IMPROVED APPROXIMATIONS FOR THE DISTRIBUTIONS OF MULTINOMIAL GOODNESS-OF-FIT STATISTICS BASED ON ϕ -DIVERGENCE UNDER NONLOCAL ALTERNATIVES

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Zografos *et al.* (1990) introduced the ϕ -divergence family of statistics C_ϕ to the goodness-of-fit test. The ϕ -divergence family of statistics C_ϕ includes the power divergence family of statistics proposed by Cressie and Read (Cressie and Read (1984) and Read and Cressie (1988)) as a special case. Sekiya and Taneichi (2004) derived the multivariate Edgeworth expansion assuming a continuous distribution for the distributions of power divergence statistics under a nonlocal alternative hypothesis. In this paper, we consider an expansion for the family of general ϕ -divergence statistics C_ϕ . We derive the multivariate Edgeworth expansion assuming a continuous distribution for the distribution of C_ϕ under a nonlocal alternative hypothesis. By using the expansion, we propose a new approximation for the power of the statistic C_ϕ . We numerically investigate the accuracy of the approximation when two types of concrete ϕ -divergence statistics are applied. By the numerical investigation, we show that the present approximation is a good approximation especially when alternative hypotheses are distant from the null hypothesis.

Key words and phrases: Goodness-of-fit test, multinomial distribution, multivariate Edgeworth expansion, nonlocal alternative hypothesis, ϕ -divergence statistic.

1. Introduction

Csiszár (1967) and Ali and Silvey (1966) proposed that the ϕ -divergence measure between discrete distributions $\mathbf{p} = (p_1, \dots, p_k)'$ and $\mathbf{q} = (q_1, \dots, q_k)'$ is defined by

$$(1.1) \quad D_\phi(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^k q_j \phi\left(\frac{p_j}{q_j}\right),$$

where

$$\begin{aligned} \phi(0) &= \lim_{t \rightarrow +0} \phi(t), \\ 0\phi\left(\frac{x}{0}\right) &= \lim_{y \rightarrow +0} y\phi\left(\frac{x}{y}\right) = x \lim_{t \rightarrow +0} t\phi\left(\frac{1}{t}\right), \quad \text{if } x > 0, \end{aligned}$$

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$0\phi(0/0) = 0$ and $\phi(t)$ being a real convex function for $t > 0$.

Let $\mathbf{X} = (X_1, \dots, X_k)'$ be distributed according to a multinomial distribution $\text{Mult}_k(n, \boldsymbol{\pi})$, where $\sum_{j=1}^k X_j = n$, $\sum_{j=1}^k \pi_j = 1$, $0 < \pi_j < 1$, $(j = 1, \dots, k)$ and $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)'$ is an unknown probability vector. In order to test the null hypothesis

$$H_0 : \boldsymbol{\pi} = \mathbf{p}$$

for some completely specified probability vector $\mathbf{p} = (p_1, \dots, p_k)'$, Zografos *et al.* (1990) introduced the ϕ -divergence family of statistics

$$C_\phi = \frac{2n}{\phi''(1)} \left\{ D_\phi \left(\frac{\mathbf{X}}{n}, \mathbf{p} \right) - \phi(1) \right\}, \quad (\phi''(1) \neq 0).$$

We consider a twice continuously differentiable convex function $\phi(t)$ for $t > 0$ satisfying $\phi(1) = \phi'(1) = 0$ and $\phi''(1) = 1$ (Pardo *et al.* (1999)) without loss of generality. The ϕ -divergence family of statistics is thus defined by

$$C_\phi = 2n D_\phi \left(\frac{\mathbf{X}}{n}, \mathbf{p} \right).$$

When we choose the following convex function ϕ_a^P as ϕ :

$$\phi_a^P(t) = \begin{cases} \{a(a+1)\}^{-1} \{t^{a+1} - t + a(1-t)\} & (a \neq 0, -1) \\ t \ln t + 1 - t & (a = 0) \\ -\ln t - 1 + t & (a = -1), \end{cases}$$

the ϕ -divergence statistics C_ϕ become the power divergence statistics $C_{\phi_a^P}$ (Cressie and Read (1984) and Read and Cressie (1988)). It is well known that the family of statistics $C_{\phi_a^P}$ includes Pearson's X^2 statistic ($a = 1$), the log-likelihood ratio statistic ($a = 0$), the Freeman-Tukey statistic ($a = -1/2$), the modified log-likelihood ratio statistic ($a = -1$), the Neyman modified X^2 statistic ($a = -2$), and the statistic recommended by Cressie and Read ($a = 2/3$). Rukhin (1994) considered the family of statistics given by the following convex function ϕ_a^R as ϕ :

$$\phi_a^R(t) = \frac{(t-1)^2}{2\{a+t(1-a)\}}, \quad a \in [0, 1].$$

Then the statistic based on ϕ_a^R is defined as

$$(1.2) \quad C_{\phi_a^R} = n \sum_{j=1}^k \frac{(\hat{p}_j - p_j)^2}{ap_j + (1-a)\hat{p}_j}, \quad a \in [0, 1],$$

where $\hat{p}_j = X_j/n$, $(j = 1, \dots, k)$. The family of the statistics $C_{\phi_a^R}$ includes Pearson's X^2 statistic ($a = 1$) and the Neyman modified X^2 statistic ($a = 0$). Furthermore, Lin (1991) considered the family of divergences given by the following convex function ϕ_a^L as ϕ .

$$\phi_a^L(t) = \{a(1-a)\}^{-1} \{at \ln t - (at+1-a) \ln(at+1-a)\}, \quad a \in [0, 1].$$

Then the statistic based on ϕ_a^L is defined as

$$(1.3) \quad C_{\phi_a^L} = \frac{2n}{a(1-a)} \left\{ - \sum_{j=1}^k s_j \ln s_j + (1-a) \sum_{j=1}^k p_j \ln p_j + a \sum_{j=1}^k \hat{p}_j \ln \hat{p}_j \right\}, \quad a \in [0, 1],$$

where $s_j = a\hat{p}_j + (1-a)p_j$, ($j = 1, \dots, k$). The family of the statistics $C_{\phi_a^L}$ includes the log-likelihood ratio statistic ($a = 0$) and the modified log-likelihood ratio statistic ($a = 1$) by considering continuity in a . Under the null hypothesis H_0 , statistics C_ϕ are asymptotically distributed according to a central chi-square distribution with $k - 1$ degrees of freedom. Yarnold (1972) obtained an approximation based on asymptotic expansion for the null distribution of Pearson's X^2 statistic. In a similar fashion to X^2 , asymptotic expansions for the null distributions of the log-likelihood ratio statistic and the Freeman-Tukey statistic were obtained by Siotani and Fujikoshi (1984), that of the power divergence statistic was obtained by Read (1984), and that of C_ϕ was obtained by Menéndez *et al.* (1997).

Against H_0 , consider the local alternative hypothesis

$$(1.4) \quad H_{1,n} : \pi_j = p_j + \frac{c_j}{\sqrt{n}} \quad \text{for all } j = 1, \dots, k,$$

where $\sum_{j=1}^k c_j = 0$. It is known that C_ϕ are asymptotically distributed according to a noncentral chi-square distribution with $k - 1$ degrees of freedom and noncentrality parameter

$$\delta = \sum_{j=1}^k \frac{c_j^2}{p_j}$$

under $H_{1,n}$. Taneichi *et al.* (2001) derived the local Edgeworth expansion under $H_{1,n}$ and obtained an expression of approximation for the distribution of C_ϕ under $H_{1,n}$. That expression consists of a term of multivariate Edgeworth expansion for a continuous distribution and a discontinuous term to account for the discontinuity in $\mathbf{X}$. The discontinuous term is expressed in a very complicated form, and it is too complicated to use in practice. Thus, using only the term of the multivariate Edgeworth expansion for a continuous distribution, Taneichi *et al.* (2001) proposed an approximation to the power of C_ϕ . Based on numerical investigation, it was concluded that omission of the discontinuous term does not lead to a serious error in approximating the power of the test based on C_ϕ .

