ROBUSTNESS OF A TWO-STAGE ESTIMATION PROCEDURE WHEN VARIANCES ARE UNEQUAL

Yoshikazu Takada* and Katuya Miyama*

This paper considers properties of the fixed-width confidence interval of the difference of two normal means constructed assuming equal variances when the variances are unequal. We explore the coverage probability of the interval and the sample size. Furthermore, we compare the expected total sample size with that of the fixed width confidence interval constructed without such an assumption.

Key words and phrases: Coverage probability, fixed-width confidence interval, sample size, two-stage procedure.

1. Introduction

We consider two normal populations Π_1 and Π_2 with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 , respectively, where μ_1 , μ_2 , σ_1^2 , and σ_2^2 are unknown. Having observed $X_{11}, \dots, X_{1r}$ from Π_1 and $X_{21}, \dots, X_{2s}$ from Π_2 , it is required to estimate $\mu_1 - \mu_2$ by $\bar{X}_{1(r)} - \bar{X}_{2(s)}$ within $\pm d$, where $\bar{X}_{1(r)} = \frac{1}{r} \sum_{i=1}^r X_{1i}$, $\bar{X}_{2(s)} = \frac{1}{s} \sum_{i=1}^s X_{2i}$, and $d(> 0)$ is a given constant. In order to meet such a requirement, we construct a confidence interval

$$I_{(r,s)} = (\bar{X}_{1(r)} - \bar{X}_{2(s)} - d, \bar{X}_{1(r)} - \bar{X}_{2(s)} + d)$$

with

$$P(\mu_1 - \mu_2 \in I_{(r,s)}) \geq 1 - \alpha$$

where $1 - \alpha$ ($0 < \alpha < 1$) is a confidence level. Such an interval is called a fixed-width confidence interval. It is well known that it is impossible to construct such an interval if the sample sizes r and s are beforehand fixed. For example, see Problem 5.22 of Lehmann and Romano (2005, p. 197). Hence sequential methods are needed to construct such an interval. Two two-stage procedures originated in Stein(1945) were proposed to determine the sample sizes. The first procedure was proposed when the variances are equal, while the second was without such an assumption.

We first consider the procedure assuming equal variances. At the first stage, let $X_{11}, \dots, X_{1m}$ and $X_{21}, \dots, X_{2m}$ be the observations from populations Π_1 and Π_2 where m is the first sample size. We calculate the sample variances $S_{1(m)}^2 = \frac{1}{m-1} \sum_{i=1}^m (X_{1i} - \bar{X}_{1(m)})^2$ and $S_{2(m)}^2 = \frac{1}{m-1} \sum_{i=1}^m (X_{2i} - \bar{X}_{2(m)})^2$. Then

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*Department of Mathematics and Engineering, Kumamoto University, Kurokami 2-39-1, Kumamoto 860-8555, Japan. Email: takada@kumamoto-u.ac.jp

the total sample size from each population is given by

$$(1.1) \quad N = \max \left\{ m, \left\lceil \frac{2a_m^2 U_m^2}{d^2} \right\rceil + 1 \right\},$$

where $[u]$ denotes the largest integer less than u , $U_m^2 = \frac{1}{2}(S_{1(m)}^2 + S_{2(m)}^2)$, and a_m denotes the upper $100\alpha/2$ percentage point of the t distribution with $2m - 2$ degrees of freedom. If $N > m$, we proceed to the second stage and take additional observation $X_{1m+1}, \dots, X_{1N}$ from Π_1 and $X_{2m+1}, \dots, X_{2N}$ from Π_2 . Then

$$P(\mu_1 - \mu_2 \in I_{(N,N)}) \geq 1 - \alpha$$

when the variances are equal. For example, see Theorem 13.2.1 of Mukhopadhyay and de Silva (2009).

