ON THE EFFICIENCIES OF SECOND ORDER RESPONSE SURFACE DESIGNS FOR THE ESTIMATION OF RESPONSES AND SLOPES USING SYMMETRICAL UNEQUAL BLOCK ARRANGEMENTS WITH TWO UNEQUAL BLOCK SIZES

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In this paper, the efficiencies of various second order response surface designs, like second order rotatable designs (SORD), second order slope rotatable designs (SOSRD), SORDs with an equi-spaced doses design, and SOSRDs with an equi-spaced doses design using symmetrical unequal block arrangements (SUBA) with two unequal block sizes, are studied for the estimation of responses and slopes at different points (central, axial, cube corner points) on second order response surface designs. The study is done because in some cases a SORD, and SOSRD constructed by using a SUBA with two unequal block sizes have fewer design points compared to those constructed by other methods.

Key words and phrases: Second order response surface designs, second order rotatable designs, second order slope rotatable designs.

1. Introduction

Box and Hunter (1957) introduced a series of response surface designs with the property that the variances of estimates of responses at points equi-distance from the centre of the design are all equal. They called these designs rotatable designs when the relationship between the response variable and several input variables is a quadratic or cubic or higher order polynomial. Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). Raghavarao (1963) constructed second order rotatable designs (SORD) using incomplete block designs. Dey (1970) studied response surface designs with equi-spaced doses.

The study of rotatable designs mainly emphasizes the estimation of the absolute response. Estimation of differences in responses at two different points in the factor space will often be of great importance. In the case of differences at two points close together, estimation of the local slope (rate of change) of the response is of interest. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate the rate of reaction in a chemical experiment, the rate of change in the yield of a crop to

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various fertilizer doses, the rate of disintegration of a radioactive material in an animal etc.

Hader and Park (1978) introduced slope rotatable central composite designs. Victorbabu and Narasimham (1991) constructed second order slope rotatable designs (SOSRD) using BIBD. Victorbabu *et al.* (1994) constructed SOSRDs with equi-spaced levels using incomplete block designs with unequal block sizes. Victorbabu (2002a) studied SOSRDs with equi-spaced levels. Victorbabu (2002b) constructed SOSRDs using symmetrical unequal block arrangements with two unequal block sizes. Victorbabu and Vasundharadevi (2004) studied the efficiencies of second order response surface designs for the estimation of responses and slopes using BIBD.

In this paper, the efficiencies of various second order response surface designs like SORDs, SOSRDs, SORDs with an equi-spaced doses design, and SOSRDs with an equi-spaced doses design using symmetrical unequal block arrangement with two unequal block sizes are studied for the estimation of responses and slopes at different points (central, axial, cube corner points) on second order response surface designs.

Some Preliminaries Regarding Response Surface Designs

A design for fitting a response surface consists of a number of suitable combinations of levels of several input factors. We shall use ' v ' for the number of factors and ' N ' for the number of combinations in the design, each factor having a constant number of levels. A response surface design can be written as v rows and N columns each. Each row is a combination of v -level codes, one from each of v -ordered factors. A combination of level codes is also called a design point. The combination with 0-code for each factor is called a central point.

Suppose we want to use the second order response surface design $D = ((x_{iu}))$ to fit the surface,

$$(1.1) \quad Y_u = b_0 + \sum_{i=1}^v b_i x_{iu} + \sum_{i=1}^v b_{ii} x_{iu}^2 + \sum_{i < j} b_{ij} x_{iu} x_{ju} + e_u$$

where x_{iu} denotes the level of the i -th factor ($i = 1, 2, \dots, v$) in the u -th run ($u = 1, 2, \dots, N$) of the experiment, and e_u 's are uncorrelated random errors with mean zero and variance σ^2 .

Here b_0, b_i, b_{ii}, b_{ij} are the parameters of the model and Y_u is the response observed at the u -th design point. The parameters in the response relation are estimated using the least squares technique. Further we impose the following simple symmetry conditions on the design points to simplify the solutions of the normal equations.

$$(1.2) \quad \sum_{u=1}^N \prod_{i=1}^v x_{iu}^{\alpha_i} = 0, \quad \text{if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4,$$

$$(1.3) \quad \begin{aligned} \text{(i)} \quad & \sum_{u=1}^N x_{iu}^2 = \text{constant} = N\lambda_2, \\ \text{(ii)} \quad & \sum_{u=1}^N x_{iu}^4 = \text{constant} = cN\lambda_4, \quad \text{for all } i, \end{aligned}$$

$$(1.4) \quad \sum_{u=1}^N x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4, \quad \text{for } i \neq j,$$

where c , λ_2 and λ_4 are constants. Using these symmetry conditions we obtain the estimates of the parameters as

$$\begin{aligned} \hat{b}_0 &= \frac{\lambda_4(c+v-1) \sum_u Y_u - \lambda_2 \sum_i (\sum_u x_{iu}^2 Y_u)}{N[\lambda_4(c+v-1) - v\lambda_2^2]}, \\ \hat{b}_i &= \frac{\sum_u x_{iu} Y_u}{N\lambda_2} \quad (i = 1, 2, \dots, v), \\ \hat{b}_{ij} &= \frac{\sum_u x_{iu} x_{ju} Y_u}{N\lambda_4} \quad (i \neq j = 1, 2, \dots, v), \\ \hat{b}_{ii} &= \frac{\sum_u x_{iu}^2 Y_u}{(c-1)N\lambda_4} - \frac{\lambda_2 \lambda_4 (c-1) \sum_u Y_u - \sum_i (\sum_u x_{iu}^2 Y_u)(\lambda_2^2 - \lambda_4)}{(c-1)N\lambda_4[\lambda_4(c+v-1) - v\lambda_2^2]}. \end{aligned}$$