On the other hand, consider a nonlocal (fixed) alternative hypothesis

$$(1.5) \quad H_1 : \boldsymbol{\pi} = \mathbf{q},$$

where $\mathbf{q} = (q_1, \dots, q_k)'$ is a completely specified probability vector satisfying $0 < q_j < 1$, ($j = 1, \dots, k$), $\sum_{j=1}^k q_j = 1$, $\mathbf{q} \neq \mathbf{p}$ and $\mathbf{q}$ does not depend

on n . It is known that each C_ϕ is asymptotically distributed according to a normal distribution under H_1 . We summarize here the results of studies on the limiting distribution of C_ϕ under H_1 . Broffitt and Randles (1977) showed that an appropriate normalized Pearson's X^2 statistic is distributed according to the standard normal distribution for fixed k and nonlocal alternatives. Read and Cressie (1988) extended the asymptotic normality to the power divergence statistics $C_{\phi_a^P}$. For the ϕ -divergence statistic C_ϕ , it was shown that

$$(1.6) \quad \frac{C_\phi - 2nD_\phi(\mathbf{q}, \mathbf{p})}{2\sqrt{n\sigma_\phi^2}} \xrightarrow{L} N(0, 1) \quad \text{as } n \rightarrow \infty,$$

where

$$(1.7) \quad \sigma_\phi^2 = \sum_{j=1}^k q_j (\phi'(r_j))^2 - \left(\sum_{j=1}^k q_j \phi'(r_j) \right)^2,$$

$$(1.8) \quad r_j = \frac{q_j}{p_j}, \quad (j = 1, \dots, k),$$

and $\xrightarrow{L}$ denotes convergence in distribution. Sekiya and Taneichi (2004) described a local Edgeworth expansion under H_1 and derived a multivariate Edgeworth expansion assuming a continuous distribution for the distribution of power divergence statistics $C_{\phi_a^P}$.

Recently, for power divergence statistics $C_{\phi_a^P}$, Ulyanov and Zubov (2009) evaluated the order of a discrete term of the distribution of the statistics under the simple null hypothesis H_0 . By using the evaluation, they obtained an evaluation for the error of approximating the distribution of statistics to a chi-square distribution under the simple null hypothesis H_0 . Furthermore, Assylbekov (2010) refined the evaluation. However, the method used to obtain their results is difficult to apply to the case of a distribution of statistics under alternative hypotheses $H_{1,n}$ or H_1 and for a more general multinomial model such as a contingency table.

In this paper, for practical consideration, we derive the expansion assuming a continuous distribution for the general ϕ -divergence statistic C_ϕ . In Section 2, we describe a local Edgeworth expansion under H_1 and derive the multivariate Edgeworth expansion assuming a continuous distribution for the distribution of C_ϕ under H_1 . In Section 3, we apply the expansion to the power approximation for C_ϕ against H_1 . Performance of the power approximation is numerically compared with that of other power approximations in the case of statistic $C_{\phi_a^R}$ given by (1.2) and statistic $C_{\phi_a^L}$ given by (1.3).

2. Improvement of the limiting normal approximation for the distribution of C_ϕ under H_1

In this section, we consider the distribution of C_ϕ under H_1 . Let $\mathbf{X} = (X_1, \dots, X_k)'$ be distributed according to $\text{Mult}_k(n, \mathbf{q})$. Let $r = k - 1$,

$$(2.1) \quad \begin{aligned} Y_j &= \frac{X_j - nq_j}{\sqrt{n}}, \quad (j = 1, \dots, r), \\ \mathbf{Y} &= (Y_1, \dots, Y_r)', \\ \tilde{\mathbf{q}} &= (q_1, \dots, q_r)', \end{aligned}$$

and

$$\Lambda = \text{diag}(q_1, \dots, q_r) - \tilde{\mathbf{q}}\tilde{\mathbf{q}}'.$$

Then

$$E(\mathbf{Y}) = \mathbf{0}, \quad \text{Cov}(\mathbf{Y}) = \Lambda,$$

and the sample space of $\mathbf{Y}$ is the set

$$L = \left\{ \mathbf{y} = (y_1, \dots, y_r)' : \mathbf{y} = \frac{\mathbf{m} - n\tilde{\mathbf{q}}}{\sqrt{n}}, \mathbf{m} \in M \right\},$$

where

$$M = \left\{ \mathbf{m} = (m_1, \dots, m_r)' : \right. \\ \left. m_j (j = 1, \dots, r) \text{ is a nonnegative integer, } \sum_{j=1}^r m_j \leq n \right\}.$$

The local Edgeworth expansion of $\mathbf{Y}$ is given by the following theorem, which follows from Theorem 22.1 in Bhattacharya and Ranga Rao (1976) (see also Lemma 2.1 in Siotani and Fujikoshi (1984)).

THEOREM 1. *For each $\mathbf{m} \in M$, let $\mathbf{y} = (\mathbf{m} - n\tilde{\mathbf{q}})/\sqrt{n}$. Then*

$$\Pr\{\mathbf{Y} = \mathbf{y} \mid H_1\} = n^{-r/2} g(\mathbf{y}) \left\{ 1 + \frac{1}{\sqrt{n}} h_1(\mathbf{y}) + \frac{1}{n} h_2(\mathbf{y}) + O(n^{-3/2}) \right\},$$

where

$$g(\mathbf{y}) = (2\pi)^{-r/2} |\Lambda|^{-1/2} \exp\left(-\frac{1}{2} \mathbf{y}' \Lambda^{-1} \mathbf{y}\right),$$

$$h_1(\mathbf{y}) = -\frac{1}{2} \sum_{j=1}^r \frac{y_j}{q_j} + \frac{1}{6} \sum_{j=1}^r \frac{y_j^3}{q_j^2},$$

$$h_2(\mathbf{y}) = \frac{1}{2}\{h_1(\mathbf{y})\}^2 + \frac{1}{12} - \frac{1}{12} \sum_{j=1}^k \frac{1}{q_j} + \frac{1}{4} \sum_{j=1}^k \frac{y_j^2}{q_j^2} - \frac{1}{12} \sum_{j=1}^k \frac{y_j^4}{q_j^3},$$

and $y_k = -\sum_{j=1}^r y_j$.

By (2.1), the ϕ -divergence statistics C_ϕ can be written as

$$C_\phi(\mathbf{Y}) = 2n \sum_{j=1}^k p_j \phi \left(\frac{\sqrt{n}Y_j + nq_j}{np_j} \right),$$

where $Y_k = -\sum_{j=1}^r Y_j$. Let

$$S_\phi(\mathbf{Y}) = \frac{C_\phi(\mathbf{Y}) - 2nD_\phi(\mathbf{q}, \mathbf{p})}{2\sqrt{n}}.$$

From the asymptotic normality of C_ϕ under H_1 , we know that the limiting distribution of $S_\phi(\mathbf{Y})$ is $N(0, \sigma_\phi^2)$. We consider the random variable $\tilde{\mathbf{Y}}$ which has the continuous probability density function

$$g(\mathbf{y}) \left\{ 1 + \frac{1}{\sqrt{n}} h_1(\mathbf{y}) + \frac{1}{n} h_2(\mathbf{y}) \right\}.$$

We also consider the distribution of $S_\phi(\tilde{\mathbf{Y}})$, which is given by substituting $\tilde{\mathbf{Y}}$ for $\mathbf{Y}$ in $S_\phi(\mathbf{Y})$. Let $F(x)$ be the cumulative distribution function of $S_\phi(\tilde{\mathbf{Y}})$. The multivariate Edgeworth expansion for $F(x)$ is given as follows.

THEOREM 2. *Let*

$$\begin{aligned} F(x) &= \Pr\{S_\phi(\tilde{\mathbf{Y}}) \leq x \mid H_1\} \\ &= \int \cdots \int_{S_\phi(\mathbf{y}) \leq x} g(\mathbf{y}) \left\{ 1 + \frac{1}{\sqrt{n}} h_1(\mathbf{y}) + \frac{1}{n} h_2(\mathbf{y}) \right\} dy_1 \cdots dy_r. \end{aligned}$$