Next we consider the procedure without such an assumption. Based on the first observations $X_{11}, \dots, X_{1m}$ and $X_{21}, \dots, X_{2m}$, the total sample sizes N_1 and N_2 from Π_1 and Π_2 are given by

$$(1.2) \quad N_1 = \max \left\{ m, \left\lceil \frac{u_m V_{1(m)} (V_{1(m)} + V_{2(m)})}{d^2} \right\rceil + 1 \right\}$$

and

$$(1.3) \quad N_2 = \max \left\{ m, \left\lceil \frac{u_m V_{2(m)} (V_{1(m)} + V_{2(m)})}{d^2} \right\rceil + 1 \right\},$$

where $V_{1(m)} = \sqrt{S_{1(m)}^2}$ and $V_{2(m)} = \sqrt{S_{2(m)}^2}$. The constant u_m is the solution of the equation

$$2P(|T_{m-1}| < \sqrt{u}) - P\left(|T_m| < \sqrt{\frac{m}{m-1}}u\right) = 1 - \alpha,$$

where T_v denotes a t random variable with v degrees of freedom. If $N_i > m$, we proceed to the second stage and take additional observation $X_{i1}, \dots, X_{iN_i}$ from Π_i , $i = 1, 2$. Then

$$P(\mu_1 - \mu_2 \in I_{(N_1, N_2)}) \geq 1 - \alpha.$$

See Theorem 3 of Schwabe (1995) and Theorem 1 of Takada and Aoshima (1996). Takada (2004) and Aoshima and Yata (2010) considered its extensions to k sample problems. Aoshima *et al.* (2011) treated the problem from a different perspective.

However, the implementation of the second procedure is troublesome due to the constant u_m which is not generally available, though Schwabe (1995) and Takada and Aoshima (1996) provided some values of u_m in their tables. Contrary to this, the first procedure is easy to implement because the constant a_m is easy to obtain. So it is preferable to use the first procedure if the performance is good even though the variances are unequal.

In this paper, we consider the two-stage procedure assuming equal variances and examine the coverage probability of the fixed-width confidence interval and

the expected sample size when the variances are unequal. Furthermore, we compare the total sample size between two procedures.

In Section 2, we consider the coverage probability of the fixed-width confidence interval based on the sample size (1.1). In Section 3, we consider the performance of the sample size (1.1) and compare the total sample size with that of the sample sizes (1.2) and (1.3). The comparison of two procedures through an example is considered in Section 4. Section 5 gives brief concluding remarks.

2. Coverage probability

In this section, we consider the coverage probability of the fixed-width confidence interval based on the sample size (1.1) when the variances are unequal. We obtain the following lower bound for the coverage probability.

THEOREM 1. $P(\mu_1 - \mu_2 \in I_{(N,N)}) \geq P(|T_{m-1}| < a_m)$

PROOF. Since the event $\{N = n\}$ and $T_{(n,n)}$ are independent, we have

$$\begin{aligned}
 P(\mu_1 - \mu_2 \in I_{(N,N)}) &= \sum_{n=m}^{\infty} P(|\bar{X}_{1(n)} - \bar{X}_{2(n)} - (\mu_1 - \mu_2)| \leq d, N = n) \\
 &= \sum_{n=m}^{\infty} P(|\bar{X}_{1(n)} - \bar{X}_{2(n)} - (\mu_1 - \mu_2)| \leq d) P(N = n) \\
 &= \sum_{n=m}^{\infty} G\left(\frac{d^2 n}{\sigma_1^2 + \sigma_2^2}\right) P(N = n) \\
 (2.1) \qquad &= E\left[G\left(\frac{d^2 N}{\sigma_1^2 + \sigma_2^2}\right)\right],
 \end{aligned}$$

where $G(x)$ is the cumulative distribution function of a chi-squared random variable with one degree of freedom. From (1.1)

$$d^2 N \geq 2a_m^2 U_m^2.$$

Using this inequality in the right hand side of (2.1), we have

$$(2.2) \qquad P(\mu_1 - \mu_2 \in I_{(N,N)}) \geq E\left[G\left(\frac{2a_m^2 U_m^2}{\sigma_1^2 + \sigma_2^2}\right)\right].$$