Using these solutions, variances and covariances of these estimates are given below:

$$\begin{aligned} (1.5) \quad V(\hat{b}_0) &= \frac{\lambda_4(c+v-1)\sigma^2}{N[\lambda_4(c+v-1) - v\lambda_2^2]}, \\ V(\hat{b}_i) &= \frac{\sigma^2}{N\lambda_2}, \\ V(\hat{b}_{ij}) &= \frac{\sigma^2}{N\lambda_4}, \\ V(\hat{b}_{ii}) &= \frac{\sigma^2}{(c-1)N\lambda_4} \left[\frac{\lambda_4(c+v-2) - (v-1)\lambda_2^2}{\lambda_4(c+v-1) - v\lambda_2^2} \right], \\ \text{Cov}(\hat{b}_0, \hat{b}_{ii}) &= \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1) - v\lambda_2^2]}, \\ \text{Cov}(\hat{b}_{ii}, \hat{b}_{jj}) &= \frac{(\lambda_2^2 - \lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1) - v\lambda_2^2]}, \end{aligned}$$

and other covariances are zero.

An inspection of the variances shows that a necessary condition for the existence of a non-singular second order design is $\lambda_4(c+v-1) - v\lambda_2^2 > 0$, which leads to the condition

$$(1.6) \quad \frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1} \text{ (Non-singularity condition).}$$

Using these variances and covariances, the variance of the estimated response at any point can be obtained. Let $\hat{Y}_u$ denote the response at the point

$(x_{1u}, x_{2u}, \dots, x_{vu})$ as estimated using the response relation. Then

$$(1.7) \quad V(\hat{Y}_u) = V(\hat{b}_0) + d_u^2[V(\hat{b}_i) + 2\text{Cov}(\hat{b}_0, \hat{b}_{ii})] \\ + d_u^4 V(\hat{b}_{ii}) + \sum_{i \neq j} x_{iu}^2 x_{ju}^2 [(c-3)\sigma^2/(c-1)N\lambda_4].$$

From equation (1.7), it is clear that if $c = 3$ by a suitable choice of the design, the coefficient of $\sum x_{iu}^2 x_{ju}^2$ vanishes and the variance becomes a function of the distance ($d_u^2 = \sum_{i=1}^v x_{iu}^2$) (the distance of the u -th design point from the origin) only, viz.

$$(1.8) \quad V(\hat{Y}_u) = V(\hat{b}_0) + d_u^2[V(\hat{b}_i) + 2\text{Cov}(\hat{b}_0, \hat{b}_{ii})] + d_u^4 V(\hat{b}_{ii}).$$

Definition of Second Order Rotatable Design (SORD):

A second order response surface design D is said to be an SORD, if in this design $c = 3$, $\sum_{u=1}^N x_{iu}^4 = 3 \sum_{u=1}^N x_{iu}^2 x_{ju}^2$ and all the other conditions (1.2), (1.3), (1.4) and (1.6) hold. Hence the variance of the estimated response of Y_u from the fitted surface is only a function of the distance ($d^2 = \sum_{i=1}^v x_{iu}^2$) of the point $(x_{1u}, x_{2u}, \dots, x_{vu})$ from the origin (center) of the design (cf. Box and Hunter (1957)).

Slope Rotatable Designs:

For the second order response surface design given in equation (1.1), we have

$$(1.9) \quad \frac{\partial \hat{Y}_u}{\partial x_i} = \hat{b}_i + 2\hat{b}_{ii}x_{iu} + \sum_{j \neq i} \hat{b}_{ij}x_{ju}$$

and

$$(1.10) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4x_{iu}^2 V(\hat{b}_{ii}) + \sum_{j \neq i} x_{ju}^2 V(\hat{b}_{ij}).$$

The condition for the right hand side of equation (1.10) to be a function of the distance ($d_u^2 = \sum_{i=1}^v x_{iu}^2$) alone (for the slope rotatability) is

$$(1.11) \quad 4V(\hat{b}_{ii}) = V(\hat{b}_{ij}).$$

Therefore, conditions (1.2) to (1.6) and (1.11) lead to the condition

$$(1.12) \quad [v(5-c) - (c-3)^2]\lambda_4 + [v(c-5) + 4]\lambda_2^2 = 0.$$

Definition of Second Order Slope Rotatable Design (SOSRD):

A second order response surface design D is said to be an SOSRD if the variance of the estimate of the first order partial derivative ($\partial \hat{Y}_u / \partial x_i$) with respect to each of the independent variables (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^v x_{iu}^2$) of the point $(x_{1u}, x_{2u}, \dots, x_{vu})$ from the origin (center) of the design. Such a spherical variance function for the estimation of slopes in the second order response surface is achieved if the design points satisfy the conditions (1.2), (1.3), (1.4), (1.6) and (1.12) (cf. Hader and Park (1978), Victorbabu and Narasimham (1991)).

SUBA with two unequal block sizes (cf. Raghavarao, 1962):

Let v -treatments $1, 2, \dots, v$ satisfy the following relations:

(a) Any two treatments are either the 1st or 2nd block associates, the relation of the block association being symmetrical, i.e., if treatment θ is the i -th block associate of ϕ , then ϕ is the i -th block associate of θ .

(b) Each treatment θ has μ_i , i -th block associates, the number μ_i being independent of θ .

(c) If any two treatments θ and ϕ are i -th block associates, then the number of treatments which are j -th block associates of θ and k -th block associates of ϕ is p_{ijk} , and is independent of the pair of i -th block associates θ and ϕ . Also $p_{ijk} = p_{ikj}$.