If we assume that $\phi(t)$ is third time differentiable and $\phi'''(t)$ is continuous, then

$$\begin{aligned} (2.2) \quad F(x) &= \Phi(z) - \frac{1}{\sqrt{n}\sigma_\phi^2} \varphi(z) \left\{ G_1^\phi + \frac{G_2^\phi}{\sigma_\phi^2} (z^2 - 1) \right\} \\ &\quad - \frac{z}{n\sigma_\phi^2} \varphi(z) \left\{ G_3^\phi + \frac{G_4^\phi}{\sigma_\phi^2} (z^2 - 3) + \frac{G_5^\phi}{\sigma_\phi^4} (z^4 - 10z^2 + 15) \right\} \\ &\quad + O(n^{-3/2}), \end{aligned}$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ denote the standard normal distribution function and its first derivative, respectively,

$$G_1^\phi = \frac{1}{2} \left\{ \sum_{j=1}^k \phi''(r_j) r_j - \sum_{j=1}^k \phi''(r_j) q_j r_j \right\},$$

$$\begin{aligned}
G_2^\phi &= \frac{1}{3} \left(\sum_{j=1}^k \phi'(r_j) q_j \right)^3 + \frac{1}{2} \left(\sum_{j=1}^k \phi'(r_j) q_j \right)^2 \left(\sum_{j=1}^k \phi''(r_j) q_j r_j \right) \\
&\quad - \left(\sum_{j=1}^k \phi'(r_j) q_j \right) \left\{ \frac{1}{2} \sum_{j=1}^k (\phi'(r_j))^2 q_j + \sum_{j=1}^k \phi''(r_j) \phi'(r_j) q_j r_j \right\} \\
&\quad + \frac{1}{6} \sum_{j=1}^k (\phi'(r_j))^3 q_j + \frac{1}{2} \sum_{j=1}^k \phi''(r_j) (\phi'(r_j))^2 q_j r_j, \\
G_3^\phi &= \frac{1}{2} (G_1^\phi)^2 + \frac{1}{4} \left(\sum_{j=1}^k \phi''(r_j) q_j r_j \right)^2 \\
&\quad + \left(\sum_{j=1}^k \phi'(r_j) q_j \right) \left\{ -\frac{1}{2} \sum_{j=1}^k \phi''(r_j) r_j + \sum_{j=1}^k \phi''(r_j) q_j r_j \right. \\
&\quad \quad \left. - \frac{1}{2} \sum_{j=1}^k \phi'''(r_j) r_j^2 + \frac{1}{2} \sum_{j=1}^k \phi'''(r_j) q_j r_j^2 \right\} \\
&\quad + \frac{1}{2} \sum_{j=1}^k \phi''(r_j) \phi'(r_j) r_j - \sum_{j=1}^k \phi''(r_j) \phi'(r_j) q_j r_j \\
&\quad + \frac{1}{4} \sum_{j=1}^k (\phi''(r_j))^2 r_j^2 - \frac{1}{2} \sum_{j=1}^k (\phi''(r_j))^2 q_j r_j^2 \\
&\quad + \frac{1}{2} \sum_{j=1}^k \phi'''(r_j) \phi'(r_j) r_j^2 - \frac{1}{2} \sum_{j=1}^k \phi'''(r_j) \phi'(r_j) q_j r_j^2, \\
G_4^\phi &= G_1^\phi G_2^\phi - \frac{1}{4} \left(\sum_{j=1}^k \phi'(r_j) q_j \right)^4 \\
&\quad - \left(\sum_{j=1}^k \phi'(r_j) q_j \right)^3 \left\{ \sum_{j=1}^k \phi''(r_j) q_j r_j + \frac{1}{6} \sum_{j=1}^k \phi'''(r_j) q_j r_j^2 \right\} \\
&\quad + \left(\sum_{j=1}^k \phi'(r_j) q_j \right)^2 \left\{ -\frac{1}{2} \left(\sum_{j=1}^k \phi''(r_j) q_j r_j \right)^2 + \frac{1}{2} \sum_{j=1}^k (\phi'(r_j))^2 q_j \right. \\
&\quad \quad + 2 \sum_{j=1}^k \phi''(r_j) \phi'(r_j) q_j r_j \\
&\quad \quad \left. + \frac{1}{2} \sum_{j=1}^k (\phi''(r_j))^2 q_j r_j^2 + \frac{1}{2} \sum_{j=1}^k \phi'''(r_j) \phi'(r_j) q_j r_j^2 \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{j=1}^k \phi'(r_j) q_j \right) \left\{ \left(\sum_{j=1}^k \phi''(r_j) q_j r_j \right) \right. \\
& \quad \times \left(\frac{1}{2} \sum_{j=1}^k (\phi'(r_j))^2 q_j + \sum_{j=1}^k \phi''(r_j) \phi'(r_j) q_j r_j \right) \\
& \quad - \frac{1}{6} \sum_{j=1}^k (\phi'(r_j))^3 q_j - \frac{3}{2} \sum_{j=1}^k \phi''(r_j) (\phi'(r_j))^2 q_j r_j \\
& \quad - \sum_{j=1}^k (\phi''(r_j))^2 \phi'(r_j) q_j r_j^2 \\
& \quad \left. - \frac{1}{2} \sum_{j=1}^k \phi'''(r_j) (\phi'(r_j))^2 q_j r_j^2 \right\} \\
& - \frac{1}{8} \left\{ \sum_{j=1}^k (\phi'(r_j))^2 q_j + 2 \sum_{j=1}^k \phi''(r_j) \phi'(r_j) q_j r_j \right\}^2 \\
& + \frac{1}{6} \sum_{j=1}^k \phi'''(r_j) (\phi'(r_j))^3 q_j r_j^2 \\
& + \frac{1}{24} \sum_{j=1}^k (\phi'(r_j))^4 q_j + \frac{1}{2} \sum_{j=1}^k \phi''(r_j) (\phi'(r_j))^3 q_j r_j \\
& + \frac{1}{2} \sum_{j=1}^k (\phi''(r_j))^2 (\phi'(r_j))^2 q_j r_j^2,
\end{aligned}$$

and

$$G_5^\phi = \frac{1}{2} (G_2^\phi)^2,$$

with σ_ϕ^2 and r_j being defined by (1.7) and (1.8), respectively, and $z = x/\sigma_\phi$.

PROOF. The characteristic function of $S_\phi(\tilde{\mathbf{Y}})$ is

$$c(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\{itS_\phi(\mathbf{y})\} g(\mathbf{y}) \left\{ 1 + \frac{1}{\sqrt{n}} h_1(\mathbf{y}) + \frac{1}{n} h_2(\mathbf{y}) \right\} dy_1 \cdots dy_r.$$

We can expand $S_\phi(\mathbf{y})$ as

$$S_\phi(\mathbf{y}) = \sum_{j=1}^r \Theta_j^\phi y_j + \frac{1}{2\sqrt{n}} \sum_{j=1}^k \phi''(r_j) \frac{y_j^2}{p_j} + \frac{1}{6n} \sum_{j=1}^k \phi'''(r_j) \frac{y_j^3}{p_j^2} + O(n^{-3/2}),$$

where

$$\Theta_j^\phi = \phi'(r_j) - \phi'(r_k), \quad (j = 1, \dots, r).$$

Therefore, we have

$$(2.3) \quad c(t) = \exp \left(-\frac{t^2}{2} (\Theta^\phi)' \Lambda \Theta^\phi \right) \\ \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g^*(\mathbf{y}) \gamma(\mathbf{y}) dy_1 \cdots dy_r + O(n^{-3/2}),$$

where

$$g^*(\mathbf{y}) = (2\pi)^{-r/2} |\Lambda|^{-1/2} \exp \left\{ -\frac{1}{2} (\mathbf{y} - it\Lambda\Theta^\phi)' \Lambda^{-1} (\mathbf{y} - it\Lambda\Theta^\phi) \right\}, \\ \gamma(\mathbf{y}) = 1 + \frac{1}{\sqrt{n}} \left\{ \frac{it}{2} \sum_{j=1}^k \phi''(r_j) \frac{y_j^2}{p_j} + h_1(\mathbf{y}) \right\} \\ + \frac{1}{n} \left\{ \frac{it}{6} \sum_{j=1}^k \phi'''(r_j) \frac{y_j^3}{p_j^2} + \frac{(it^2)}{8} \left(\sum_{j=1}^k \phi''(r_j) \frac{y_j^2}{p_j} \right)^2 \right. \\ \left. + \frac{it}{2} h_1(\mathbf{y}) \sum_{j=1}^k \phi''(r_j) \frac{y_j^2}{p_j} + h_2(\mathbf{y}) \right\}$$

and $\Theta^\phi = (\Theta_1^\phi, \dots, \Theta_r^\phi)'$. Carrying out the integral (2.3) with the aid of the moment formulae for a multivariate normal variate, we have

$$c(t) = \exp \left(-\frac{1}{2} t^2 \sigma_\phi^2 \right) \left[1 + \frac{1}{\sqrt{n}} \{ (it) G_1^\phi + (it)^3 G_2^\phi \} \right. \\ \left. + \frac{1}{n} \{ (it)^2 G_3^\phi + (it)^4 G_4^\phi + (it)^6 G_5^\phi \} \right] + O(n^{-3/2}).$$

Inverting the equation above, we obtain the probability density function $f(x)$ of $S_\phi(\tilde{\mathbf{Y}})$ as

$$f(x) = \frac{1}{\sqrt{\sigma_\phi^2}} \varphi(z) + \frac{1}{\sqrt{n}} \varphi(z) \left\{ \frac{G_1^\phi}{\sigma_\phi^2} H_1(z) + \frac{G_2^\phi}{\sigma_\phi^4} H_3(z) \right\} \\ + \frac{1}{n \sqrt{\sigma_\phi^2}} \varphi(z) \left\{ \frac{G_3^\phi}{\sigma_\phi^2} H_2(z) + \frac{G_4^\phi}{\sigma_\phi^4} H_4(z) + \frac{G_5^\phi}{\sigma_\phi^6} H_6(z) \right\} + O(n^{-3/2}),$$

where $H_j(\cdot)$ is the j th Hermitian polynomial

$$H_j(z) = (-1)^j \frac{1}{\varphi(z)} \left(\frac{d^j}{dz^j} \varphi(z) \right).$$

Consequently, the cumulative distribution function $F(x)$ of $S_\phi(\tilde{\mathbf{Y}})$ is obtained as (2.2). We have completed the proof of Theorem 2.