Writing $Y_1 = (m-1)S_{1(m)}^2/\sigma_1^2$ and $Y_2 = (m-1)S_{2(m)}^2/\sigma_2^2$, we have

$$\begin{aligned}
 \frac{2a_m^2 U_m^2}{\sigma_1^2 + \sigma_2^2} &= \frac{a_m^2(\sigma_1^2 Y_1 + \sigma_2^2 Y_2)}{(m-1)(\sigma_1^2 + \sigma_2^2)} \\
 (2.3) \qquad &= \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \left(\frac{a_m^2 Y_1}{m-1}\right) + \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \left(\frac{a_m^2 Y_2}{m-1}\right),
 \end{aligned}$$

so that

$$G\left(\frac{2a_m^2 U_m^2}{\sigma_1^2 + \sigma_2^2}\right) \geq \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} G\left(\frac{a_m^2 Y_1}{m-1}\right) + \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} G\left(\frac{a_m^2 Y_2}{m-1}\right).$$

The last inequality follows from the fact that $-G(x)$ is convex. Since Y_1 and Y_2 are chi-squared random variables with $m - 1$ degrees of freedom, it follows from (2.2) that

$$P(\mu_1 - \mu_2 \in I_{(N,N)}) \geq E \left[G \left(\frac{a_m^2 Y_1}{m-1} \right) \right] = P(|T_{m-1}| < a_m),$$

which proves the result.

Table 1 gives numerical values of the lower bound (LB) for $m = 10(2)30$ and $1 - \alpha = 0.95$. The table shows that the coverage probability is not so bad compared with the nominal value if we properly choose the first sample size. For example, when the nominal value is 95%, if we choose 20 observations as the first sample, it is guaranteed that the coverage probability is at least greater than 94.3% even though the variances are unequal. The following result shows that the lower bound is exact.

Table 1. The lower bound of the coverage probability when $1 - \alpha = 0.95$.

m	10	12	14	16	18	20	22	24	26	28	30
LB	0.935	0.937	0.939	0.941	0.942	0.943	0.943	0.944	0.944	0.945	0.945

THEOREM 2. *If one of the variances is fixed and the other goes to infinity, then*

$$\lim P(\mu_1 - \mu_2 \in I_{(N,N)}) = P(|T_{m-1}| < a_m).$$

PROOF. Assume that σ_1^2 is fixed and σ_2^2 goes to infinity. It follows from (1.1) that

$$\frac{2a_m^2 U_m^2}{\sigma_1^2 + \sigma_2^2} < \frac{d^2 N}{\sigma_1^2 + \sigma_2^2} < \frac{2a_m^2 U_m^2}{\sigma_1^2 + \sigma_2^2} + \frac{md^2}{\sigma_1^2 + \sigma_2^2}.$$

Hence from (2.3)

$$(2.4) \quad \frac{a_m^2}{m-1} \left(\frac{1}{1+r} Y_1 + \frac{r}{1+r} Y_2 \right) < \frac{d^2 N}{\sigma_1^2 + \sigma_2^2} < \frac{a_m^2}{m-1} \left(\frac{1}{1+r} Y_1 + \frac{r}{1+r} Y_2 \right) + \frac{ma_m^2}{\lambda},$$

where $r = \sigma_2^2/\sigma_1^2$ and $\lambda = a_m^2(\sigma_1^2 + \sigma_2^2)/d^2$, so that as σ_2^2 goes to infinity, we have

$$\lim \frac{d^2 N}{\sigma_1^2 + \sigma_2^2} = \frac{a_m^2}{m-1} Y_2.$$

Then the result follows from (2.1).

Since $P(|T_{m-1}| < a_m) < 1 - \alpha$, Theorem 2 implies that the coverage probability become less than the nominal value when the difference between the variances

becomes large. In order to further explore the coverage probability, we need the following lemma, the proof of which is immediately obtained from (2.1) and (2.4).

LEMMA 1. *If σ_1^2 and σ_2^2 go to infinity with $r = \sigma_2^2/\sigma_1^2$ fixed, then*

$$(2.5) \quad \lim P(\mu_1 - \mu_2 \in I_{(N,N)}) \\ = E \left[G \left(\frac{1}{1+r} \left(\frac{a_m^2 Y_1}{m-1} \right) + \frac{r}{1+r} \left(\frac{a_m^2 Y_2}{m-1} \right) \right) \right],$$

where Y_1 and Y_2 are independent chi-squared random variables with $m-1$ degrees of freedom.

