{\rho}_h(m, x)$ (with probability one, for all $x > 0$), which implies $\delta_\pi(x) \leq m\bar{\rho}_h(m, x) \leq 2m\bar{\rho}_h(m, x) \leq m \wedge x$, for all $x > 0$.

Example 3 (continued). For Student densities in (3.8), $\nu \geq 1$, it is immediate from the above that dominance holds universally for **all** symmetric Bayes estimators as soon as $m \leq \sigma \frac{1}{\sqrt{\nu+2}}$.

Now, concerning the applicability of Corollary 1 to Bayesian estimators, the results of Lemma 3 lead to the following.

COROLLARY 4. *For models in (2.1), with logconcave densities on (θ, ∞) having a first derivative which is logconvex on $(\theta, \theta + 2m)$, with $m \leq G_h^{-1}(1)$, and for prior densities π which are symmetric, continuous, logconcave, and nondecreasing on $(0, m)$, condition (3.5) is sufficient for δ_π to dominate δ_{mle} .*

PROOF. It suffices to check that conditions **(i)** and **(ii)** of Corollary 1 are satisfied. On one hand, **(i)** holds given part (d, **ii**) of Lemma 3 and part (b) of Lemma 2. On the other hand, part (e) of Lemma 3 tells us that **(ii)** is satisfied with the given assumptions on the prior π .

Example 4. (Uniform and other symmetric, logconcave priors). Corollary 4 applies to the important and benchmark choice of a fully uniform prior on the parameter space $\Theta(m)$ and the corresponding Bayes estimator δ_U . The universal dominance result of Corollary 3 already applies, but the conditions here of Corollary 1 (or Corollary 4) give us weaker conditions (on m) for δ_U to dominate δ_{mle} . Hence, we proceeded to evaluate, for a selection of models in (2.1) with logconcave densities having logconvex derivatives on $(0, \infty)$ (i.e., $h \in H^*(\infty)$), the set:

$$C_U = \{m : m \leq G_h^{-1}(1) \text{ and (3.5) is satisfied for } \delta_U\}.$$

In Table 1, these sets C_U are reported on and contrasted with the universal cutoff points, as well as the dominance cutoffs points (i.e., $m \leq G_h^{-1}(1)$) for the boundary uniform prior. For other continuous, symmetric, logconcave priors π , which are also nondecreasing on $(0, m)$, it follows from part (f) of Lemma 3 and Corollary 4 that the dominance set

$$C_\pi = \{m : m \leq G_h^{-1}(1) \text{ and (3.5) is satisfied for } \delta_\pi\}$$

will contain the set C_U . Furthermore, again by virtue of part (f) of Lemma 3, the steeper π is on $(0, m)$, the larger C_π is, with the steepest π being boundary uniform in the limit with corresponding maximal C_π given by the interval $[0, G_h^{-1}(1)]$. Finally, the above elements are partially illustrated in Table 2 for various models and the logconcave priors $\pi(\theta) \propto |\theta|^k I_{(-m, m)}(\theta)$; $k = 0, 1, 2, 4$.

These results above apply to logconcave densities. However, we saw earlier with Example 3 that the non logconcave Cauchy or Student densities could be handled with a useful upper envelope bound for $\bar{\rho}_h(m, x)$, $x > 0$, and making use of part **(i)** of Corollary 2. Here, we do the same for a general logconvex on (θ, ∞)

Table 1. Sufficient conditions for dominance of δ_{mle} by different Bayesian estimators.

<i>Density</i> ($f(x \mid \theta) \propto$)	<i>Type of Prior</i>		
	<i>Symmetric</i> $(m \leq G_h^{-1}(\frac{1}{2}) \wedge \frac{1}{2}F_h^{-1}(\frac{5}{6}))$	<i>Boundary Uniform</i> $(m \leq G_h^{-1}(1))$	<i>Uniform</i> $(\{m : m \leq G_h^{-1}(1) \text{ and } (3.5) \text{ is satisfied for } \delta_U\})$
Normal ($e^{-(1/2)(x-\theta /\sigma)^2}$)	$m \leq 0.4837\sigma$	$m \leq \sigma$	$m \leq 0.5230\sigma$
Laplace ($e^{- x-\theta /\sigma}$)	$m \leq 0.5\sigma$	$m \leq \sigma$	$m \leq 0.59720\sigma$
Logistic ($\frac{e^{(x-\theta /\sigma)}}{(1+e^{(x-\theta /\sigma)})^2}$)	$m \leq 0.8047\sigma$	$m \leq 1.5434\sigma$	$m \leq 0.9004\sigma$
Hyp. Secant ($\text{sech}((x-\theta)/\sigma)$)	$m \leq 0.6585\sigma$	$m \leq 1.1997\sigma$	$m \leq 0.7640\sigma$

Table 2. Dominance set C_π for different models and priors.

