GENERALIZED CLASS OF MEAN ESTIMATORS FOR TWO PHASE SAMPLING IN THE PRESENCE OF NONRESPONSE

Zahoor Ahmad*, IrsaZafar* and Zakia Bano*

A lot of work has been done in two-phase sampling for estimating the population mean of a study variable while considering the non-response at the second phase, see e.g. Khare and Srivastava (1993, 1995), Tabasum and Khan (2004), Singh and Kumar (2008) and Singh *et al.* (2010). Most authors have used information of a single auxiliary variable while estimating the mean of the study variable, whereas in a practical situation we may require/have auxiliary information on multiple characters. Khare and Sinha (2009, 2011) have used multi-auxiliary characters to estimate the population mean in the presence of nonresponse in the case of simple random sampling. In two-phase sampling, when auxiliary information is obtained at both phases, the nonresponse may occur at both phases as well. In this paper we have proposed a generalized class of estimators for estimating the population mean of a study variable under a two-phase sampling scheme using multi-auxiliary variable(s) in the presence of nonresponse at both phases. The information on all auxiliary variable(s) is not known for the population. The bias and mean square error have been derived for the suggested class. Special cases of the class have also been identified. An empirical study has been conducted for comparing the efficiency of the proposed estimators with a modified version of existing ones.

Key words and phrases: Generalized class of estimators, multi-auxiliary variables, non-response, two phase sampling.

1. Introduction

The use of auxiliary information can increase the precision of an estimator when study variable y is highly correlated with the auxiliary variable x (see Singh *et al.* (2010)). In single phase sampling, while estimating the population mean by ratio, regression etc. methods of estimation, the auxiliary information in the form of a population parameter is required; when such information is not available, a two-phase sampling scheme can be used where a large first-phase sample measured over the auxiliary variable x is used to obtain a good estimate of population parameters. A second-phase sample can then be taken and the study variable y with an auxiliary variable x can be observed. Wu and Luan (2003) discussed that the major advantage of using two-phase sampling is the gain in high precision without a substantial increase in cost.

The presence or absence of population auxiliary information plays an important role in the efficiency of estimators for two-phase sampling. Samiuddin and Hanif (2007) first discussed that there are three possible cases for the use of auxiliary information; no information case (when population information about none

of the auxiliary variables is available), partial information case (when population information on some auxiliary variables is available) and full information case (when population information on all auxiliary variables is available). Ahmad and Hanif (2010) emphasize that the estimators developed by using the same technique, for example regression or ratio, should be compared on the basis of the mean square error within these three cases because it is obvious that an estimator will be more efficient for the full information case as compared to partial and no. The same is true for partial and no information cases. The trend of using multi-auxiliary variables is increasing due to practical requirements. The above three cases can be considered for the use of multi-auxiliary information and in this manner the precision of the estimates can also be improved. Recently under these three cases, several classes of univariate and multivariate ratio and regression estimators have been developed in two-phase sampling by Ahmad *et al.* (2009a, 2009b, 2010) and Ahmad and Hanif (2010).

The most common method of data collection in survey research is sending questionnaires through the mail. The reason may be due to the minimum cost involved in this method. But this method has a major disadvantage in that, a high rate of non-response may occur. This may result in an unknown bias at any assumption because of the fact that the estimate based only on responding units is representative of both the responding and non-responding units. Personal interview is another method of data collection which generally may result in a complete response, but the cost involved in personal interviews is much higher than the mail questionnaire method. We may conclude from the above discussion that the advantage of one method is the disadvantage of the other and vice versa. Hansen and Hurwitz (1946) combined the advantages of both procedures. They considered a problem to determine the number of mail questionnaires along with the number of personal interviews to conduct in following up to non-response to the mail questionnaire, in order to attain the required precision at minimum cost.