Remark. The result of Lemma 1 (and Theorems in the next section) also follows from the condition that d goes to zero with σ_1^2 and σ_2^2 fixed.

Let the right hand side of (2.5) be

$$(2.6) \quad F(\omega) = E \left[G \left(\frac{a_m^2}{m-1} Y \right) \right],$$

where $Y = \omega Y_1 + (1-\omega)Y_2$ with $\omega = 1/(1+r)$. We approximate the distribution of Y by that of $c\chi_v^2$ where χ_v^2 is a chi-squared random variable with v degrees of freedom and c is a positive constant, which is known as Satterthwaite's method. Then

$$(2.7) \quad c = \omega^2 + (1-\omega)^2, \quad v = \frac{m-1}{c}.$$

See the Appendix for the derivation of (2.7). Hence from (2.6)

$$(2.8) \quad F(\omega) \cong E \left[G \left(\frac{a_m^2 \chi_v^2}{v} \right) \right] = P(|T_v| < a_m).$$

From the property of the t-distribution, the right hand side of (2.8) is maximized when $\omega = 1/2$, which corresponds to the equal variances. Hence it is anticipated that the coverage probability is maximized when the variances are equal. In order to confirm this anticipation, we conducted simulations. We chose that $m = 10$, $d = 1.0$, and $1 - \alpha = 0.95$. Table 2 gives the simulation results based on 10,000 replications for $\sigma_1^2 + \sigma_2^2 = 10$ and 20, in which CP denotes the estimates of the coverage probability. The simulation results seem to support what was anticipated above.

Table 2. Estimates of the coverage probability ($m = 10, d = 1.0, 1 - \alpha = 0.95$).

$\sigma_1^2 + \sigma_2^2 = 10$					
(σ_1^2, σ_2^2)	(2,8)	(4,6)	(5,5)	(6,4)	(8,2)
CP	0.944	0.950	0.952	0.950	0.948

$\sigma_1^2 + \sigma_2^2 = 20$					
(σ_1^2, σ_2^2)	(4,16)	(8,12)	(10,10)	(12,8)	(16,4)
CP	0.947	0.951	0.953	0.950	0.947

3. Sample size

In this section, we evaluate the expected sample size of each procedure and compare the total sample sizes. In order to derive the expected sample size, we need the following lemma, which is maybe well known, but for the completeness the proof is given in the Appendix.

LEMMA 2. *Let X be a non-negative random variable with a density function. Then*

$$\lim_{\lambda \rightarrow \infty} E(\lambda X - [\lambda X]) = \frac{1}{2}.$$

We first give an asymptotic evaluation of $E(N)$.

THEOREM 3. *If σ_1 and σ_2 go to infinity with $\phi = \sigma_2/\sigma_1$ fixed, then*

$$\lim E(N - \lambda) = \frac{1}{2},$$

where $\lambda = \frac{a_m^2(\sigma_1^2 + \sigma_2^2)}{d^2}$.

PROOF. Let $R = \lceil \frac{2a_m^2 U_m^2}{d^2} \rceil + 1$ and $J = \frac{2a_m^2 U_m^2}{d^2} - \lfloor \frac{2a_m^2 U_m^2}{d^2} \rfloor$. Then

$$R = \frac{2a_m^2 U_m^2}{d^2} - J + 1,$$

so that

$$(3.1) \quad E(R) = \lambda - E(J) + 1.$$

Since $N = \max(m, R)$,

$$(3.2) \quad 0 \leq E(N - R) \leq mP(N = m).$$

It follows from (1.1) and (2.3) that

$$P(N = m) = P\left(\frac{2a_m^2 U_m^2}{d^2} \leq m\right) = P\left(\frac{1}{m-1}(\omega Y_1 + (1-\omega)Y_2) \leq \frac{m}{\lambda}\right),$$

where $\omega = 1/(1 + \phi^2)$. Hence

$$\lim P(N = m) = 0,$$

so that from (3.2)

$$\lim E(N - R) = 0,$$

which implies

$$(3.3) \quad \lim E(N - \lambda) = \lim E(R - \lambda).$$

Since

$$\frac{2a_m^2 U_m^2}{d^2} = \frac{\lambda(\omega Y_1 + (1-\omega)Y_2)}{m-1},$$

it follows from Lemma 2 that

$$\lim E(J) = \frac{1}{2},$$

so that from (3.1) and (3.3)

$$\lim E(N - \lambda) = \frac{1}{2},$$

which completes the proof.