Definition of SUBA with two unequal block sizes:

The arrangement of v -treatments, satisfying the above conditions, in b blocks where b_1 blocks of size k_1 and b_2 blocks of size k_2 ($b_1 + b_2 = b$), is said to be a symmetrical unequal block arrangement with two unequal block sizes if

- (i) every treatment occurs in $\frac{b_i k_i}{v}$ blocks of size k_i ($i = 1, 2$), and
- (ii) every pair of first associate treatments occurs together in u blocks of size k_1 and in $(\lambda - u)$ blocks of size k_2 , while every pair of second associate treatments occurs together in λ blocks of size k_2 .

From (i) each treatment occurs in $(\frac{b_1 k_1}{v}) + (\frac{b_2 k_2}{v}) = r$ blocks in all. $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ are known as the parameters of the SUBA with two unequal block sizes.

Let there be a SUBA with two unequal block sizes with the parameters $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ and $k = \sup(k_1, k_2)$. Let us write the design in the form of a $b \times v$ matrix, the elements of which are zero and α . If in any block a particular treatment occurs, the element in that block corresponding to that treatment will be α , otherwise, zero. We denote these design points generated from the transpose of the incidence matrix of an SUBA with two unequal block sizes by $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]$. Let $2^{t(k)}$ denote the number of design points of a fractional factorial design of 2^k in ± 1 levels, such that no interaction with less than five factors is confounded. $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from the SUBA with two unequal block sizes by ‘multiplication’ (cf. Raghavarao (1971), pp. 298–300), $(\pm\beta, 0, 0, \dots, 0)2^1$ denotes the design points generated from $(\pm\beta, 0, 0, \dots, 0)$ point set, $\cup$ denotes the union of the design points generated from different sets of points and (n_0) is the number of central points in the design.

RESULT. Construction of SOSRD using a SUBA with two unequal block sizes (Victorbabu (2002b)): Consider the SUBA $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ with two unequal block sizes. Then the design points $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup (\beta, 0, 0, \dots, 0)2^1 \cup (n_0)$ give a v -dimensional SOSRD in $N = b2^{t(k)} + 2v + n_0$ design points, where $(\beta/\alpha)^2$ is a positive real root of the equation

$$\begin{aligned}
(1.13) \quad & (8v - 4N)(\beta/\alpha)^8 + 8vr2^{t(k)}(\beta/\alpha)^6 \\
& + [2vr^22^{2t(k)} + \{((12 - 2v)\lambda - 4r)N \\
& \quad + (16\lambda - 20v\lambda + 4vr)\}2^{t(k)}](\beta/\alpha)^4 \\
& + [4vr^2 + (16 - 20v)r\lambda]2^{2t(k)}(\beta/\alpha)^2 \\
& + [(5v - 9)\lambda^2 + (6 - v)r\lambda - r^2]N2^{2t(k)} \\
& + (vr + 4\lambda - 5v\lambda)r^22^{3t(k)} = 0.
\end{aligned}$$

If at least one positive real root exists for equation (1.13) then the design exists.

Equi-spaced doses designs:

The SORD/SOSRD is not generally available with equi-spaced doses (levels). Though designs with equi-spaced levels are not necessary, they are likely to be preferred in view of the ease in handling the doses. Further in SORD/SOSRD, the calculation of the actual doses often requires approximations and hence actual dose levels can not be applied in practice. For example, in Agriculture experiments, the factors may have equi-spaced doses. The response surface designs with v -factors each having equi-spaced doses may be obtained by using a SUBA with two unequal block sizes as follows:

Consider the following sets of points: (i) $b2^{t(k)}$ points obtained through the incidence matrix of a SUBA with two unequal block sizes $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)}$, (ii) $2v$ axial points $(\pm\beta, 0, 0, \dots, 0)$ etc., The corresponding equi-spaced doses design is obtained by changing the axial points from $(\pm\beta, 0, 0, \dots, 0)$ etc., to $(\pm 2\alpha, 0, 0, \dots, 0)$ —repeated ‘ y ’ times and (iii) n_0 -central points. Thus we note the levels of factors used in these design points are $-2\alpha, -\alpha, 0, \alpha, 2\alpha$.

2. Estimation of responses and slopes using a SUBA with two unequal block sizes

Consider a SUBA $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ with two unequal block sizes and $k = \sup(k_1, k_2)$. Then the design points $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup (\beta, 0, 0, \dots, 0)2^1 \cup (n_0)$ give a v -dimensional SORD/SOSRD in $N = b2^{t(k)} + 2v + n_0$ design points. The unknown levels are obtained by using conditions (1.2) to (1.6), $\sum x_{iu}^4 = 3 \sum x_{iu}^2 x_{ju}^2$ for Box and Hunter (1957) rotatability, and (1.2) to (1.6), (1.12) for Hader and Park (1978) slope rotatability.

2.1. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in SORD at the central point

From equations (1.3) and (1.4), we have

$$(2.1) \quad \sum x_{iu}^2 = r2^{t(k)}\alpha^2 + 2\beta^2 = N\lambda_2,$$

$$(2.2) \quad \sum x_{iu}^4 = r2^{t(k)}\alpha^4 + 2\beta^4 = 3N\lambda_4,$$

$$(2.3) \quad \sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)}\alpha^4 = N\lambda_4.$$

Taking $\lambda_2 = 1$ (called a scaling condition for comparison of two designs for the same 'v') and on simplification, we get

$$(2.4) \quad \alpha = \sqrt{\frac{N}{r2^{t(k)} + 2^{(t(k)+1)/2}(3\lambda - r)^{1/2}}},$$

$$(2.5) \quad \beta^4 = 2^{t(k)-1}(3\lambda - r)\alpha^4,$$

$$(2.6) \quad V(\hat{Y}_u) = V(\hat{b}_0) = \frac{\lambda_4(c + v - 1)\sigma^2}{N[\lambda_4(c + v - 1) - v]} \quad (\text{from equation (1.8)}).$$