<i>Density</i> ($f(x \mid \theta) \propto$)	<i>Type of Prior</i> $\pi(\theta) \propto$			
	$I_{(-m,m)}(\theta)$	$ \theta I_{(-m,m)}(\theta)$	$\theta^2 I_{(-m,m)}(\theta)$	$\theta^3 I_{(-m,m)}(\theta)$
Normal ($e^{(1/2)(x-\theta /\sigma)^2}$)	$(0, 0.5230]$	$(0, 0.6213]$	$(0, 0.6624]$	$(0, 0.7189]$
Laplace ($e^{- x-\theta /\sigma}$)	$(0, 0.5972]$	$(0, 0.7270]$	$(0, 0.7805]$	$(0, 0.8441]$
Logistic ($\frac{e^{(x-\theta /\sigma)}}{(1+e^{(x-\theta /\sigma)})^2}$)	$(0, 0.9004]$	$(0, 0.9783]$	$(0, 1.0222]$	$(0, 1.0760]$
Hyp. Secant ($\text{sech}((x-\theta)/\sigma)$)	$(0, 0.7640]$	$(0, 0.8938]$	$(0, 0.9534]$	$(0, 1.0237]$

density³ and conclude with applications for scale mixtures of Laplace densities. Interestingly, the Laplace density in (2.1), with both a logconcave and logconvex density on (θ, ∞) fits into both frameworks.

THEOREM 5. *For models in (2.1) with logconvex densities on (θ, ∞) , and with a finite right derivative of h at 0 (i.e. $h'(0+) < \infty$), the condition $m \leq \frac{1}{2h'(0+)}$ is sufficient for:*

- (i) *an equivariant estimator δ of θ , such that $\delta \neq \delta_{\text{mle}}$ and $0 \leq \delta(x) \leq (m \wedge x)$ for all $x > 0$, to dominate δ_{mle} ;*
- (ii) *an arbitrary Bayes estimator δ_π with respect to a symmetric about 0 prior density π on $\Theta(m)$ to dominate δ_{mle} .*

PROOF. We apply part (i) of Corollary 2 for (i), and Corollary 3(b) for (ii), requiring only the condition

$$(4.1) \quad 2m\bar{\rho}_h(m, x) \leq (m \wedge x), \quad \text{for all } x > 0.$$

Observe that, in view of the assumed symmetry, increasing and concavity properties of h on $(0, \infty)$ in (2.1), as well as the monotone increasing property of the hyperbolic tangent function on $(0, \infty)$, for $x > 0$:

- (A) both $h(\theta + x) - h(\theta - x)$ and $\rho_h(\theta, x)$ increase in θ on $[0, x]$;
- (B) both $h(\theta + x) - h(\theta - x)$ and $\rho_h(\theta, x)$ decrease in θ on (x, ∞) .

Given the definitions of $\rho_h(\cdot, \cdot)$ and $\bar{\rho}_h(\cdot, \cdot)$, these properties imply that $\bar{\rho}_h(m, x) = \rho_h(m \wedge x, x)$, for $x > 0$. Similarly, or as seen as a consequence of (B) above since $\rho_h(mx) = \rho_h(x, m)$, we have that $\rho_h(m, x)$ decreases in x on $[m, \infty)$, for $m > 0$ fixed. Therefore $\bar{\rho}_h(m, x)$ decreases in x on $[m, \infty)$, and (4.1) becomes equivalent to:

$$(4.2) \quad \begin{aligned} 2m\rho_h(x, x) &\leq x, \quad \text{for } x \in [0, m] \\ \Leftrightarrow 2m \tanh\left(\frac{h(2x) - h(0)}{2}\right) &\leq x, \quad \text{for } x \in [0, m]. \end{aligned}$$

Finally, observe that the left-hand side of (4.2) is a concave function of x on $[0, m]$; given that we have a composition of the concave hyperbolic tangent function on $[0, \infty)$ with the concave and increasing on $[0, m]$ function given by $\frac{h(2x) - h(0)}{2}$; so that (4.1) and (4.2) hold if and only if

$$\begin{aligned} \frac{\partial}{\partial x} \left\{ 2m \tanh\left(\frac{h(2x) - h(0)}{2}\right) \right\}_{x=0+} &\leq 1, \quad \text{for } x \in [0, m] \\ \Leftrightarrow 2mh'(0+) &\leq 1 \\ \Leftrightarrow m &\leq \frac{1}{2h'(0+)}. \end{aligned}$$

³ This also implies that such a density is logconvex on $(-\infty, \theta)$; without being logconvex on $(-\infty, \infty)$ however.