From Theorem 3 we can approximate the value of $E(N)$ by $\lambda + 1/2$ for large σ_1^2 and σ_2^2 . In order to see how good this approximation is, we conducted simulations when $m = 10$, $d = 1.0$, and $1 - \alpha = 0.95$ for $\sigma_1^2 + \sigma_2^2 = 10$ and 20. Table 3 gives the estimates of $E(N)$ and its standard error (SE) based on 10,000 replications.

Table 3. Estimates of $E(N)$ ($m = 10, d = 1.0, 1 - \alpha = 0.95$).

$\sigma_1^2 + \sigma_2^2 = 10$						
(σ_1^2, σ_2^2)	(2, 8)	(4, 6)	(5, 5)	(6, 4)	(8, 2)	$\lambda + \frac{1}{2}$
E(N)	44.5	44.7	44.4	44.5	44.6	44.6
SE	0.17	0.15	0.15	0.15	0.17	

$\sigma_1^2 + \sigma_2^2 = 20$						
(σ_1^2, σ_2^2)	(4, 16)	(8, 12)	(10, 10)	(12, 8)	(16, 4)	$\lambda + \frac{1}{2}$
E(N)	88.9	88.5	88.5	88.6	88.9	88.8
SE	0.35	0.30	0.29	0.30	0.35	

From Table 3 it turns out that $E(N)$ seems to depend on σ_1^2 and σ_2^2 through $\sigma_1^2 + \sigma_2^2$, and $\lambda + \frac{1}{2}$ gives a good approximation to $E(N)$.

The next result gives the asymptotic evaluations of $E(N_1)$ and $E(N_2)$, the proof of which is given in the Appendix.

THEOREM 4. *It σ_1 and σ_2 go to infinity with $\phi = \sigma_2/\sigma_1$ fixed, Then*

$$\lim E(N_1 - \lambda_1) = \frac{1}{2}, \quad \lim E(N_2 - \lambda_2) = \frac{1}{2},$$

where $\lambda_1 = \frac{u_m \sigma_1^2}{d^2} (1 + \phi \rho^2)$ and $\lambda_2 = \frac{u_m \sigma_2^2}{d^2} (1 + \frac{\rho^2}{\phi})$ with $\rho = \sqrt{\frac{2}{m-1}} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m-1}{2})}$, and $\Gamma(x)$ is the gamma function.

In order to see how good the approximations $\lambda_1 + 1/2$ and $\lambda_2 + 1/2$ to $E(N_1)$ and $E(N_2)$ are, we conducted simulations when $m = 15(1)25$, $d = 1.0$, $1 - \alpha = 0.95$, $\sigma_1 = 3.0$, and $\sigma_2 = 1.5$. Table 4 gives the results. Table 4 shows that the approximations $\lambda_1 + 1/2$ and $\lambda_2 + 1/2$ of $E(N_1)$ and $E(N_2)$ are fairly good.

Table 4. Estimates of $E(N_1)$ and $E(N_2)$ ($d = 1.0$, $1 - \alpha = 0.95$, $\sigma_1 = 3.0$, $\sigma_2 = 1.5$).