If we use only the leading term of the right-hand side of (2.2), we have

$$F(x) = \Phi(z) + O(n^{-1/2}).$$

3. Application and numerical investigation

By considering the distribution of $S_\phi(\tilde{\mathbf{Y}})$ as the distribution of $S_\phi(\mathbf{Y})$, we derive the following two approximations of power $\Pr\{C_\phi(\mathbf{Y}) > c_0^\phi | H_1\}$ of $C_\phi(\mathbf{Y})$ against H_1 by using (2.2). The first approximation is given by

$$(3.1) \quad 1 - \Phi(v) + \frac{1}{\sqrt{n\sigma_\phi^2}}\varphi(v) \left\{ G_1^\phi + \frac{G_2^\phi}{\sigma_\phi^2}(v^2 - 1) \right\},$$

where

$$v = \frac{c_0^\phi - 2nD_\phi(\mathbf{q}, \mathbf{p})}{2\sqrt{n\sigma_\phi^2}}$$

and c_0^ϕ is the critical value for C_ϕ . This approximation is obtained by considering (2.2) up to order of $n^{-1/2}$. We refer to this power approximation method as the $AN(m)$ approximation. The second approximation is given by

$$(3.2) \quad 1 - \Phi(v) + \frac{1}{\sqrt{n\sigma_\phi^2}}\varphi(v) \left\{ G_1^\phi + \frac{G_2^\phi}{\sigma_\phi^2}(v^2 - 1) \right\} \\ + \frac{v}{n\sigma_\phi^2}\varphi(v) \left\{ G_3^\phi + \frac{G_4^\phi}{\sigma_\phi^2}(v^2 - 3) + \frac{G_5^\phi}{\sigma_\phi^4}(v^4 - 10v^2 + 15) \right\}.$$

This approximation is obtained by considering (2.2) up to order of n^{-1} . We refer to this power approximation method as the $AN(h)$ approximation.

We compare the performance of the $AN(m)$ and $AN(h)$ approximations with that of other power approximations. Drost *et al.* (1989) proposed two power approximations for the power divergence statistic C_{ϕ^a} , A^a and B^a . The A^a approximation is constructed from a linear combination of mutually independent noncentral chi-square random variables, and the B^a approximation is constructed from a linear expression of a single noncentral chi-square random variable that has the same first moment as A^a . Taneichi *et al.* (2001) derived A^ϕ and B^ϕ approximations on the basis of A^a and B^a approximations for the ϕ -divergence family of statistics C_ϕ .

Let

$$A_j = \phi'(r_j), \quad (j = 1, \dots, k), \\ B_j = \frac{1}{p_j}\phi''(r_j), \quad (j = 1, \dots, k), \\ \mathbf{q}^{1/2} = (q_1^{1/2}, \dots, q_k^{1/2})', \\ Q = \text{diag}((B_1q_1)^{1/2}, \dots, (B_kq_k)^{1/2}),$$

$\gamma_1, \dots, \gamma_r$ are non-zero eigenvalues and S is the $k \times k$ orthogonal matrix of eigenvectors of $Q(E - \mathbf{q}^{1/2}(\mathbf{q}^{1/2})')Q$:

$$S'Q(E - \mathbf{q}^{1/2}(\mathbf{q}^{1/2})')QS = \text{diag}(\gamma_1, \dots, \gamma_r, 0),$$

where E denotes the k -dimensional unit matrix.

Then the A^ϕ approximation to the power of C_ϕ against the alternative hypothesis H_1 given by (1.5) is represented as

$$(3.3) \quad 1 - \Pr \left\{ \sum_{j=1}^{k-1} \gamma_j \chi_1^2 \left(\frac{(\xi_j)^2}{\gamma_j} \right) < c_0^\phi - 2n\nu + \sum_{j=1}^{k-1} (\xi_j)^2 \right\},$$

where

$$\begin{aligned} (\xi_1, \dots, \xi_k)' &= S' Q \boldsymbol{\mu}, \\ \boldsymbol{\mu} &= n^{1/2} \left(\frac{A_1}{B_1 \sqrt{q_1}}, \dots, \frac{A_k}{B_k \sqrt{q_k}} \right)', \\ \nu &= D_\phi(\mathbf{q}, \mathbf{p}), \end{aligned}$$

where $\chi_1^2(\delta)$ is a chi-square random variable with 1 degree of freedom and non-centrality parameter δ , and $D_\phi(\cdot, \cdot)$ is given by (1.1).

The B^ϕ approximation to the power of C_ϕ against the alternative hypothesis H_1 given by (1.5) is represented as

$$(3.4) \quad 1 - \Pr \left\{ \chi_r^2(\omega) < \frac{c_0^\phi - 2n\nu + \bar{B}\omega}{\bar{B}} \right\},$$

where

$$\bar{B} = \frac{1}{k-1} \sum_{j=1}^k B_j q_j (1 - q_j)$$

and

$$\omega = n\bar{B}^{-2} \left\{ \sum_{j=1}^k A_j^2 q_j - \left(\sum_{j=1}^k A_j q_j \right)^2 \right\}.$$

Sekiya *et al.* (1999) proposed NT approximation, which is a normal approximation based on the normalizing transformation of a power divergence statistic. Taneichi *et al.* (2001) derived an N^ϕ approximation based on the method of the NT approximation for the power divergence statistic for the ϕ -divergence family of statistics C_ϕ . The N^ϕ approximation to the power of C_ϕ against the alternative hypothesis H_1 given by (1.5) is represented as

$$(3.5) \quad 1 - \Phi \left(\sqrt{\frac{n}{\nu_2}} \left\{ \psi \left(\frac{c_0^\phi}{2n} \right) - \frac{1}{n} \left(\nu_1 - \frac{\nu_3}{6\nu_2} \right) \right\} \right),$$

where

$$\psi(x) = \begin{cases} \frac{\nu}{\zeta + 1} \left(\left(\frac{x}{\nu} \right)^{\zeta+1} - 1 \right) & (\zeta \neq -1) \\ \nu \ln \frac{x}{\nu} & (\zeta = -1), \end{cases}$$

Table 1. Values of Er for the $R_{0.5}$ statistic when $k = 4$, $\pi = p^{(1)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.042267	<u>0.008041</u>	0.008728	0.102908	0.016269	0.064929	0.043094
$6k$	0.083914	0.040939	0.039016	0.104563	<u>0.027471</u>	0.076088	0.036797
$8k$	0.091101	0.027192	0.024616	0.080194	<u>0.024519</u>	0.056578	0.030508
$10k$	0.079879	0.012802	0.010821	0.083912	<u>0.010804</u>	0.045919	0.025671
Q_2							
$5k$	0.158130	<u>0.019257</u>	0.033956	0.033969	0.042920	0.052703	0.037563
$6k$	0.184125	0.037407	0.038586	0.021644	0.060576	0.026920	<u>0.019488</u>
$8k$	0.133935	0.010371	0.011988	<u>0.009698</u>	0.029118	0.009838	0.010215
$10k$	0.087939	0.009338	0.010270	0.005938	0.009060	0.012111	<u>0.002858</u>
Q_3							
$5k$	0.086395	0.015851	0.020009	<u>0.009033</u>	0.026678	0.021775	0.011980
$6k$	0.076164	0.009338	0.010582	0.013161	0.027859	0.008709	<u>0.008145</u>
$8k$	0.030895	0.007392	0.010024	0.003709	0.017181	0.012948	<u>0.001153</u>
$10k$	0.013837	0.004755	0.008947	0.002229	0.016993	0.013377	<u>0.000258</u>
Arbitrary							
$5k$	0.102145	<u>0.014416</u>	0.023351	0.075076	0.028649	0.056071	0.037001
$6k$	0.126202	0.036814	0.036425	0.083851	0.041674	0.056762	<u>0.028994</u>
$8k$	0.102344	0.021184	<u>0.019793</u>	0.065830	0.025039	0.043270	0.023788
$10k$	0.079303	0.011516	0.010654	0.070156	<u>0.010263</u>	0.037348	0.019960

According to the numerical values in Tables 1–5, we have the following results for the $R_{0.5}$ statistic.