Simplifying equation (2.6) by taking $c = 3$, we get

$$(2.7) \quad V(\hat{Y}_u) = \frac{\lambda_4(v + 2)\sigma^2}{N[\lambda_4(v + 2) - v]},$$

$$(2.8) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) = \frac{\sigma^2}{N} \quad (\text{from equation (1.10)}).$$

2.2. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in SOSRD at the central point

The estimate of the slope in the second order model and its variances are obtained by equations (1.9) and (1.10). The required levels α, β in SOSRD using a SUBA with two unequal block sizes can be obtained as follows (cf. Victorbabu (2002b)):

From the conditions (1.3) and (1.4), we have

$$(2.9) \quad \sum x_{iu}^2 = r2^{t(k)}\alpha^2 + 2\beta^2 = N\lambda_2,$$

$$(2.10) \quad \sum x_{iu}^4 = r2^{t(k)}\alpha^4 + 2\beta^4 = cN\lambda_4,$$

$$(2.11) \quad \sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)}\alpha^4 = N\lambda_4.$$

From equations (2.10) and (2.11), we get $c = \frac{r2^{t(k)}\alpha^4 + 2\beta^4}{\lambda 2^{t(k)}\alpha^4}$. Substituting for λ_2, λ_4 and c in the slope rotatability condition (1.12) and on simplification, we get the biquadratic equation given in equation (1.13). We obtain $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ at the central point as follows:

$$(2.12) \quad V(\hat{Y}_u) = V(\hat{b}_0) = \frac{\lambda_4(c + v - 1)\sigma^2}{N[\lambda_4(c + v - 1) - v\lambda_2^2]} \\ = \frac{\lambda_4(c + v - 1)\sigma^2}{N[\lambda_4(c + v - 1) - v]} \quad (\text{taking } \lambda_2 = 1),$$

$$(2.13) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2} = \frac{\sigma^2}{N}.$$

2.3. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in an equi-spaced doses SORD at the central point

Following Dey (1970), an equi-spaced doses second order rotatable design (EDSORD) using a SUBA with two unequal block sizes can be obtained by

considering the following design points, (i) $b2^{t(k)}$ points obtained through the incidence matrix of a SUBA with two unequal block sizes, (ii) $2v$ axial points viz., $(\pm 2\alpha, 0, 0, \dots, 0)$ etc., repeated ‘ y ’ times and (iii) n_0 -central points. Then the design points $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup y(\pm 2\alpha, 0, 0, \dots, 0)2^1 \cup (n_0)$ give a v -dimensional SORD with equi-spaced levels in $N = b2^{t(k)} + 2yv + n_0$ design points. These design points are determined so that they satisfy conditions (1.2), (1.3) and (1.4). From equations (1.3) and (1.4), we have

$$(2.14) \quad \sum x_{iu}^2 = r2^{t(k)}\alpha^2 + 8y\alpha^2 = N\lambda_2,$$

$$(2.15) \quad \sum x_{iu}^4 = r2^{t(k)}\alpha^4 + 32y\alpha^4 = 3N\lambda_4,$$

$$(2.16) \quad \sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)}\alpha^4 = N\lambda_4.$$

From equations (2.15) and (2.16), we get $y = (3\lambda - r)2^{t(k)-5}$. Taking $\lambda_2 = 1$ in equation (2.14), we can find α . We thus get the variances of estimates of responses and slopes at the central point as $V(\hat{Y}_u) = V(\hat{b}_0)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i}) = V(\hat{b}_i)$.

2.4. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in an equi-spaced doses SOSRD at the central point

Following Victorbabu *et al.* (1994) and Victorbabu (2002a), an equi-spaced doses second order slope rotatable design (EDSOSRD) can be obtained by considering the following design points, (i) $b2^{t(k)}$ points obtained through the incidence matrix of an SUBA with two unequal block sizes, (ii) $2v$ axial points, viz., $(\pm 2\alpha, 0, 0, \dots, 0)$ -repeated ‘ y ’-times, and (iii) n_0 -central points. Then the design points $[\alpha - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{t(k)} \cup y(\pm 2\alpha, 0, 0, \dots, 0)2^1 \cup (n_0)$ give a v -dimensional SOSRD with equi-spaced levels in $N = b2^{t(k)} + 2yv + n_0$ design points, if

$$(2.17) \quad y = (c\lambda - r)2^{t(k)-5},$$

$$(2.18) \quad n_0 = \frac{(c\lambda + 3r)^2[v(5 - c) - 4]2^{t(k)-4}}{\lambda[v(5 - c) - (c - 3)^2]} - (b2^{t(k)} + 2yv)$$

and n_0 turns out to be an integer.

For the design points generated from SUBA with two unequal block sizes, conditions (1.2), (1.3), and (1.4) hold. Hence,

$$(2.19) \quad \sum x_{iu}^2 = r2^{t(k)}\alpha^2 + 8y\alpha^2 = N\lambda_2,$$

$$(2.20) \quad \sum x_{iu}^4 = r2^{t(k)}\alpha^4 + 32y\alpha^4 = cN\lambda_4,$$

$$(2.21) \quad \sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)}\alpha^4 = N\lambda_4.$$

We have $\sum x_{iu}^4 = c \sum x_{iu}^2 x_{ju}^2$ which leads to ‘ y ’ given in (2.17). From the slope rotatability condition (1.12), we get n_0 . The value of ‘ α ’ can be obtained from the scaling condition. We thus get the variances of estimates of responses and slopes at the central point as $V(\hat{Y}_u) = V(\hat{b}_0)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i}) = V(\hat{b}_i)$.