m	$E(N_1)$	SE	$E(N_2)$	SE	$\lambda_1 + 1/2$	$\lambda_2 + 1/2$
15	66.4	0.21	33.2	0.09	66.6	33.2
16	65.3	0.20	32.7	0.09	65.5	32.6
17	64.3	0.20	32.0	0.08	64.6	32.2
18	63.5	0.18	31.8	0.08	63.7	31.8
19	63.2	0.18	31.5	0.08	63.0	31.5
20	62.5	0.17	31.1	0.07	62.4	31.2
21	61.6	0.16	31.0	0.07	61.8	30.9
22	61.2	0.16	30.9	0.07	61.3	30.7
23	60.9	0.15	30.9	0.06	60.9	30.5
24	60.5	0.15	30.7	0.06	60.5	30.3
25	59.9	0.15	30.7	0.06	60.1	30.1

Table 5. The interval of $\phi = \sigma_2/\sigma_1$ in which $\lambda_1 + \lambda_2 - 2\lambda > 0$ ($1 - \alpha = 0.95$).

m	15	16	17	18	19	20	21	22	23	24	25
ϕ_L	0.42	0.44	0.45	0.47	0.48	0.49	0.50	0.51	0.52	0.53	0.54
ϕ_U	2.36	2.27	2.20	2.13	2.07	2.02	1.98	1.94	1.91	1.88	1.85

The following result gives the comparison of total sample sizes between two procedures, which is easy to follow from Theorems 3 and 4.

THEOREM 5. *It σ_1 and σ_2 go to infinity with $\phi = \sigma_2/\sigma_1$ fixed, then*

$$\lim E(N_1 + N_2 - 2N - (\lambda_1 + \lambda_2 - 2\lambda)) = 0$$

and

$$\lambda_1 + \lambda_2 - 2\lambda = \frac{\sigma_1^2}{d^2} f(\phi),$$

where

$$f(\phi) = (u_m - 2a_m^2)\phi^2 + 2u_m\rho^2\phi - 2a_m^2.$$

Theorem 5 shows that $E(N_1 + N_2) - 2E(N)$ is approximated by $\lambda_1 + \lambda_2 - 2\lambda$. Table 5 gives the interval (ϕ_L, ϕ_U) of ϕ in which $\lambda_1 + \lambda_2 - 2\lambda > 0$ for $m = 15(1)25$ and $1 - \alpha = 0.95$. Each interval in Table 5 contains one corresponding to the equal variances, which shows that when the difference between the variances is small (large) the procedure assuming equal variances needs fewer (more) observations than the procedure without such an assumption.

4. Example

We use the data of Example 2 in Aoshima and Mukhopadhyay (2002) to see the performance of two procedures when applied to real data. The example concerns two organic diets (diet 1 and diet 2) fed to raise a large number of chicks in a poultry farm. The weight (ounce) of a chick is approximately distributed

Table 6. diet 1.

81.2	71.6	80.5	78.5	89.5	81.3	81.0	84.9	82.2	82.6
77.9	82.4	79.8	78.1	79.3	82.5	84.1	77.6	82.6	78.9

Table 7. diet 2.

79.7	81.9	81.7	88.9	79.8	77.6	88.9	84.6	84.8	82.0
82.0	83.3	77.9	71.1	81.2	78.1	76.1	83.0	86.8	77.0

according to a normal distribution. We are interested in the problem of estimating the difference of the population means by a fixed-width confidence interval with $d = 1.5$ and $1 - \alpha = 0.95$. We take the first sample size m equal to 20. Tables 6 and 7 are the data of the first 20 observations from Tables 10 (diet 1) and 11 (diet 2) in Aoshima and Mukhopadhyay (2002).

We first consider the sample size (1.1). Since $S_{1(20)}^2 = 12.66$ and $S_{2(20)}^2 = 19.32$,

$$U_{20}^2 = \frac{1}{2}(12.66 + 19.32) = 15.99$$

Substituting $m = 20$, $d = 1.5$, $a_{20} = 2.024$ and $U_{20} = 15.99$ into (1.1), we have

$$N = \max \left\{ 20, \left\lceil \frac{2 \times 2.024^2 \times 15.99}{1.5^2} \right\rceil + 1 \right\} = 59.$$