Considering the above approach, let us assume that the population of size N is divided into two groups, one group consisting of the units who respond, called the response class, and the other group consisting of the units who do not respond, called the non-response class. Let N_1 and N_2 be the sizes of the responding class (denoted by U_1) and the non-responding class (denoted by U_2) of the population respectively. A sample of size n_1 (denoted by u_1) is selected from the population of N units and it is assumed that n_{11} units (denoted by v'_1) respond and n_{12} (denoted by v'_2) do not, where $v'_1 = u_1 \cap U_1$ and $v'_2 = u_1 \cap U_2$. A sub-sample (denoted by v'_{2m}) of r_1 units out of n_{12} is re-contacted such that $r_1 = \frac{n_{12}}{k_1}$, $k_1 > 1$. Under this sampling scheme, Hansen and Hurwitz (1946) constructed the following unbiased estimator for population mean $\bar{Y}$ of variable of interest y as

$$(1.1) \quad \bar{y}_1^* = w_1 \bar{y}_1 + w'_1 \bar{y}_{1r},$$

where $\bar{y}_1$ and $\bar{y}_{1r}$ denote the means of the study variable, for respondent and re-contacted samples respectively, and the weights $w_1 = n_{11}/n_1$ and $w'_1 = n_{12}/n_1$.

The variance of (1.1) is

$$(1.2) \quad \text{Var}(\bar{y}_1^*) = (N - n_1)(n_1 N)^{-1} S_y^2 + \frac{W_2(k_1 - 1)}{n_1} S_{y_2}^2,$$

where $S_y^2 = (N - 1)^{-1} \sum_{j=1}^N (y_j - \bar{Y})^2$ and $S_{y_2}^2 = (N_2 - 1) \sum_{j=1}^{N_2} (y_j - \bar{Y}_2)^2$ are population variances for responding and nonresponding portions with means $\bar{Y} = (N)^{-1} \sum_{i=1}^N y_i$ and $\bar{Y}_2 = (N_2)^{-1} \sum_{i=1}^{N_2} y_i$, $W_2 = N_2(N)^{-1}$.

In the case of nonresponse for simple random sampling, Tripathi and Khare (1997) considered the estimation of mean vector for partial and complete non-response. Khare and Sinha (2009) proposed two classes of estimators of mean in the presence of non-response using multi-auxiliary variables. Further, Khare and Sinha (2011) considered class of estimators for population mean using multi-auxiliary characters with unknown means in the presence of non-response.

After introducing the concept of two-phase sampling, use of auxiliary information and non-response in Section 1, Section 2 contains the two-phase sampling scheme with important developments in estimating population mean using single auxiliary information in the presence of non-response in the second phase. For the multi-auxiliary information case, the same sampling scheme, some important notations and results are discussed in Section 3. In Section 4, we have proposed a very general class of regression-cum-ratio-exponential estimator for estimating the mean of a study variable in the presence of non-response at both phases, and its special cases are also given in this section. Detailed empirical study has been conducted and discussed in Section 5. Finally some conclusions are provided.

2. Two-phase sampling in the presence of non-response in the second phase using a single auxiliary variable

Consider the population of N units, and further it is assumed that from N , N_1 units (denoted by U_1) belong to the respondent group and N_2 units (denoted by U_2) belong to non-respondents. From N units, a first phase sample (denoted by u_1) of n_1 units is drawn by simple random sampling without replacement (SRSWOR). A second phase sample (denoted by u_2) of n_2 units (i.e. $n_2 < n_1$) is drawn from n_1 by simple random sampling without replacement (SRSWOR) and the variable of interest y is measured on it. In the second phase let n_{21} units supply information which is denoted by v_1 and n_{22} units refuse to respond which is denoted by v_2 , where $v_1 = u_2 \cap U_1$ and $v_2 = u_2 \cap U_2$. A sub-sample (denoted by v_{2m}) of r_2 units is randomly taken from the n_{22} non-respondents by applying the method of Hansen and Hurwitz (1946) and this sub sample is specified by $r_2 = n_{22}/k_2$, $k_2 > 1$. Assume that no non response is observed in this sub sample. Following Hansan and Hurwitz (1946), let $\bar{y}_2^*$ and $\bar{x}_2^*$ denote the estimate of population mean $\bar{Y}$ and $\bar{X}$ respectively in the second phase with non-response and let $\bar{x}_1$ denote the sample mean of the auxiliary variable in the first phase. Under this sampling scheme the following are the wellknown estimators for estimating the population mean of a study variable using auxiliary information in the presence of non-response.