2.5. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in SORD at axial points

Here we evaluate $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ at the axial points in SORD as follows:

$$(2.22) \quad V(\hat{Y}_u) = V(\hat{b}_0) + p^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + p^4V(\hat{b}_{ii}),$$

$$(2.23) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4p^2V(\hat{b}_{ii}),$$

where in the axial point $x_i = \pm p$, other coordinates are zero.

2.6. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in SOSRD at axial points

Here variances of estimates of responses and slopes at axial points like $(\pm p, 0, \dots, 0), (0, \pm p, 0, \dots, 0), \dots, (0, 0, \dots, \pm p)$ are obtained as follows:

$$(2.24) \quad V(\hat{Y}_u) = V(\hat{b}_0) + p^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + p^4V(\hat{b}_{ii}),$$

$$(2.25) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = \frac{1}{N} \left[\frac{\lambda_4 + \lambda_2 p^2}{\lambda_2 \lambda_4} \right] \sigma^2 = \frac{1}{N} \left[1 + \frac{p^2}{\lambda_4} \right] \sigma^2 \quad (\text{taking } \lambda_2 = 1).$$

2.7. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ for an equi-spaced doses SORD at axial points

Here we find the variances of estimates of responses and slopes as follows:

$$(2.26) \quad V(\hat{Y}_u) = V(\hat{b}_0) + p^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + p^4V(\hat{b}_{ii}),$$

$$(2.27) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4x_{iu}^2 V(\hat{b}_{ii}) + \sum_{j \neq i} x_{ju}^2 V(\hat{b}_{ij}),$$

at $(0, 0, \dots, \pm p, 0, 0)$ with $x_i = \pm p$.

2.8. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ for an equi-spaced doses SOSRD at axial points

Here we find the variances of estimates of responses and slopes as follows:

$$(2.28) \quad V(\hat{Y}_u) = V(\hat{b}_0) + p^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + p^4V(\hat{b}_{ii}),$$

$$(2.29) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4p^2V(\hat{b}_{ii}) \\ = \frac{\sigma^2}{N\lambda_2} + \frac{4p^2\sigma^2}{(c-1)N\lambda_4} \left[1 + \frac{(\lambda_2^2 - \lambda_4)}{\lambda_4(c+v-1) - v\lambda_2^2} \right] \\ = \frac{\sigma^2}{N} + \frac{4p^2\sigma^2}{(c-1)N\lambda_4} \\ \times \left[1 + \frac{(1 - \lambda_4)}{\lambda_4(c+v-1) - v} \right] \quad (\text{taking } \lambda_2 = 1).$$

2.9. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in SORD at cube corner points

Here we evaluate $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ at cube corner points like $(\pm p, \pm p, \dots, \pm p)$ in SORD as follows:

$$(2.30) \quad V(\hat{Y}_u) = V(\hat{b}_0) + vp^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + v^2p^4V(\hat{b}_{ii}),$$

$$(2.31) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4p^2V(\hat{b}_{ii}).$$

2.10. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in SOSRD at cube corner points

Here we obtain $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ at cube corner points in SOSRD as follows:

$$(2.32) \quad V(\hat{Y}_u) = V(\hat{b}_0) + p^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + p^4V(\hat{b}_{ii}),$$

$$(2.33) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = \frac{1}{N} \left[\frac{\lambda_4 + \lambda_2 p^2}{\lambda_2 \lambda_4} \right] \sigma^2 = \frac{1}{N} \left[1 + \frac{p^2}{\lambda_4} \right] \sigma^2 \quad (\text{taking } \lambda_2 = 1).$$

2.11. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in equi-spaced doses SORD at cube corner points

Here we find the variances of estimates of responses and slopes at cube corner points.

$$(2.34) \quad V(\hat{Y}_u) = V(\hat{b}_0) + p^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + p^4V(\hat{b}_{ii}),$$

$$(2.35) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4x_{iu}^2 V(\hat{b}_{ii}) + \sum_{j \neq i} x_{ju}^2 V(\hat{b}_{ij})$$

$$= \frac{\sigma^2}{N\lambda_2} + \frac{p^2\sigma^2}{N\lambda_4}$$

$$\times \left[\frac{4}{c-1} \left\{ \frac{\lambda_4(c+v-2) - (v-1)\lambda_2^2}{\lambda_4(c+v-1) - v\lambda_2^2} \right\} + (v-1) \right]$$

$$= \frac{\sigma^2}{N} + \frac{p^2\sigma^2}{N\lambda_4} \left[\frac{4}{c-1} \left\{ \frac{\lambda_4(c+v-2) - (v-1)}{\lambda_4(c+v-1) - v} \right\} + (v-1) \right]$$

(taking $\lambda_2 = 1$).

2.12. $V(\hat{Y}_u)$ and $V(\frac{\partial \hat{Y}_u}{\partial x_i})$ in equi-spaced doses SOSRD at cube corner points

Here we find the variances of estimates of responses and slopes at cube corner points.

$$(2.36) \quad V(\hat{Y}_u) = V(\hat{b}_0) + vp^2[V(\hat{b}_i) + 2 \text{Cov}(\hat{b}_0, \hat{b}_{ii})] + v^2p^4V(\hat{b}_{ii}),$$

$$(2.37) \quad V\left(\frac{\partial \hat{Y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4p^2V(\hat{b}_{ii}).$$

3. Comparison of efficiencies of different second order designs for the estimation of responses and slopes

The variances of estimates of responses and slopes at (i) Central point $(0, 0, \dots, 0)$, (ii) Axial points $(\pm p, 0, \dots, 0), (0, \pm p, 0, \dots, 0), \dots, (0, 0, \dots, \pm p)$ and (iii) Cube Corner points $(\pm p, \pm p, \dots, \pm p)$ are computed for $6 \leq v \leq 15$ using an SUBA with two unequal block sizes in the second order response surface design. The results are given in Tables 1 and 2 in the Appendix.