Example 5 (Laplace distributions and their scale mixtures). Laplace distributions with location parameter θ and scale parameter $\sigma > 0$; with densities $\frac{1}{2\sigma}e^{-((x-\theta)/\sigma)}$; are both logconcave and logconvex on (θ, ∞) . We saw previously how our earlier results (namely Corollary 3, Corollary 4) apply for $\theta \in \Theta(m)$ (e.g., Tables 1, 2). But, we also can apply Theorem 5 with $h'(y) = \frac{1}{\sigma}$, for $y > 0$, which leads to the condition $m \leq \frac{1}{2\sigma}$ for results (i) and (ii) of Theorem 5 to hold. Interestingly these universal dominance results match the logconcave results obtained earlier (see Table 1) in part (ii) of Corollary 2 and part (a) of Corollary 3 (although the derivations differ). Consider now scale mixtures of Laplace distributions where X admits the representation:

$$X - \theta \mid V = v \sim \text{Laplace with scale parameter } \frac{1}{v}, \quad \theta \in \Theta(m),$$

for some positive mixing random variable V having cumulative distribution function G . Such distributions have densities on $\Re$ as in (2.1) with:

$$h(y) = -\log \int_0^\infty \frac{v}{2} e^{-v|y|} dG(v).$$

A direct evaluation tells us that, for $y > 0$,

$$(4.3) \quad h'(y) = \frac{\int_0^\infty v^2 e^{-vy} dG(v)}{\int_0^\infty v e^{-vy} dG(v)} = E_y[W],$$

where W has density proportional to $we^{-yw}dG(w)$ on $(0, \infty)$. It is easy to verify that these densities form a family with decreasing monotone likelihood ratio in W , with y viewed as the parameter. Therefore, $h'(y)$ decreases in y on $(0, \infty)$, h is concave on $(0, \infty)$, and scale mixtures of Laplace densities in (2.1) are always logconvex on (θ, ∞) . Finally, applying Theorem 5 in cases of mixtures such that $E(V^2) < \infty$, we obtain directly from (4.3): $h'(0+) = \frac{E(V^2)}{E(V)}$, and the condition $m \leq \frac{E(V)}{2E(V^2)}$ for the universal dominance results (i) and (ii) of Theorem 5 to hold.

Appendix

The property that the covariance $\text{Cov}(f_1(X), f_2(X))$ is nonnegative for increasing f_1 and f_2 is used below.

PROOF OF PART (E) OF LEMMA 1. In order to establish the result in (e), it will suffice to show that, for any fixed $\theta \in [0, m]$, the function $\rho_h(\theta, x)$ is concave in x on $(0, m)$, given that $\rho_h(\theta, 0) = 0$ for all θ . Since the hyperbolic tangent function is concave on $[0, \infty)$, and that a composition $\tanh \circ \varphi$ is necessarily concave for nondecreasing and concave φ , it will suffice to show that $\varphi(x) = h(\theta + x) - h(\theta - x)$ is nondecreasing and concave in x on $(0, m)$ and takes its values on $[0, \infty)$. Since h is nondecreasing on $(0, m)$ and even and convex on $(-2m, 2m)$, we have $\varphi(0) = 0$ and φ nondecreasing on $(0, m)$. Also, we have for (x_0, x_1) such that $0 < x_0 < x_1 < m$:

$$(i) \quad \varphi'(x_1) - \varphi'(x_0) = [\{h'(\theta + x_1) - h'(\theta + x_0)\} - \{h'(\theta - x_0) - h'(\theta - x_1)\}]$$

$$\leq 0 \quad \text{if } x_1 \leq \theta$$

and

$$\begin{aligned} \text{(ii)} \quad \varphi'(x_1) - \varphi'(x_0) &= [\{h'(x_1 + \theta) - h'(x_1 - \theta)\} - \{h'(x_0 + \theta) - h'(x_0 - \theta)\}] \\ &\leq 0 \quad \text{if } \theta \leq x_0, \end{aligned}$$

because h' is concave on $(0, 2m)$. Therefore, φ is concave on both the intervals $(0, \theta)$ and (θ, m) . Finally, φ is continuous at θ and $\varphi'(\theta-) = h'(2\theta) - h'(0-) \geq h'(2\theta) - h'(0+) = \varphi'(\theta+)$, implying that φ is concave on $(0, m)$.