1. The $AN(h)$ approximation is good for Q_3 , which is a set of alternatives distant from the null hypothesis.

2. The $AN(h)$ approximation is good for $p^{(2)}$, $p^{(3)}$, $p^{(4)}$ and $p^{(5)}$ in an arbitrary case.

3. The $AN(h)$ approximation performs better than the $AN(m)$ approximation.

According to Tables 6–10, we have the following results for the $L_{0.1}$ statistic.

1. The $AN(h)$ approximation is better than the other six approximations for Q_3 .

2. The $AN(h)$ approximation performs better than other approximations for $p^{(2)}$, $p^{(3)}$ and $p^{(5)}$ in an arbitrary case.

3. The $AN(h)$ approximation performs better than the $AN(m)$ approximation.

$$\zeta = -\frac{\nu\nu_3}{3(\nu_2)^2},$$

$$\nu_1 = \frac{1}{2} \sum_{j=1}^k B_j M_{jj}^{(2)},$$

$$\nu_2 = \sum_{j=1}^k \sum_{l=1}^k A_j A_l M_{jl}^{(2)},$$

$$\nu_3 = \sum_{j=1}^k \sum_{l=1}^k \sum_{m=1}^k (A_j A_l A_m M_{jlm}^{(3)} + 3A_j A_l B_m M_{jm}^{(2)} M_{lm}^{(2)}),$$

$$M_{jl}^{(2)} = \delta_{jl} q_j - q_j q_l,$$

Table 2. Values of Er for the $R_{0.5}$ statistic when $k = 4$, $\pi = p^{(2)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.080773	0.051033	0.051318	0.051034	<u>0.037789</u>	0.088908	0.065634
$6k$	0.058833	<u>0.010078</u>	0.020124	0.053461	0.018312	0.052311	0.028333
$8k$	0.080078	0.023952	0.023210	0.043796	0.022870	0.046254	<u>0.019546</u>
$10k$	0.081118	0.019932	0.019092	0.037204	0.019786	0.036316	<u>0.017424</u>
Q_2							
$5k$	0.142714	0.058404	0.059582	0.035187	0.068475	0.036568	<u>0.028485</u>
$6k$	0.104642	0.011780	0.025846	<u>0.011537</u>	0.030836	0.023342	0.016560
$8k$	0.087366	0.013979	0.014404	0.012550	0.022144	0.010755	<u>0.008544</u>
$10k$	0.062798	0.007885	0.007509	0.009393	0.013466	0.007468	<u>0.004594</u>
Q_3							
$5k$	0.096097	0.013989	0.015151	0.013016	0.039734	0.011002	<u>0.009252</u>
$6k$	0.056412	0.010579	0.011162	<u>0.004012</u>	0.025147	0.013589	0.006687
$8k$	0.030006	0.005964	0.006718	0.006950	0.026769	0.008966	<u>0.003498</u>
$10k$	0.022518	0.003655	0.007001	0.003642	0.024328	0.010603	<u>0.001088</u>
Arbitrary							
$5k$	0.111639	0.048204	0.049067	0.033441	0.052186	0.057493	<u>0.028521</u>
$6k$	0.079680	<u>0.010740</u>	0.018410	0.030521	0.025845	0.036102	0.020310
$8k$	0.081544	0.018905	0.018597	0.024694	0.024081	0.031730	<u>0.013769</u>
$10k$	0.073006	0.015513	0.014854	0.032629	0.017986	0.026174	<u>0.012384</u>

For R_a ($a = 0.0 \sim 1.0$) statistics, we have the following results.

1. The $AN(h)$ and N^ϕ approximations are better than other approximations for $a = 0.0, 0.1, 0.2$ in an arbitrary case.
2. For $a = 0.3, 0.4$, $AN(h)$ and B^ϕ approximations are better than other approximations in an arbitrary case.
3. In the seven power approximations, $AN(h)$ and A^ϕ approximations are better than other approximations for $a = 0.6, 0.7$ in an arbitrary case.
4. For $a = 0.8, 0.9, 1.0$, the A^ϕ approximation is better than the other six approximations in an arbitrary case.
5. The $AN(h)$ approximation is a good approximation, especially for $a = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ in the Q_3 case.
6. The $AN(h)$ approximation performs better than the $AN(m)$ approximation.

and

$$M_{jlm}^{(3)} = \delta_{jl}\delta_{jm}q_j - (\delta_{jl}q_jq_m + \delta_{jm}q_jq_l + \delta_{lm}q_jq_l) + 2q_jq_lq_m.$$

By (3.1), (3.2), (3.3), (3.4) and (3.5), we obtained the power approximation against H_1 .

On the other hand, Taneichi *et al.* (2002) proposed the AE approximation, which is based on multivariate Edgeworth expansion assuming a continuous distribution for the distribution of $C_{\phi_a^P}$ under $H_{1,n}$ given by (1.4). Taneichi *et al.* (2001) derived the AE approximation for the distribution of a ϕ -divergence statistic. The expression of the AE approximation for the power of C_ϕ against

Table 3. Values of Er for the $R_{0.5}$ statistic when $k = 4$, $\pi = p^{(3)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.048615	0.029430	<u>0.026263</u>	0.083458	0.044519	0.062808	0.035057
$6k$	0.049613	0.023083	<u>0.020387</u>	0.062194	0.033961	0.055609	0.032460
$8k$	0.078913	0.022438	<u>0.021182</u>	0.054360	0.022162	0.053890	0.027135
$10k$	0.107247	0.034363	0.034932	0.060677	0.039536	0.053451	<u>0.026454</u>
Q_2							
$5k$	0.152525	0.032091	0.032347	<u>0.014656</u>	0.065019	0.018980	0.015949
$6k$	0.136704	0.022260	0.025188	0.013632	0.035946	0.020456	<u>0.013070</u>
$8k$	0.134928	0.011430	0.014668	0.011254	0.028610	<u>0.008107</u>	0.008340
$10k$	0.118579	0.009481	0.007639	0.005586	0.029159	0.011006	<u>0.005320</u>
Q_3							
$5k$	0.118788	0.009312	0.013883	0.006203	0.104387	0.011812	<u>0.005688</u>
$6k$	0.098506	0.010161	0.013299	0.011132	0.091359	0.009141	<u>0.008646</u>
$8k$	0.080790	0.004135	0.006478	0.006406	0.076357	0.007961	<u>0.003801</u>
$10k$	0.067067	0.004301	0.009396	<u>0.001680</u>	0.053925	0.012614	0.002245
Arbitrary							
$5k$	0.118868	0.024126	0.024248	0.045743	0.079670	0.033202	<u>0.020579</u>
$6k$	0.101999	<u>0.018509</u>	0.019515	0.037989	0.060586	0.032759	0.019828
$8k$	0.103064	<u>0.015742</u>	0.016298	0.043464	0.041232	0.034565	0.018362
$10k$	0.104862	0.024746	0.025154	0.043005	0.036153	0.038284	<u>0.018876</u>

For L_a ($a = 0.0 \sim 1.0$) statistics, we have the following results.

1. The $AN(h)$ and A^ϕ approximations are better than the other five approximations for $a = 0.0, 0.1, 0.2$ and 0.3 in an arbitrary case.
2. For $a = 0.4, 0.5, 0.6$ and 0.7 , the $AN(h)$ approximation is a good approximation in an arbitrary case.
3. The $AN(h)$ and N^ϕ approximations perform better than the other five approximations for $a = 0.8$ and 0.9 in an arbitrary case.
4. The $AN(h)$ approximation is a good approximation, especially for $a = 0.0, 0.1, 0.2, 0.3$ and 0.4 in the Q_3 case.
5. The $AN(h)$ approximation performs better than the $AN(m)$ approximation.