The simple random sampling version of mean estimator in the presence of non-response suggested by Hansen and Hurwitz (1946) may be modified in the case of two-phase sampling when non-response is considered at the second phase as:

$$(2.1) \quad \bar{y}_2^* = w_2 \bar{y}_2 + w_2' \bar{y}_{2r},$$

where $\bar{y}_2$ and $\bar{y}_{2r}$ denote the means of study variables for respondent and re-contacted samples in the second phase, and $w_2 = n_{21}/n_2$ and $w_2' = n_{22}/n_2$ where n_2 and n_{22} are respondent and non-respondent respectively, sample sizes in the second phase. The expression of variance is

$$(2.2) \quad \text{Var}(\bar{y}_2^*) = (N - n_2)(n_2 N)^{-1} S_y^2 + \frac{W_2(k_2 - 1)}{n_2} S_{y_2}^2.$$

Khare and Srivastava (1993, 1995) proposed ratio, product and regression estimators in the presence of non-response at the second phase. Their ratio estimator is

$$(2.3) \quad T_{R1d} = \frac{\bar{y}_2^*}{\bar{x}_2^*} \bar{x}_1,$$

$$(2.4) \quad \text{Bias}(T_{R1d}) \approx (\bar{X}^{-1})[(n_2)^{-1} - (n_1)^{-1}(RS_x^2 - S_{xy}) + W_2(k_2 - 1)(n_2)^{-1}(RS_{x_2}^2 - S_{xy(2)})],$$

and

$$(2.5) \quad \text{MSE}(T_{R1d}) \approx (N - n_1)(n_1 N)^{-1} S_y^2 + (n_1 - n_2)(n_1 n_2)^{-1} S_r^2 + W_2(k_2 - 1)(n_2)^{-1} S_{r_2}^2,$$

where $S_r^2 = \{S_y^2 + R^2 S_x^2(1 - 2C)\}$, $S_{r_2}^2 = \{S_{y_2}^2 + R^2 S_{x_2}^2(1 - 2C_2)\}$, $C = \frac{\beta}{R}$ and $C_{(2)} = \frac{\beta_{(2)}}{R}$.

The product estimator is

$$(2.6) \quad T_{P1d} = \frac{\bar{y}_2^*}{\bar{x}_1} \bar{x}_2^*,$$

$$(2.7) \quad \text{Bias}(T_{P1d}) \approx \frac{1}{\bar{X}}[(n_1 - n_2)(n_1 n_2)^{-1} S_{xy} + W_2(k_2 - 1)(n_2)^{-1} S_{xy(2)}],$$

and

$$(2.8) \quad \text{MSE}(T_{P1d}) \approx (N - n_1)(n_1 N)^{-1} S_y^2 + (n_1 - n_2)(n_1 n_2)^{-1} S_p^2 + W_2(k_2 - 1)(n_2)^{-1} S_{p_2}^2,$$

where $S_p^2 = \{S_y^2 + R^2 S_x^2(1 + 2C)\}$, $S_{p_2}^2 = \{S_{y_2}^2 + R^2 S_{x_2}^2(1 + 2C_2)\}$.