PROOF OF THEOREM 1. Decomposing the difference in conditional risks for $\theta \in [0, m]$, we obtain

$$\begin{aligned} &E_\theta[|\theta - \delta_{\text{mle}}(X)|^2 - |\theta - \delta(X)|^2 \mid |X| > m] \\ &= E_\theta[\{(m - \delta(|X|)) \operatorname{sgn}(X)\} \{((m + \delta(|X|) \operatorname{sgn}(X)) - 2\theta)\} \mid |X| > m] \\ &= E_\theta[(m - \delta(|X|))(\delta(|X|) - \{2\theta\rho_h(\theta, |X|) - m\}) \mid |X| > m] \\ &\geq E_\theta[(m - \delta(|X|))(\delta(|X|) - \{2\theta\alpha_h(m, \theta) - m\}) \mid |X| > m] \\ &\geq E_\theta[(m - \delta(|X|))(\delta(|X|) - \{2m\bar{\alpha}_h(m) - m\}) \mid |X| > m] \\ &= E_\theta \left[(m - \delta(|X|)) \left(\delta(|X|) - \left\{ m \left(\frac{1 - 6F_{0,h}(-2m)}{1 + 2F_{0,h}(-2m)} \right) \right\} \right) \mid |X| > m \right] \\ &> 0, \end{aligned}$$

where **(i)** both the second equality and inequality exploit property (c) of Lemma 1 as well as the definitions of ρ_h and $\bar{\alpha}_h$, **(ii)** the first inequality holds with the increasing properties of $\rho_h(\theta, \cdot)$ (Lemma 1) and $\delta(\cdot)$, **(iii)** the strict inequality follows with the given bounds on δ , and **(iv)** the last equality follows from part (a) of Lemma 2.

PROOF OF THEOREM 2. Decomposing the difference in conditional risks for $\theta \in [0, m]$, we obtain

$$\begin{aligned} &E_\theta[|\theta - \delta_{\text{mle}}(X)|^2 - |\theta - \delta(X)|^2 \mid |X| \leq m] \\ &= E_\theta[\{(|X| - \delta(|X|)) \operatorname{sgn}(X)\} \{((|X| + \delta(|X|) \operatorname{sgn}(X)) - 2\theta)\} \mid |X| \leq m] \\ &= E_\theta \left[(|X| - \delta(|X|)) \left(\delta(|X|) - \left\{ 2\theta \frac{\rho_h(\theta, |X|)}{|X|} - 1 \right\} |X| \right) \mid |X| \leq m \right] \\ &\geq E_\theta[(|X| - \delta(|X|))(\delta(|X|) - \{2\beta_h(m, \theta) - 1\}|X|) \mid |X| \leq m] \\ &\geq E_\theta[(|X| - \delta(|X|))(\delta(|X|) - \{2\bar{\beta}_h(m) - 1\}|X|) \mid |X| \leq m] \\ &> 0, \end{aligned}$$

where **(i)** both the second equality and inequality exploit property (c) of Lemma 1 as well as the definitions of ρ_h and $\bar{\beta}_h$, **(ii)** the first inequality holds given that $\delta(x)$, $x(x - \delta(x))$ increase in x ; $x \in (0, m)$; and $\frac{\rho_h(\theta, x)}{x}$ decreases in x ; $x > 0$; (part (e) of Lemma 1), and **(iii)** the strict inequality follows with the given assumptions on δ .

PROOF OF THEOREM 4. It follows from Marchand and Perron (2005) (Lemma 2, Theorem 2) that a sufficient condition for a shrinkage estimator δ (i.e., $\|\delta\| \leq \|\delta_{\text{mle}}\|$) to dominate δ_{mle} is:

$$\frac{2m^2 r \gamma}{m^2 + r^2 + d} \leq m \wedge r, \quad \text{for all } r > 0, \quad \text{with } \gamma = 1 + \left(1 \vee \frac{d}{p}\right).$$

For $r \leq m$, the above requires: $\frac{2m^2 \gamma}{m^2 + d} \leq 1 \Leftrightarrow m^2 \leq \frac{d}{2\gamma - 1}$.

For $r > m$, we require:

$$\begin{aligned} \frac{2m^2 r \gamma}{m^2 + r^2 + d} \leq m \text{ for all } r > m &\Leftrightarrow \frac{2m^2 \gamma \sqrt{m^2 + d}}{m^2 + (m^2 + d) + d} \leq m \\ &\Leftrightarrow m^2 \leq \frac{d}{\gamma^2 - 1}. \end{aligned}$$

Finally, the result follows since $\gamma^2 - 1 \geq 2\gamma - 1$ given that $\gamma \geq 2$, and by verifying that $\frac{d}{\gamma^2 - 1} = \frac{p^2}{d + 2p} \wedge \frac{d}{3}$.

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