The variances of estimates of responses $\hat{Y}_u$ and slopes $\frac{\partial \hat{Y}_u}{\partial x_i}$ are further computed for special values of $p = 0.25, 0.50, 1.00, 2.00$ in the case of axial and cube corner points besides the central points in all the above designs, and the numerical results regarding the comparative efficiencies of the above designs are given in Table 2 in the Appendix.

On the basis of this analysis certain broad conclusions are drawn regarding the efficiencies of the above designs for the estimation of responses and slopes, and certain practical recommendations are made in this connection regarding the choice of a suitable design (design with lower variance) for the estimation of responses and slopes at central, axial and cube corner points.

4. Conclusion

There is no uniform best design for the estimation of responses and slopes at central points, axial points and cube corner points on the second order response surface for different values of v and p . However, the following broad conclusions can be drawn from the analysis of Table 2 in the Appendix.

1. For the estimation of responses at the central point, axial points and cube corner points, SOSRD using a SUBA with two unequal block sizes seems to be the preferable design.

2. For the estimation of slopes, (i) at the central points all designs are equally good, (ii) at axial points, EDSOSRD using a SUBA with two unequal block sizes seems to be generally the preferable design, and (iii) at cube corner points, SOSRD using an SUBA with two unequal block sizes seems to be generally the preferable design.

Thus for simultaneous estimation of responses and slopes at central, axial and cube corner points, EDSOSRD using an SUBA with two unequal block sizes seems to be generally the preferable design from this limited study.

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Appendix

Table 1. Variance of estimates of responses and slopes in different second order designs.