$H_{1,n}$ is represented as

$$\begin{aligned}
 (3.6) \quad & 1 - \Pr\{\chi_r^2(\delta) < c_0^\phi\} - \frac{1}{6\sqrt{n}} \sum_{j=0}^3 w_j \Pr\{\chi_{r+2j}^2(\delta) < c_0^\phi\} \\
 & - \frac{1}{72n} \sum_{j=0}^6 v_j \Pr\{\chi_{r+2j}^2(\delta) < c_0^\phi\},
 \end{aligned}$$

where

$$\begin{aligned}
 w_0 &= 2S_2^3, \\
 w_1 &= -3\{(S_1^1 + S_2^3) + T^3 S_1^1\},
 \end{aligned}$$

Table 4. Values of Er for the $R_{0.5}$ statistic when $k = 4$, $\pi = p^{(4)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.038166	0.027559	<u>0.026861</u>	0.061536	0.041470	0.071714	0.035245
$6k$	0.064714	0.027334	<u>0.026560</u>	0.052533	0.030146	0.067045	0.034706
$8k$	0.106397	0.041042	0.042707	0.040404	0.042263	0.061131	<u>0.029081</u>
$10k$	0.106827	0.013065	<u>0.012078</u>	0.055586	0.028429	0.037834	0.024681
Q_2							
$5k$	0.111375	0.032688	0.035081	<u>0.019449</u>	0.042811	0.029333	0.021235
$6k$	0.118166	0.025078	0.026249	0.018130	0.037964	0.017923	<u>0.016549</u>
$8k$	0.118273	0.013974	0.015600	0.012979	0.026790	0.011034	<u>0.008888</u>
$10k$	0.101244	0.007702	<u>0.006403</u>	0.006722	0.022649	0.010561	0.007652
Q_3							
$5k$	0.098476	0.016156	0.018490	0.016172	0.074393	<u>0.010734</u>	0.013211
$6k$	0.089007	0.008875	0.010221	0.011517	0.066224	0.007769	<u>0.007746</u>
$8k$	0.070454	0.005430	0.006216	0.007491	0.056739	0.007701	<u>0.004002</u>
$10k$	0.051043	0.004895	0.009297	<u>0.002945</u>	0.035825	0.012418	0.003794
Arbitrary							
$5k$	0.094951	0.024126	0.025508	0.031530	0.059836	0.033816	<u>0.020239</u>
$6k$	0.095339	0.018969	0.019553	0.027492	0.054523	0.028944	<u>0.017488</u>
$8k$	0.099198	0.022437	0.023374	0.023038	0.041761	0.032261	<u>0.016064</u>
$10k$	0.091979	0.009375	<u>0.009054</u>	0.038540	0.027072	0.023412	0.015543

$$\begin{aligned}
w_2 &= 3S_1^1 + T^3(3S_1^1 - S_2^3), \\
w_3 &= (1 + T^3)S_2^3, \\
v_0 &= 6(1 - S_1^0) - 18S_3^4 + 4(S_2^3)^2, \\
v_1 &= 9(-k^2 - 2k + 3S_1^0) + 18T^3(-k^2 + S_1^0) + 9T^4(2k - 1 - S_1^0) \\
&\quad + 3(T^3)^2(-3k^2 - 6k + 4 + 5S_1^0) + 3\{-6kS_1^2 + 6S_2^2 + 11S_3^4 \\
&\quad - 3(S_1^2)^2 - 4(S_2^3)^2 - 4S_1^1S_2^3\} + 6T^3\{-3kS_1^2 + 3S_2^2 - 2S_1^1S_2^3\}, \\
v_2 &= 18(k^2 + 2k - 1 - 2S_1^0) + 12T^3(3k^2 + 3k - 2 - 4S_1^0) + 9T^4(-2k + 1 + S_1^0) \\
&\quad + 6(T^3)^2(3k^2 + 6k - 4 - 5S_1^0) + 3\{6kS_1^2 + 3(S_1^1)^2 + 3(S_2^3)^2 + 10S_1^1S_2^3\} \\
&\quad + 2T^3\{9S_3^4 + 9(S_1^1)^2 - 9(S_1^2)^2 - 2(S_2^3)^2 + 15S_1^1S_2^3\} + 18T^4\{S_1^2 - S_2^2\} \\
&\quad + 9(T^3)^2\{-2(k + 2)S_1^2 + 4S_2^2 + (S_1^1)^2\}, \\
v_3 &= 3(1 + T^3)^2(-3k^2 - 6k + 4 + 5S_1^0) + 2\{9(k + 2)S_1^2 - 27S_2^2 - 9S_3^4 \\
&\quad - 9(S_1^1)^2 + 9(S_1^2)^2 + 2(S_2^3)^2 - 9S_1^1S_2^3\} + 2T^3\{9(3k + 4)S_1^2 - 45S_2^2 - 9S_3^4 \\
&\quad - 18(S_1^1)^2 + 9(S_1^2)^2 + 5(S_2^3)^2 - 6S_1^1S_2^3\} + 3T^4\{-6S_1^2 + 6S_2^2 - S_3^4\} \\
&\quad + 3(T^3)^2\{12(k + 2)S_1^2 - 24S_2^2 + 3S_3^4 - 6(S_1^1)^2 - 3(S_1^2)^2 + 2S_1^1S_2^3\}, \\
v_4 &= 3\{-6(k + 2)S_1^2 + 12S_2^2 - 2S_3^4 + 3(S_1^1)^2 - 2(S_2^3)^2 - 2S_1^1S_2^3\} \\
&\quad + 6T^3\{-6(k + 2)S_1^2 + 12S_2^2 - 3S_3^4 + 3(S_1^1)^2 + 3(S_1^2)^2 - (S_2^3)^2 - 3S_1^1S_2^3\}
\end{aligned}$$

Table 5. Values of Er for the $R_{0.5}$ statistic when $k = 4$, $\pi = p^{(5)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.044482	0.024932	<u>0.022576</u>	0.070262	0.032252	0.061835	0.032815
$6k$	0.067414	0.032518	0.030674	0.056282	<u>0.028858</u>	0.059027	0.030262
$8k$	0.100328	0.040601	0.040705	0.045488	0.039802	0.059119	<u>0.028959</u>
$10k$	0.086140	0.015541	<u>0.014511</u>	0.036950	0.018041	0.039054	0.018991
Q_2							
$5k$	0.124811	0.029449	0.028948	<u>0.019071</u>	0.047091	0.025758	0.019959
$6k$	0.126329	0.026115	0.026612	0.015109	0.039806	0.015802	<u>0.014386</u>
$8k$	0.110150	0.011854	0.012767	0.010918	0.026571	0.008754	<u>0.007362</u>
$10k$	0.083496	0.006406	0.006401	0.007542	0.018744	0.007963	<u>0.004071</u>
Q_3							
$5k$	0.098434	0.009743	0.012487	<u>0.009239</u>	0.064349	0.013256	0.009443
$6k$	0.082612	0.007421	0.009612	0.008095	0.060910	0.008729	<u>0.004875</u>
$8k$	0.061500	0.005606	0.007686	0.004031	0.051555	0.010201	<u>0.001715</u>
$10k$	0.044984	0.003303	0.007845	0.002759	0.039167	0.011103	<u>0.001046</u>
Arbitrary							
$5k$	0.095221	0.022473	<u>0.021812</u>	0.042556	0.049499	0.038507	0.021853
$6k$	0.095942	0.025626	0.025145	0.034438	0.044181	0.036501	<u>0.019354</u>
$8k$	0.097214	0.027151	0.027341	0.031606	0.037262	0.039334	<u>0.019453</u>
$10k$	0.081747	0.011908	<u>0.011362</u>	0.026651	0.020498	0.028663	0.013898

$$\begin{aligned}
& + 3T^4 S_3^4 + (T^3)^2 \{-18(k+2)S_1^2 + 36S_2^2 - 18S_3^4 + 9(S_1^1)^2 \\
& + 18(S_1^2)^2 + (S_2^3)^2 - 12S_1^1 S_2^3\}, \\
v_5 &= 3\{3S_3^4 - 3(S_1^2)^2 + 2S_1^1 S_2^3\} + 2T^3\{9S_3^4 - 9(S_1^2)^2 - (S_2^3)^2 + 6S_1^1 S_2^3\} \\
& + (T^3)^2\{9S_3^4 - 9(S_1^2)^2 - 2(S_2^3)^2 + 6S_1^1 S_2^3\}, \\
v_6 &= (1 + T^3)^2 (S_2^3)^2, \\
S_m^l &= \sum_{j=1}^k (c_j)^l / (p_j)^m,
\end{aligned}$$

and $T^l = \phi^{(l)}(1)$. If we use only the first two terms of (3.6), then we have

$$(3.7) \quad 1 - \Pr\{\chi_r^2(\delta) < c_0^\phi\},$$

which is a noncentral chi-square approximation. When we put $c_j = \sqrt{n}(q_j - p_j)$, ($j = 1, \dots, k$), expressions (3.6) and (3.7) represent the AE and noncentral χ^2 approximations to the power of C_ϕ against H_1 given by (1.5).