The regression estimator is

$$(2.9) \quad T_{LR1d} = \bar{y}_2^* + b^*(\bar{x}_1 - \bar{x}_2^*), \quad \text{where } b^* = \frac{S_{xy}}{S_x^2},$$

$$\begin{aligned}
 (2.10) \quad \text{Bias}(T_{LR1d}) \approx & \beta_{xy} \left(\frac{N(N-n_1)}{(N-1)(N-2)} \frac{\mu_{21}}{n_1 \bar{X} S_{yx}} \right. \\
 & - \frac{N(N-n_1)}{(N-1)(N-2)} \frac{\mu_{30}}{n_1 \bar{X} S_x^2} + \theta_2 \frac{\mu_{30(2)}}{\bar{X} S_x^2} \\
 & + \frac{N(N-n_2)}{(N-1)(N-2)} \frac{\mu_{30}}{n_2 \bar{X} S_x^2} \\
 & \left. - \frac{N(N-n_2)}{(N-1)(N-2)} \frac{\mu_{21}}{n_2 \bar{X} S_{yx}} - \theta_2 \frac{\mu_{21(2)}}{\bar{X} S_{yx}} \right)
 \end{aligned}$$

where

$$\mu_{vs} = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{X})^v (y_i - \bar{Y})^s$$

and

$$\mu_{vs(2)} = \frac{1}{(N_2-1)} \sum_{i=1}^{N_2} (x_i - \bar{X}_2)^v (y_i - \bar{Y}_2)^s,$$

(v, s) being nonnegative integers.

Also

$$\begin{aligned}
 (2.11) \quad \text{MSE}(T_{LR1d}) = & (N-n_1)(n_1 N)^{-1} S_y^2 + (n_1-n_2)(n_1 n_2)^{-1} S_y^2 (1-\rho^2) \\
 & + W_2(k_2-1)(n_2)^{-1} S_{\beta_2}^2,
 \end{aligned}$$

where $S_{\beta_2}^2 = [S_{y_2}^2 + \beta S_{x_2}^2 (\beta - 2\beta_{(2)})]$.

Tabasum and Khan (2004) revisited the Khare and Srivastava (1993) ratio estimator T_{R1d} and found that the cost of this estimator is less than the cost calculated by Hansen-Hurwitz (1946) estimators under certain conditions. Singh *et al.* (2010) suggested the following generalized version of modified exponential ratio and exponential product type estimators using two-phase sampling.

$$(2.12) \quad t_{(a)}^{(1)} = \bar{y}_2^* \exp \left[\frac{a(\bar{x}_2^* - \bar{x}_1)}{(\bar{x}_2^* + \bar{x}_1)} \right],$$

with

$$\begin{aligned}
 (2.13) \quad \text{Bias}(t_{(a)}^{(1)}) = & \frac{a}{2\bar{X}} \left[(n_1-n_2)(n_1 n_2)^{-1} \left(S_{xy} - \frac{R}{2} S_x^2 \right) \right. \\
 & \left. + W_2(k_2-1)(n_2)^{-1} \left(S_{xy_{(2)}} - \frac{R}{2} S_{x_2}^2 \right) \right]
 \end{aligned}$$

and

$$(2.14) \quad \text{MSE}(t_{(a)}^{(1)}) = (n_1-n_2)(n_1 n_2)^{-1} \left[S_y^2 + \left(\frac{aR^2 S_x^2}{4} \right) (1+4C) \right]$$

$$\begin{aligned}
& + (N - n_1)(n_1 N)^{-1} S_y^2 \\
& + W_2(k_2 - 1)(n_2)^{-1} \left[S_{y_2}^2 + \left(\frac{a R^2 S_{x_2}^2}{4} \right) (1 + 4C_{(2)}) \right].
\end{aligned}$$

When $a = -1$ and 1 , the expression given in (2.12) becomes an exponential-ratio-estimator and exponential-product estimator along with their bias and mean square error respectively.

3. Two-phase sampling using multi-auxiliary variables in the presence of non-response at both phases

In two-phase sampling, when auxiliary information is obtained from the first phase sample that is relatively larger than the second phase, the non-response rate will be larger in the first phase sample. A lot of literature is devoted to the case when nonresponse occurs in the second phase. Hence there is a need to develop estimators when non-response is considered in both phases. In this paper, a class of generalized estimators in the presence of nonresponse in both phases is developed. It is also considered that information on all auxiliary variables is not known (No Information Case) that usually occurs in most practical situations.