Hence $N - m = 59 - 20 = 39$, so that 39 additional observations from each population are needed. We took the next 39 observations from Tables 10 and 11 in Aoshima and Mukhopadhyay (2002). Then

$$\bar{X}_{1(59)} = 80.94, \quad \bar{X}_{2(59)} = 81.51,$$

so that $T_{(51,51)} = 80.94 - 81.51 = -0.57$, and the fixed-width confidence interval becomes

$$(-0.57 - 1.5, -0.57 + 1.5) = (-2.07, 0.93).$$

Next we consider the sample sizes (1.2) and (1.3) using Tables 6 and 7 as the first sample observations. From Table 1 in Schwabe (1995), we find $u_{20} = 2.15^2$. Since $V_{1(20)} = \sqrt{S_{1(20)}^2} = 3.56$ and $V_{2(20)} = \sqrt{S_{2(20)}^2} = 4.40$, it follows from (1.2) and (1.3) that

$$N_1 = \max \left\{ 20, \left\lceil \frac{2.15^2 \times 3.56 \times (3.56 + 4.40)}{1.5^2} \right\rceil + 1 \right\} = 59$$

and

$$N_2 = \max \left\{ 20, \left\lceil \frac{2.15^2 \times 4.40 \times (3.56 + 4.40)}{1.5^2} \right\rceil + 1 \right\} = 72.$$

Hence $N_1 - m = 59 - 20 = 39$ and $N_2 - m = 72 - 20 = 52$, so that we took the next 39 observations from Table 10 and 52 observations from Table 11 in

Aoshima and Mukhopadhyay (2002). Then

$$\bar{X}_{1(59)} = 80.94, \quad \bar{X}_{2(72)} = 81.43,$$

so that $T_{(59,72)} = 80.94 - 81.43 = -0.49$ and the fixed-width confidence interval becomes

$$(-0.49 - 1.5, -0.49 + 1.5) = (-1.99, 1.01).$$

Now let us compare the total sample sizes between the two procedures. Since $2N = 2 \times 59 = 118$ and $N_1 + N_2 = 59 + 72 = 131$, the total sample size of the procedure assuming equal variances is less than that of the procedure without such an assumption. One reason is that based on the first stage observations, the estimate of σ_1 is $V_{1(20)} = 3.56$ and that of σ_2 is $V_{2(20)} = 4.40$, so that the estimate of $\phi = \sigma_2/\sigma_1$ is $4.40/3.56 = 1.24$, which is included in the interval of Table 5. The other reason is that the P-value of the hypothesis test regarding equal variances based on the first stage observations is 0.365, so that the difference between the variances are considered small.

5. Concluding remarks

We considered the performance of the coverage probability and the expected sample size of the two-stage procedure assuming equal variances when the variances are unequal. It turns out that the reduction of the coverage probability is permissible compared with the nominal value for a moderate first sample size, and that the expected sample size is superior to that of the procedure without such an assumption when the difference between the variances is not large. So the choice between two procedures depends on the difference between the variances, which leads us to consider the following hybrid two-stage procedure. At the first stage, we perform the hypothesis test regarding the equal variances based on the first stage observations. If the hypothesis is not rejected, we use the procedure assuming equal variances in the second stage, otherwise, we use the procedure without such an assumption in the second stage. In the near future we will study the performance of this hybrid procedure. Furthermore, we will extend the results to the k -sample problem.

Appendix

PROOF OF (2.7). It is easy to see that

$$E(Y) = m - 1, \quad V(Y) = 2(m - 1)(\omega^2 + (1 - \omega)^2).$$

On the other hand,

$$E(c\chi_v^2) = cv, \quad V(c\chi_v^2) = 2c^2v.$$

Hence

$$m - 1 = cv, \quad 2(m - 1)(\omega^2 + (1 - \omega)^2) = 2c^2v,$$

which gives

$$c = \omega^2 + (1 - \omega)^2, \quad v = \frac{m - 1}{\omega^2 + (1 - \omega)^2} = \frac{m - 1}{c}.$$