We consider $C_{\phi_a^R}$ given by (1.2) and $C_{\phi_a^L}$ given by (1.3) as concrete ϕ -divergence statistics. Hereafter, we call them R_a statistics and L_a statistics, respectively. If we put $\phi'(x) = 2^{-1}(x-1)(a+x(1-a)+1)(a+x(1-a))^{-2}$, $\phi''(x) = (a+x(1-a))^{-3}$ and $\phi'''(x) = -3(1-a)(a+x(1-a))^{-4}$ in (3.1) and

Table 6. Values of Er for the $L_{0,1}$ statistic when $k = 4$, $\pi = p^{(1)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.036495	<u>0.004295</u>	0.005243	0.100749	0.012843	0.064217	0.047649
$6k$	0.080365	0.041302	0.040255	0.097540	<u>0.036550</u>	0.078985	0.037054
$8k$	0.086773	0.029153	<u>0.028244</u>	0.090201	0.031834	0.059757	0.030650
$10k$	0.075125	0.012070	<u>0.010902</u>	0.085371	0.016466	0.045351	0.025972
Q_2							
$5k$	0.148746	<u>0.029657</u>	0.072152	0.036234	0.051725	0.059631	0.044640
$6k$	0.063679	0.177310	0.035668	<u>0.020435</u>	0.072654	0.029027	0.021475
$8k$	0.130002	0.014000	0.025253	<u>0.010989</u>	0.037138	0.014941	0.012599
$10k$	0.085631	0.013124	0.020861	<u>0.003547</u>	0.012969	0.016971	0.005073
Q_3							
$5k$	0.080064	0.024691	0.044568	0.009813	0.023640	0.032719	<u>0.007970</u>
$6k$	0.071363	0.011411	0.018118	<u>0.009854</u>	0.024337	0.014648	0.010511
$8k$	0.028245	0.008573	0.008775	0.003131	0.013605	0.013569	<u>0.002953</u>
$10k$	0.013377	0.006404	0.007271	<u>0.000821</u>	0.011682	0.012823	0.001885
Arbitrary							
$5k$	0.095318	<u>0.020708</u>	0.102225	0.071197	0.032211	0.058040	0.041337
$6k$	0.103939	<u>0.025182</u>	0.065664	0.069101	0.033111	0.053641	0.026445
$8k$	0.090088	<u>0.017449</u>	0.047478	0.072134	0.022837	0.043131	0.021865
$10k$	0.076466	<u>0.012836</u>	0.025895	0.070792	0.015491	0.037543	0.020208

(3.2), we can obtain the coefficients $\sigma_\phi^2, G_1^\phi, \dots, G_5^\phi$ of $AN(m)$ and $AN(h)$ approximations for R_a statistics. If we put

$$\phi'(x) = \begin{cases} (1-a)^{-1} \ln(x/(ax+1-a)) & (a \neq 1, 0) \\ \ln x & (a = 0) \\ -x^{-1} + 1 & (a = 1), \end{cases}$$

$$\phi''(x) = \begin{cases} (1-a)^{-1}(x^{-1} - a(ax+1-a)^{-1}) & (a \neq 1, 0) \\ x^{-1} & (a = 0) \\ x^{-2} & (a = 1), \end{cases}$$

and

$$\phi'''(x) = \begin{cases} (1-a)^{-1}(-x^{-2} + a^2(ax+1-a)^{-2}) & (a \neq 1, 0) \\ -x^{-2} & (a = 0) \\ -2x^{-3} & (a = 1) \end{cases}$$

in (3.1) and (3.2), we can obtain the coefficients of $AN(m)$ and $AN(h)$ approximations for L_a statistics. Similarly, if we put $T^3 = 3(a-1)$ and $T^4 = 12(a-1)^2$ in (3.6), we obtain the coefficients of the AE approximation for the R_a statistic, and if we put $T^3 = -(a+1)$ and $T^4 = 2(a^2 + a + 1)$ in (3.6), we obtain the

Table 7. Values of Er for the $L_{0,1}$ statistic when $k = 4$, $\pi = p^{(2)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.092420	0.064645	0.063841	0.065054	0.063427	0.099946	<u>0.053759</u>
$6k$	0.054942	<u>0.011614</u>	0.026262	0.061178	0.018912	0.053618	0.030464
$8k$	0.080382	0.026706	0.035874	0.049150	0.033238	0.047222	<u>0.020727</u>
$10k$	0.070747	<u>0.015968</u>	0.034751	0.044895	0.019547	0.034900	0.016839
Q_2							
$5k$	0.155924	0.066879	0.066192	0.044740	0.096565	0.048234	<u>0.038333</u>
$6k$	0.101113	0.016570	0.039577	<u>0.010770</u>	0.036602	0.027041	0.014226
$8k$	0.088277	0.012942	0.019476	0.011595	0.028619	0.012351	<u>0.009637</u>
$10k$	0.056979	0.010098	0.012611	0.009230	0.012823	0.012049	<u>0.005433</u>
Q_3							
$5k$	0.097303	0.016797	0.025714	<u>0.012189</u>	0.043469	0.017548	0.013469
$6k$	0.053102	0.014168	0.022316	0.005414	0.019946	0.018227	<u>0.005016</u>
$8k$	0.027371	0.007356	0.007040	0.004979	0.019090	0.011241	<u>0.002952</u>
$10k$	0.016179	0.003878	0.004113	0.004032	0.018571	0.009233	<u>0.001682</u>
Arbitrary							
$5k$	0.119433	0.057972	0.064664	0.042642	0.072775	0.068856	<u>0.037773</u>
$6k$	0.075613	<u>0.014261</u>	0.065762	0.030564	0.027683	0.038013	0.019896
$8k$	0.081503	0.020691	0.041172	0.022929	0.031145	0.032772	<u>0.014551</u>
$10k$	0.064233	0.013220	0.031754	0.036793	0.017311	0.025713	<u>0.011932</u>

coefficient of the AE approximation for the L_a statistic. If we consider that ϕ_a^R and ϕ_a^L are ϕ in (3.3), (3.4), (3.5), we obtain A^ϕ , B^ϕ and N^ϕ approximations for R_a and L_a statistics, respectively.

For $k = 4$, we consider the equiprobable null hypothesis $H_0^{(1)} : \pi = \mathbf{p}^{(1)}$, where $\mathbf{p}^{(1)} = (p_1^{(1)}, \dots, p_k^{(1)})' = (1/k, \dots, 1/k)'$. We also consider four null hypotheses $H_0^{(l)} : \pi = \mathbf{p}^{(l)}$, ($l = 2, \dots, 5$), where

$$\begin{aligned}
 \mathbf{p}^{(2)} &= (0.2, 0.2, 0.2, 0.4)', \\
 \mathbf{p}^{(3)} &= (0.1, 0.3, 0.3, 0.3)', \\
 \mathbf{p}^{(4)} &= (0.125, 0.125, 0.375, 0.375)' \quad \text{and} \\
 \mathbf{p}^{(5)} &= (0.125, 0.25, 0.25, 0.375)'.
 \end{aligned}$$

For each null hypothesis, we randomly generate $M = 1000$ alternative hypotheses $H_1^{(i)} : \pi = \mathbf{q}^{(i)}$, ($i = 1, \dots, M$). Under each alternative $H_1^{(i)}$, we compute the true power $TP(i)$ of the randomized size- α test and approximate power $AP(i)$ of each approximation. For computing all approximate powers, we use the exact critical value obtained by direct enumeration. We investigate the performance of