After discussing the sampling scheme of Section 2, from the first phase sample (denoted by u_1) of n_1 units we assume that n_{11} units supply information which is denoted by, and n_{12} units refuse to respond which is denoted by v'_2 , where $v'_1 = u_1 \cap U_1$ and $v'_2 = u_1 \cap U_2$. A sub-sample (denoted by v'_{2m}) of r_1 units is randomly taken from the n_{12} non-respondents by applying the method of Hansen and Hurwitz (1946) and this sub sample is specified by $r_1 = n_{12}/k_1$, $k_1 > 1$.

In the Hansen and Hurwitz (1946) approach it is assumed that no non response is observed in the sub sample of non-respondents. But it is not a realistic assumption in recent sampling and it is possible that non-response may appear in re-contracted samples.

Let, x_i denote the i th auxiliary variable for all $i = 1, 2, 3, \dots, q$ with population mean $\bar{X}_i = N^{-1} \sum_{j=1}^N X_{ij}$, $\bar{X}_{1i} = N_1^{-1} \sum_{j'=1}^{N_1} X_{1ij'}$ and $\bar{X}_{2i} = N_2^{-1} \sum_{L=1}^{N_2} X_{2iL}$ denote the population means of the i th auxiliary variable for the response and non-response groups respectively, and $S_{x_i}^2 = (N-1)^{-1} \sum_{j=1}^N (X_{ij} - \bar{X}_i)^2$ and $S_{x_{2i}}^2 = (N_2-1)^{-1} \sum_{L=1}^{N_2} (X_{2iL} - \bar{X}_{2i})^2$ denote the variances of the i th auxiliary variable for the complete population and non-respondents respectively. $S_{x_i x_{i'}} = (N-1)^{-1} \sum_{j=1}^N (X_{ij} - \bar{X}_i)(X_{i'j} - \bar{X}_{i'})$ and $S_{x_i x_{i'(2)}} = (N_2-1)^{-1} \sum_{L=1}^{N_2} (X_{iL} - \bar{X}_{2i})(X_{i'L} - \bar{X}_{2i'})$ denote the co-variances between the two auxiliary variables for complete population and the population of non-respondents respectively, and $S_{x_i y} = (N-1)^{-1} \sum_{j=1}^N (X_{ij} - \bar{X}_i)(y_j - \bar{Y})$ and $S_{x_i y_{(2)}} = (N_2-1)^{-1} \sum_{L=1}^{N_2} (x_{iL} - \bar{X}_{2i})(y_{iL} - \bar{Y}_2)$ denote the co-variances between the study and auxiliary variable for complete population and the population of non-respondents respectively.

Let $\bar{x}_{1i} = n_{11}^{-1} \sum_{j=1}^{n_{11}} x_{1ij}$ and $\bar{x}_{2i} = n_{21}^{-1} \sum_{l=1}^{n_{21}} x_{2il}$ denote the sample means of the i th auxiliary variable for the first and second phase responding units respectively, and $\bar{x}_{r1i} = \frac{1}{r_1} \sum_{j'=1}^{r_1} x_{r1ij'}$ and $\bar{x}_{r2i} = \frac{1}{r_2} \sum_{l'=1}^{r_2} x_{r2il'}$ denote the sample

means of the i th auxiliary variable for sub-sampled units in the first phase and second phase respectively.

Following Hansen and Hurwitz (1946) approach, $\bar{x}_{1i}^* = n_1^{-1}(n_{11}\bar{x}_{1i} + n_{12}\bar{x}_{r1i})$ and $\bar{x}_{2i}^* = n_2^{-1}(n_{21}\bar{x}_{2i} + n_{22}\bar{x}_{r2i})$ denote sample means of the i th auxiliary variable for first phase and second phase respectively considering the respondents and re-contacted samples.