Number of factors v'	Designs	$\pm\alpha$	$\pm\beta$	At central points		At axial points		At cube corner points	
				V_1	V_2	V_1	V_2	V_1	V_2
6 (6, 7, 2, 3, 4, 1)	SORD	1.6956	—	517	19	667 - 29p ² + 77p ⁴	144 + 311p ²	667 - 174p ² + 2791p ⁴	144 + 311p ²
	SOSRD	1.4446	3.0752	1521	145	1521 - 314p ² + 72p ⁴	145 + 287p ²	1521 - 314p ² + 72p ⁴	145 + 287p ²
	EDSORD	—	—	—	—	—	—	—	—
	EDSOSRD	—	—	—	—	—	—	—	—
6 (6, 11, 7, 3, 4, 2, 9, 4)	SORD	1.2313	3.0967	1444	53	1444 - 411p ² + 67p ⁴	53 + 207p ²	1444 - 2466p ² + 2411p ⁴	53 + 207p ²
	SOSRD	1.1811	4.0474	299	52	299 - 29p ² + 21p ⁴	52 + 81p ²	299 - 29p ² + 21p ⁴	53 + 81p ²
	EDSORD	—	—	—	—	—	—	—	—
	EDSOSRD	—	—	—	—	—	—	—	—
6 (6, 15, 9, 2, 4, 3, 12, 5)	SORD	1.2659	3.3323	516	39	516 - 116p ² + 33p ⁴	39 + 134p ²	516 - 715p ² + 1206p ⁴	39 + 134p ²
	SOSRD	1.2232	4.3316	188	39	188 - 9p ² + 13p ⁴	39 + 56p ²	188 - 9p ² + 13p ⁴	39 + 56p ²
	EDSORD	1.2841	—	806	36	806 - 221p ² + 41p ⁴	36 + 162p ²	806 - 221p ² + 41p ⁴	36 + 162p ²
	EDSOSRD	—	—	—	—	—	—	—	—
6 (6, 15, 8, 2, 4, 6, 9, 4)	SORD	1.3476	3.2052	391	39	391 - 77p ² + 29p ⁴	39 + 118p ²	391 - 466p ² + 106p ⁴	39 + 118p ²
	SOSRD	1.3003	4.2747	178	39	178 - 6p ² + 13p ⁴	39 + 54p ²	178 - 6p ² + 13p ⁴	39 + 54p ²
	EDSORD	1.3565	—	454	39	454 - 101p ² + 31p ⁴	37 + 123p ²	454 - 101p ² + 31p ⁴	37 + 123p ²
	EDSOSRD	1.4142	—	143	26	143 - 13p ² + 9p ⁴	26 + 39p ²	143 - 78p ² + 351p ⁴	26 + 351p ²
8 (8, 24, 9, 2, 4, 12, 12, 3)	SORD	1.6688	—	181	25	181 - 14p ² + 14p ⁴	25 + 57p ²	181 - 111p ² + 907p ⁴	25 + 57p ²
	SOSRD	1.6687	—	156	24	156 - 7p ² + 8p ⁴	24 + 33p ²	156 - 7p ² + 8p ⁴	24 + 33p ²
	EDSORD	—	—	—	—	—	—	—	—
	EDSOSRD	1.5811	—	243	21	243 - 35p ² + 8p ⁴	208 + 333p ²	243 - 277p ² + 533p ⁴	21 + 533p ²
9 (9, 12, 7, 3, 6, 3, 9, 4)	SORD	1.2907	3.8602	345	25	345 - 46p ² + 16p ⁴	25 + 66p ²	345 - 416p ² + 1333p ⁴	25 + 66p ²
	SOSRD	1.2907	3.8602	155	25	155 - 4p ² + 7p ⁴	25 + 31p ²	155 - 4p ² + 7p ⁴	25 + 31p ²
	EDSORD	1.3413	—	338	21	338 - 49p ² + 15p ⁴	21 + 58p ²	338 - 49p ² + 15p ⁴	21 + 58p ²
	EDSOSRD	—	—	—	—	—	—	—	—
9 (9, 15, 6, 3, 4, 6, 9, 2)	SORD	1.6425	—	428	39	428 - 47p ² + 23p ⁴	39 + 96p ²	428 - 431p ² + 1934p ⁴	39 + 96p ²
	SOSRD	1.5266	4.2163	471	38	471 - 57p ² + 14p ⁴	38 + 57p ²	471 - 57p ² + 14p ⁴	38 + 57p ²
	EDSORD	—	—	—	—	—	—	—	—
	EDSOSRD	1.6425	—	28	31	28 - 24p ² + 11p ⁴	31 + 43p ²	28 - 221p ² + 862p ⁴	31 + 862p ²
9 (9, 15, 7, 3, 5, 6, 9, 3)	SORD	1.4691	2.9382	738	38	738 - 116p ² + 28p ⁴	38 + 114p ²	738 - 1052p ² + 2309p ⁴	38 + 114p ²
	SOSRD	1.4082	4.2943	338	38	338 - 28p ² + 13p ⁴	38 + 53p ²	338 - 28p ² + 13p ⁴	38 + 53p ²
	EDSORD	1.4691	—	738	38	738 - 116p ² + 28p ⁴	38 + 114p ²	738 - 116p ² + 28p ⁴	38 + 114p ²
	EDSOSRD	1.7321	—	75	23	75 - 11p ² + 5p ⁴	23 + 23p ²	75 - 104p ² + 468p ⁴	23 + 468p ²
10 (10, 11, 5, 4, 5, 5, 6, 2)	SORD	1.5165	2.5505	1686	51	1686 - 276p ² + 43p ⁴	51 + 172p ²	1686 - 2763p ² + 4294p ⁴	51 + 172p ²
	SOSRD	1.4271	4.1281	781	51	781 - 95p ² + 19p ⁴	51 + 75p ²	781 - 95p ² + 19p ⁴	51 + 75p ²
	EDSORD	—	—	—	—	—	—	—	—
	EDSOSRD	—	—	—	—	—	—	—	—
10 (10, 15, 8, 4, 6, 5, 10, 4)	SORD	1.3572	3.8386	517	19	517 - 79p ² + 15p ⁴	19 + 61p ²	517 - 794p ² + 1533p ⁴	19 + 61p ²
	SOSRD	1.3226	5.1574	161	19	161 - 8p ² + 6p ⁴	19 + 25p ²	161 - 8p ² + 6p ⁴	19 + 25p ²
	EDSORD	1.3957	—	476	17	476 - 74p ² + 14p ⁴	17 + 55p ²	476 - 74p ² + 14p ⁴	17 + 55p ²
	EDSOSRD	1.6383	—	36	11	36 - 5p ² + 2p ⁴	1033 + 1033p ²	36 - 52p ² + 256p ⁴	11 + 256p ²
12 (12, 13, 7, 3, 8, 4, 9, 4)	SORD	1.3456	4.7858	93	12	93 - 1p ² + 6p ⁴	12 + 24p ²	93 - 24p ² + 86p ⁴	11 + 24p ²
	SOSRD	1.3231	6.0309	62	12	62 + 3p ² + 3p ⁴	12 + 13p ²	62 + 3p ² + 3p ⁴	12 + 13p ²
	EDSORD	1.4255	—	72	9	72 - 1p ² + 5p ⁴	9 + 19p ²	72 - 1p ² + 5p ⁴	9 + 19p ²
	EDSOSRD	1.6007	—	24	6	24 + 3p ² + 1p ⁴	594 + 594p ²	24 - 35p ² + 214p ⁴	6 + 214p ²
15 (15, 16, 6, 5, 6, 6, 10, 2)	SORD	1.6817	—	287	18	287 - 17p ² + 11p ⁴	18 + 41p ²	287 - 262p ² + 2321p ⁴	18 + 41p ²
	SOSRD	1.5968	5.1674	441	18	441 - 38p ² + 4p ⁴	18 + 16p ²	441 + 38p ² + 4p ⁴	18 + 16p ²
	EDSORD	—	—	—	—	—	—	—	—
	EDSOSRD	1.8708	—	61	13	61 - 6p ² + 3p ⁴	13 + 13p ²	61 - 95p ² + 717p ⁴	13 + 717p ²

Note: $V_1 = V(\hat{Y}_u)10^4/\sigma^2$, $V_2 = V(\partial\hat{Y}_u/\partial x_i)10^4/\sigma^2$.

Table 2. Designs with lowest variances for the estimation of responses and slopes.