Table 8. Values of Er for the $L_{0,1}$ statistic when $k = 4$, $\pi = p^{(3)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.064809	0.044752	0.043892	0.079204	<u>0.031484</u>	0.087999	0.046115
$6k$	0.056915	0.025681	0.025611	0.080625	<u>0.018200</u>	0.068501	0.051985
$8k$	0.076202	0.025348	<u>0.024751</u>	0.052311	0.034605	0.058779	0.030908
$10k$	0.092600	0.032878	0.032418	0.062600	0.046879	0.055446	<u>0.026312</u>
Q_2							
$5k$	0.151771	0.047297	0.117857	<u>0.027494</u>	0.071496	0.031972	0.027772
$6k$	0.120834	0.025233	0.183083	<u>0.019043</u>	0.036847	0.034538	0.020476
$8k$	0.117913	0.010850	0.054797	0.006803	0.040770	0.012103	<u>0.005180</u>
$10k$	0.095191	0.010114	0.017228	0.008083	0.025443	0.012303	<u>0.003911</u>
Q_3							
$5k$	0.080918	0.013205	0.105940	0.014572	0.087022	0.013885	<u>0.013003</u>
$6k$	0.054208	0.010766	0.157480	0.003301	0.058391	0.015522	<u>0.003289</u>
$8k$	0.062854	0.005000	0.005223	0.004278	0.052339	0.008700	<u>0.000977</u>
$10k$	0.044398	0.003359	0.002654	0.005243	0.043260	0.006261	<u>0.001681</u>
Arbitrary							
$5k$	0.100720	0.033151	0.097989	0.047234	0.072159	0.042818	<u>0.027829</u>
$6k$	0.081140	<u>0.020773</u>	0.143315	0.038110	0.045353	0.041733	0.024476
$8k$	0.089227	0.018098	0.033112	0.035830	0.040342	0.037828	<u>0.017849</u>
$10k$	0.085549	0.024301	0.024875	0.045170	0.037272	0.039524	<u>0.018550</u>

each approximation based on the index

$$Er = \sqrt{\frac{1}{M} \sum_{i=1}^M \{AP(i) - TP(i)\}^2}.$$

We investigate $a = 0.0 \sim 1.0$ by step 0.1 for each L_a and R_a statistic and consider the alternative hypotheses that are included in the set Q_l ($l = 1, 2, 3$) defined as follows:

$$Q_1 = \left\{ \mathbf{q} = (q_1, \dots, q_k)' : \Delta(\mathbf{q}) \leq \frac{1}{k} \right\},$$

$$Q_2 = \left\{ \mathbf{q} = (q_1, \dots, q_k)' : \frac{1}{k} < \Delta(\mathbf{q}) \leq \frac{2}{k} \right\}$$

and

$$Q_3 = \left\{ \mathbf{q} = (q_1, \dots, q_k)' : \frac{2}{k} < \Delta(\mathbf{q}) \right\},$$

where

$$\Delta(\mathbf{q}) = \sum_{j=1}^k \frac{(q_j - p_j^{(l)})^2}{p_j^{(l)}}, \quad (l = 1, \dots, 5).$$

Table 9. Values of Er for the $L_{0.1}$ statistic when $k = 4$, $\pi = p^{(4)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.044910	0.033668	0.033916	0.065593	<u>0.019006</u>	0.084347	0.056746
$6k$	0.077079	0.041991	0.041883	0.069761	<u>0.037414</u>	0.085138	0.056466
$8k$	0.102990	0.049026	0.048529	0.071619	0.062548	0.068770	<u>0.035864</u>
$10k$	0.079444	<u>0.010706</u>	0.028274	0.065943	0.026717	0.040293	0.024901
Q_2							
$5k$	0.096514	0.036453	0.037146	<u>0.024316</u>	0.055979	0.032156	0.027831
$6k$	0.110231	0.028991	0.046904	<u>0.021684</u>	0.056538	0.025424	0.022004
$8k$	0.101343	0.011416	0.016756	0.011383	0.038766	0.009505	<u>0.008304</u>
$10k$	0.072468	0.006561	0.006484	0.009948	0.019562	0.008905	<u>0.005241</u>
Q_3							
$5k$	0.077116	0.015195	0.120588	0.016853	0.061929	<u>0.014246</u>	0.014676
$6k$	0.068545	0.011220	0.080301	0.009735	0.055147	0.012884	<u>0.007391</u>
$8k$	0.055114	0.004702	0.004626	0.004734	0.039554	0.007786	<u>0.001334</u>
$10k$	0.031636	0.003713	0.003054	0.005405	0.030775	0.006835	<u>0.002290</u>
Arbitrary							
$5k$	0.077271	<u>0.024336</u>	0.073689	0.036740	0.056259	0.037356	0.026686
$6k$	0.080683	0.025545	0.077422	0.030133	0.051903	0.041568	<u>0.024879</u>
$8k$	0.084361	0.025541	0.025231	0.044349	0.044489	0.034821	<u>0.019577</u>
$10k$	0.064293	0.007782	<u>0.007576</u>	0.043843	0.024291	0.023716	0.015663

Note that the alternative hypotheses in set Q_1 are those for which the distance is very close to the null hypothesis, Q_2 is a set of alternative hypotheses with moderate distance from the null hypothesis and Q_3 is a set far from the null hypothesis. We also consider arbitrary alternative hypotheses. Some values of Er for the level of significance $\alpha = 0.05$, $k = 4$, and sample size $n = 5k, 6k, 8k$ and $10k$ are listed in Tables 1–10. Tables 1–5 are for the R_a statistic when $a = 0.5$ and Tables 6–10 are for the L_a statistic when $a = 0.1$. In each table, the underlined value indicates that the corresponding approximation is the best of all for each n .

4. Concluding remarks

In this paper, we derived a multivariate Edgeworth expansion assuming a continuous distribution for the distribution of C_ϕ under H_1 and applied the expansion to the power approximation for C_ϕ against H_1 . According to the results of a numerical investigation presented in Section 3, we conclude that the $AN(h)$ approximation performs better than the other approximations when $0.3 \leq a \leq 0.6$ for R_a statistics and $0 \leq a \leq 0.9$ for L_a statistics in an arbitrary case. The $AN(h)$ approximation is good for both R_a and L_a statistics in the Q_3 case, which means that the $AN(h)$ approximation is better than the other six approximations when the alternatives are far from the null hypothesis.

Table 10. Values of Er for the $L_{0,1}$ statistic when $k = 4$, $\pi = p^{(5)}$ and alternative hypotheses are in Q_l ($l = 1, 2, 3$) and arbitrary.

n	χ^2	A^ϕ	B^ϕ	N^ϕ	AE	$AN(m)$	$AN(h)$
Q_1							
$5k$	0.050371	0.031640	0.031230	0.070705	<u>0.019022</u>	0.075182	0.046495
$6k$	0.068190	0.034617	0.034106	0.049210	0.032760	0.068103	<u>0.031619</u>
$8k$	0.087374	0.034667	0.034137	0.047933	0.045436	0.056245	<u>0.026740</u>
$10k$	0.073920	<u>0.013962</u>	0.014152	0.039243	0.021301	0.039265	0.020760
Q_2							
$5k$	0.120858	0.031825	0.111414	<u>0.018582</u>	0.052777	0.030813	0.022296
$6k$	0.115920	0.023399	0.084618	0.014429	0.052254	0.022255	<u>0.013797</u>
$8k$	0.094047	0.012667	0.028457	0.009155	0.030111	0.014416	<u>0.006898</u>
$10k$	0.069007	0.009167	0.014909	0.010236	0.016834	0.011461	<u>0.005933</u>
Q_3							
$5k$	0.080353	0.014268	0.178507	0.011646	0.054853	0.016258	<u>0.010168</u>
$6k$	0.071263	0.010393	0.050573	0.006505	0.045879	0.013670	<u>0.004814</u>
$8k$	0.046669	0.006311	0.006230	0.005254	0.036892	0.009771	<u>0.002624</u>
$10k$	0.030239	0.004190	0.003525	0.006148	0.032289	0.007149	<u>0.002980</u>
Arbitrary							
$5k$	0.088473	0.025725	0.117313	0.036187	0.046536	0.044176	<u>0.024624</u>
$6k$	0.087776	0.026180	0.081930	0.030372	0.043702	0.042844	<u>0.019436</u>
$8k$	0.081705	0.022923	0.036113	0.031512	0.037193	0.037951	<u>0.017726</u>
$10k$	0.067791	<u>0.011270</u>	0.018141	0.028748	0.020327	0.028611	0.015294

We also found in the case of alternative hypotheses, the $AN(h)$ approximation, that is, the approximation based on expansion (2.2) up to order of n^{-1} , greatly improves the $AN(m)$ approximation, that is, the approximation based on expansion (2.2), up to an order of $n^{-1/2}$. In the future, we will search for a powerful statistic among ϕ -divergence statistics based on the power approximation $AN(h)$.

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