Further, let $\lambda_1 = n_1^{-1}(1 - f_1)$ and $\lambda_2 = n_2^{-1}(1 - f_2)$ be sampling fractions for the first and second phase respectively, $\theta_1 = n_1^{-1}W_2(k_1 - 1)$ and $\theta_2 = n_2^{-1}W_2(k_2 - 1)$ are the sampling fractions for non-respondents who are recontacted at the first and second phase respectively, where $f_1 = N^{-1}n_1$ and $f_2 = N^{-1}n_2$. Also, $\lambda_3 = \lambda_2 - \lambda_1$ and $\theta_3 = \theta_2 - \theta_1$.

4. Generalized class of regression-cum-ratio-exponential estimators

Mohanty (1967) introduced a concept of combining regression and ratio estimators in simple random sampling and call it a regression-cum-ratio estimator. Similarly regression can be combined with exponential estimator. The sum of the ratio and exponential with some suitable weights can be multiplied with regression and it can help to see the role of the exponential part along with the ratio and regression parts. Also the rationale of suggesting the following general class of estimators is to search for the best member from all members of the class. Combining the regression, ratio and exponential parts, the class can be suggested as:

$$(4.1) \quad t_q = \left(\bar{y}_2^* + a \sum_{i=1}^{q_1} \alpha_i (\bar{x}_{1i}^* - \bar{x}_{2i}^*) \right) \left[b \prod_{i=q_1+1}^{q_2} \left(\frac{\bar{x}_{1i}^*}{\bar{x}_{2i}^*} \right)^{c\alpha_i} + d \exp \sum_{i=q_2+1}^{q_3} \alpha_i \left(\frac{g(\bar{x}_{2i}^* - \bar{x}_{1i}^*)}{(\bar{x}_{2i}^* + \bar{x}_{1i}^*)} \right) \right]$$

where a , b , c , d and g are suitable constants to be chosen for generating members of this class such that $b + d = 1$; and α_i 's are unknown constants to be determined by minimizing the mean square error of t_q given in (4.1).

From Hansen and Hurwitz (1946), let $\bar{e}_{\bar{y}_2^*} = \bar{y}_2^* - \bar{Y}$, $\bar{e}_{x_{1i}^*} = x_{1i}^* - \bar{X}_i$ and $\bar{e}_{x_{2i}^*} = x_{2i}^* - \bar{X}_i$ be the sampling errors of y , i th auxiliary variable in the first phase and i th auxiliary variable in the second phase respectively. Then simplifying and using binomial expansion, we can write (4.1) after ignoring the third and higher order terms as:

$$\begin{aligned} t_q \approx & \left(\bar{Y} + \bar{e}_{y_2^*} + a \sum_{i=1}^{q_1} \alpha_i (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) \right) \\ & \times \left[b \prod_{i=q_1+1}^{q_2} \left(1 + c\alpha_i \frac{\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}}{\bar{X}_i} + \frac{c\alpha_i(c\alpha_i + 1)}{2\bar{X}_i^2} \bar{e}_{x_{2i}^*}^2 \right. \right. \\ & \left. \left. + \frac{c\alpha_i(c\alpha_i - 1)}{2\bar{X}_i^2} \bar{e}_{x_{1i}^*}^2 - c^2\alpha_i^2 \frac{\bar{e}_{x_{1i}^*} \bar{e}_{x_{2i}^*}}{\bar{X}_i^2} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + d \left(1 + \sum_{i=q_2+1}^{q_3} \alpha_i \frac{g}{2\bar{X}_i} (\bar{e}_{x_{2i}^*} - \bar{e}_{x_{1i}^*}) - \sum_{i=q_2+1}^{q_3} \alpha_i \frac{g(\bar{e}_{x_{2i}^*}^2 - \bar{e}_{x_{1i}^*}^2)}{4\bar{X}_i^2} \right. \\
& \left. + \left(\sum_{i=q_2+1}^{q_3} \alpha_i \right)^2 \left(\frac{g^2}{8\bar{X}_i^2} (\bar{e}_{x_{2i}^*}^2 + \bar{e}_{x_{1i}^*}^2 - 2\bar{e}_{x_{2i}^*} \bar{e}_{x_{1i}^*}) \right) \right) \Bigg]
\end{aligned}$$