PROOF OF LEMMA 2. Let $U = \lambda x - [\lambda x]$. It is enough to show that U converges to a uniform distribution on $(0, 1)$ as $\lambda \rightarrow \infty$. Let $G(x)$ and $g(x)$ be the cumulative distribution function and the density function of X . Then for $0 < u < 1$

$$\begin{aligned}
 P(U \leq u) &= P(\lambda X - [\lambda X] \leq u) \\
 &= \sum_{n=0}^{\infty} P(\lambda X - [\lambda X] \leq u, [\lambda X] = n) \\
 &= \sum_{n=0}^{\infty} P(\lambda X - n \leq u, n \leq \lambda X < n + 1) \\
 &= \sum_{n=0}^{\infty} P(n \leq \lambda X \leq n + u) \\
 &= \sum_{n=0}^{\infty} \left(G\left(\frac{n+u}{\lambda}\right) - G\left(\frac{n}{\lambda}\right) \right) \\
 &= \sum_{n=0}^{\infty} \frac{u}{\lambda} g(\xi_n),
 \end{aligned}$$

where $n/\lambda \leq \xi_n \leq (n+u)/\lambda$ ($< (n+1)/\lambda$). Hence

$$\begin{aligned}
 \lim_{\lambda \rightarrow \infty} P(U \leq u) &= u \lim_{\lambda \rightarrow \infty} \sum_{n=0}^{\infty} \frac{1}{\lambda} g(\xi_n) \\
 &= u \int_0^{\infty} g(x) dx \\
 &= u,
 \end{aligned}$$

which completes the proof.

PROOF OF THEOREM 4. We only show the result for N_1 , because the proof for N_2 is similar. Let $W_1 = \sqrt{m-1}V_{1(m)}/\sigma_1$ and $W_2 = \sqrt{m-1}V_{2(m)}/\sigma_2$. Then W_1^2 and W_2^2 are independent chi-squared random variables with $m-1$ degrees of freedom, and

$$(A.1) \quad \frac{u_m V_{1m}(V_{1m} + V_{2m})}{d^2} = \sigma_1^2 \frac{u_m(W_1^2 + \phi W_1 W_2)}{(m-1)d^2},$$

which yields

$$\begin{aligned}
 E \left(\frac{u_m V_{1(m)}(V_{1(m)} + V_{2(m)})}{d^2} \right) &= \frac{\sigma_1^2 u_m}{(m-1)d^2} (m-1 + \phi E(W_1)^2) \\
 &= \frac{\sigma_1^2 u_m}{d^2} \left\{ 1 + \phi \left(\frac{E(W_1)}{\sqrt{m-1}} \right)^2 \right\} \\
 (A.2) \quad &= \frac{\sigma_1^2 u_m}{d^2} (1 + \phi \rho^2),
 \end{aligned}$$

since $E(W_1)/\sqrt{m-1} = \rho$. Letting

$$R_1 = \left\lceil \frac{u_m V_{1m}(V_{1m} + V_{2m})}{d^2} \right\rceil + 1$$

and

$$J_1 = \frac{u_m V_{1m}(V_{1m} + V_{2m})}{d^2} - \left\lfloor \frac{u_m V_{1m}(V_{1m} + V_{2m})}{d^2} \right\rfloor,$$

we have

$$R_1 = \frac{u_m V_{1m}(V_{1m} + V_{2m})}{d^2} - J_1 + 1,$$

so that from (A.2)

$$(A.3) \quad E(R_1) = \lambda_1 - E(J_1) + 1.$$

From (1.2), $N_1 = \max(m, R_1)$. Hence

$$(A.4) \quad 0 \leq E(N_1 - R_1) \leq mP(N_1 = m).$$

It follows from (A.1) that

$$P(N_1 = m) = P\left(\frac{u_m(W_1^2 + \phi W_1 W_2)}{(m-1)d^2} \leq \frac{m}{\sigma_1^2}\right),$$

which implies that $\lim P(N_1 = m) = 0$, so that from (A.4)

$$\lim E(N_1 - R_1) = 0,$$

and hence

$$(A.5) \quad \lim E(N_1 - \lambda_1) = \lim E(R_1 - \lambda_1).$$

It follows from Lemma 3.1 and (A.1)

$$\lim E(J_1) = \frac{1}{2},$$

so that from (A.3) and (A.5)

$$\lim E(N_1 - \lambda_1) = \frac{1}{2},$$

which completes the proof.

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