Number of factors 'v'	p	For estimation of responses			For estimation of slopes		
		Central points	Axial points	Cube corner points	Central points	Axial points	Cube corner points
6 (6, 7, 3, 2, 3, 3, 4, 1)	0.25		SORD	SOSRD		SOSRD	SOSRD
	0.50	all designs are equally efficient	SORD	SOSRD	all designs are equally efficient	SOSRD	SOSRD
	1.00		SORD	SOSRD		SOSRD	SOSRD
	2.00		SOSRD	SOSRD		SOSRD	SOSRD
6 (6, 11, 7, 3, 4, 2, 9, 4)	0.25		SOSRD	SOSRD		SOSRD	SOSRD
	0.50	all designs are equally efficient	SOSRD	SOSRD	all designs are equally efficient	SOSRD	SOSRD
	1.00		SOSRD	SOSRD		SOSRD	SOSRD
	2.00		SOSRD	SOSRD		SOSRD	SOSRD
6 (6, 15, 9, 2, 4, 3, 1, 2, 5)	0.25		SOSRD	SOSRD		SOSRD	SOSRD
	0.50	all designs are equally efficient	SOSRD	SOSRD	all designs are equally efficient	SOSRD	SOSRD
	1.00		SOSRD	SOSRD		SOSRD	SOSRD
	2.00		SOSRD	SOSRD		SOSRD	SOSRD
6 (6, 15, 8, 2, 4, 6, 9, 4)	0.25		EDSOSRD	SOSRD		EDSOSRD	SOSRD
	0.50	all designs are equally efficient	EDSOSRD	EDSOSRD	all designs are equally efficient	EDSOSRD	SOSRD
	1.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
8 (8, 24, 9, 2, 4, 12, 12, 3)	0.25		SOSRD	SOSRD		EDSOSRD	SOSRD
	0.50	all designs are equally efficient	SOSRD	SOSRD	all designs are equally efficient	SOSRD	SOSRD
	1.00		SOSRD	SOSRD		EDSOSRD	SOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
9 (9, 12, 7, 3, 6, 3, 9, 4)	0.25		SOSRD	SOSRD		EDSORD	EDSOSRD
	0.50	all designs are equally efficient	SOSRD	SOSRD	all designs are equally efficient	SOSRD	SOSRD
	1.00		SOSRD	SOSRD		SOSRD	SOSRD
	2.00		SOSRD	SOSRD		SOSRD	SOSRD
9 (9, 15, 6, 3, 4, 6, 9, 2)	0.25		EDSOSRD	EDSOSRD		EDSOSRD	EDSOSRD
	0.50	all designs are equally efficient	EDSOSRD	EDSOSRD	all designs are equally efficient	EDSOSRD	EDSOSRD
	1.00		EDSOSRD	SOSRD		EDSOSRD	EDSOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	EDSOSRD
9 (9, 15, 7, 3, 5, 6, 9, 3)	0.25		EDSOSRD	EDSOSRD		EDSOSRD	SOSRD
	0.50	all designs are equally efficient	EDSOSRD	EDSOSRD	all designs are equally efficient	EDSOSRD	SOSRD
	1.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	EDSORD
10 (10, 11, 5, 4, 5, 5, 6, 2)	0.25		SOSRD	SOSRD		SOSRD	SOSRD
	0.50	all designs are equally efficient	SOSRD	SOSRD	all designs are equally efficient	SOSRD	SOSRD
	1.00		SOSRD	SOSRD		SOSRD	SOSRD
	2.00		SOSRD	SOSRD		SOSRD	SOSRD
10 (10, 15, 8, 4, 6, 5, 10, 4)	0.25		EDSOSRD	EDSOSRD		EDSOSRD	EDSORD
	0.50	all designs are equally efficient	EDSOSRD	EDSOSRD	all designs are equally efficient	EDSOSRD	SOSRD
	1.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
12 (12, 13, 7, 3, 8, 4, 9, 4)	0.25		EDSOSRD	EDSOSRD		EDSOSRD	SOSRD
	0.50	all designs are equally efficient	EDSOSRD	EDSOSRD	all designs are equally efficient	EDSOSRD	SOSRD
	1.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	SOSRD
15 (15, 16, 6, 5, 6, 6, 10, 2)	0.25		EDSOSRD	EDSOSRD		EDSOSRD	EDSOSRD
	0.50	all designs are equally efficient	EDSOSRD	EDSOSRD	all designs are equally efficient	EDSOSRD	EDSOSRD
	1.00		EDSOSRD	SOSRD		EDSOSRD	EDSOSRD
	2.00		EDSOSRD	SOSRD		EDSOSRD	EDSOSRD

REFERENCES

- Box, G. E. P. and Hunter, J. S. (1957). Multifactor experimental designs for exploring response surfaces, *Annals of Mathematical Statistics*, **28**, 195–241.
- Das, M. N. and Narasimham, V. L. (1962). Construction of rotatable designs through balanced incomplete block designs, *Annals of Mathematical Statistics*, **33**, 1421–1439.
- Dey, A. (1970). On response surface designs with equi-spaced doses, *Calcutta Statistical Association Bulletin*, **19**, 135–144.
- Hader, R. J. and Park, S. H. (1978). Slope rotatable central composite designs, *Technometrics*, **20**, 413–417.
- Raghavarao, D. (1962). Symmetrical unequal block arrangements with two unequal block sizes, *Annals of Mathematical Statistics*, **33**, 620–633.
- Raghavarao, D. (1963). Construction of second order rotatable designs through incomplete block designs, *Journal of Indian Statistical Association*, **1**, 221–225.
- Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments*, John Wiley, New York.
- Victorbabu, B. Re. (2002a). Second order slope rotatable designs with equi-spaced levels, *Proceedings of Andhra Pradesh Akademi of Sciences*, **6**, 211–214.
- Victorbabu, B. Re. (2002b). Construction of second order slope rotatable designs using symmetrical unequal block arrangements with two unequal block sizes, *Journal of the Korean Statistical Society*, **31**, 153–161.
- Victorbabu, B. Re. (2007). On second order slope rotatable designs - A review, *Journal of the Korean Statistical Society*, **36**, 373–386.
- Victorbabu, B. Re. and Narasimham, V. L. (1991). Construction of second order slope rotatable designs through balanced incomplete block designs, *Communications in Statistics - Theory and Methods*, **20**, 2467–2478.
- Victorbabu, B. Re. and Vasundharadevi, V. (2004). On the efficiency of second order response surface designs for estimation of responses and slopes using balanced incomplete block designs, *Statistical Methods*, **6**, 210–224.
- Victorbabu, B. Re., Narayanarao, E. S. V. and Narasimham, V. L. (1994). Construction of second order slope rotatable designs with equi-spaced levels using incomplete block designs with unequal block sizes, *Proceedings of the XV Indian Society for Probability and Statistics Conference, M.S. University, Tirunelveli-627 009, India*, 116–119.