or

$$\begin{aligned}
t_q \approx & \left(\bar{Y} + \bar{e}_{y_2^*} + a \sum_{i=1}^{q_1} \alpha_i (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) \right) \\
& \times \left[1 + bc \sum_{i=q_1+1}^{q_2} \frac{\alpha_i}{\bar{X}_i} (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) + bc \sum_{i=q_1+1}^{q_2} \frac{\alpha_i (c\alpha_i + 1)}{2\bar{X}_i^2} \bar{e}_{x_{2i}^*}^2 \right. \\
& + d \sum_{i=q_2+1}^{q_3} \frac{g\alpha_i}{2\bar{X}_i} (\bar{e}_{x_{2i}^*} - \bar{e}_{x_{1i}^*}) + bc \sum_{i=q_1+1}^{q_2} \alpha_i \frac{(c\alpha_i - 1)}{2\bar{X}_i^2} \bar{e}_{x_{1i}^*}^2 \\
& - bc^2 \sum_{i=q_1+1}^{q_2} \alpha_i^2 \frac{\bar{e}_{x_{1i}^*} \bar{e}_{x_{2i}^*}}{\bar{X}_i^2} - d \sum_{i=q_2+1}^{q_3} \alpha_i \frac{g(\bar{e}_{x_{2i}^*}^2 - \bar{e}_{x_{1i}^*}^2)}{4\bar{X}_i^2} \\
& \left. + d \left(\sum_{i=q_2+1}^{q_3} \alpha_i \right)^2 \left(\frac{g^2}{8\bar{X}_i^2} (\bar{e}_{x_{2i}^*}^2 + \bar{e}_{x_{1i}^*}^2 - 2\bar{e}_{x_{2i}^*} \bar{e}_{x_{1i}^*}) \right) \right]
\end{aligned}$$

Again ignoring third and higher order terms for each expansion of the product and simplifying, we can write the expression as:

$$\begin{aligned}
t_q \approx & \bar{Y} + \bar{e}_{y_2^*} + a \sum_{i=1}^{q_1} \alpha_i (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) + \bar{Y} bc \sum_{i=q_1+1}^{q_2} \frac{\alpha_i}{\bar{X}_i} (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) \\
& + \bar{Y} d \sum_{i=q_1+1}^{q_3} \alpha_i \frac{g}{2\bar{X}_i} (\bar{e}_{x_{2i}^*} - \bar{e}_{x_{1i}^*}) \\
& + abc \sum_{i=q_1+1}^{q_2} \frac{\alpha_i}{\bar{X}_i} (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) \sum_{i=1}^{q_1} \alpha_i (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}) \\
& + \bar{Y} bc \sum_{i=q_1+1}^{q_2} \alpha_i \frac{\bar{e}_{x_{2i}^*}^2}{2\bar{X}_i^2} + \bar{Y} bc^2 \sum_{i=q_1+1}^{q_2} \alpha_i^2 \frac{\bar{e}_{x_{1i}^*}^2}{2\bar{X}_i^2} - \bar{Y} bc \sum_{i=q_1+1}^{q_2} \alpha_i \frac{\bar{e}_{x_{1i}^*}^2}{2\bar{X}_i^2} \\
& - \bar{Y} bc^2 \sum_{i=q_1+1}^{q_2} \alpha_i^2 \frac{\bar{e}_{x_{1i}^*} \bar{e}_{x_{2i}^*}}{\bar{X}_i^2} - 2\bar{Y} d \frac{g^2}{8\bar{X}_i^2} \left(\sum_{i=q_2+1}^{q_3} \alpha_i \right)^2 \bar{e}_{x_{2i}^*} \bar{e}_{x_{1i}^*} \\
& + \bar{Y} dg \sum_{i=q_2+1}^{q_3} \alpha_i \frac{\bar{e}_{x_{1i}^*}}{4\bar{X}_i^2} + \bar{Y} d \frac{g^2}{8\bar{X}_i^2} \left(\sum_{i=q_2+1}^{q_3} \alpha_i \right)^2 \bar{e}_{x_{1i}^*}^2 \\
& - \bar{Y} dg \sum_{i=q_2+1}^{q_3} \alpha_i \frac{\bar{e}_{x_{2i}^*}}{4\bar{X}_i^2} + \bar{Y} bc^2 \sum_{i=q_1+1}^{q_2} \alpha_i^2 \frac{\bar{e}_{x_{2i}^*}^2}{2\bar{X}_i^2} + bc \sum_{i=q_1+1}^{q_2} \alpha_i \frac{\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}}{\bar{X}_i} \bar{e}_{y_2^*}
\end{aligned}$$

$$\begin{aligned}
& + d \sum_{i=q_2+1}^{q_3} \alpha_i \frac{g}{2\bar{X}_i} (\bar{e}_{x_{2i}^*} - \bar{e}_{x_{1i}^*}) \bar{e}_{y_2^*} \\
& + d \sum_{i=q_2+1}^{q_3} \alpha_i \frac{g}{2\bar{X}_i} (\bar{e}_{x_{2i}^*} - \bar{e}_{x_{1i}^*}) a \sum_{i=1}^{q_1} \alpha_i (\bar{e}_{x_{1i}^*} - \bar{e}_{x_{2i}^*}).
\end{aligned}$$

We can write the above expression in matrix notation as:

$$\begin{aligned}
(4.2) \quad t_q - \bar{Y} & \approx \bar{e}_{y_2^*} - a\alpha_{q_1}^t d_{q_1}^* - \bar{Y}bc\alpha_{q_2}^t \mathbf{X}_{q_2} d_{q_2}^* \\
& + \frac{\bar{Y}}{2} dg\alpha_{q_3}^t \mathbf{X}_{q_3} d_{q_3}^* + \frac{\bar{Y}}{2} bc^2\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \alpha_{q_2} \mathbf{1}_{q_2}^t \bar{e}_{x_{2q_2}^*}^2 \\
& + \frac{\bar{Y}}{2} bc\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \bar{e}_{x_{2q_2}^*}^2 + \frac{\bar{Y}}{2} bc^2\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \alpha_{q_2} \mathbf{1}_{q_2}^t \bar{e}_{x_{1q_2}^*}^2 \\
& - \bar{Y}bc^2\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \bar{e}_{x_{1q_2}^*} \bar{e}_{x_{2q_2}^*}^t \alpha_{q_2} - \frac{\bar{Y}}{2} bc\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \bar{e}_{x_{1q_2}^*}^2 \\
& - bc\alpha_{q_2}^t \mathbf{X}_{q_2} d_{q_2}^* \bar{e}_{y_2^*} + abc\alpha_{q_2}^t \mathbf{X}_{q_2} d_{q_2}^* d_{q_1}^{*t} \alpha_{q_1} \\
& - \frac{\bar{Y}}{4} dg\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \bar{e}_{x_{2q_3}^*}^2 + \frac{\bar{Y}}{4} dg\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \bar{e}_{x_{1q_3}^*}^2 + \frac{dg}{2} \alpha_{q_3}^t \mathbf{X}_{q_3} d_{q_3}^* \bar{e}_{y_2^*} \\
& + \frac{\bar{Y}}{8} dg^2\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} \mathbf{1}_{q_3}^t \bar{e}_{x_{2q_3}^*}^2 + \frac{\bar{Y}}{8} dg^2\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} \mathbf{1}_{q_3}^t \bar{e}_{x_{1q_3}^*}^2 \\
& - \frac{\bar{Y}}{4} dg^2\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \bar{e}_{x_{1q_3}^*} \bar{e}_{x_{2q_3}^*}^t \alpha_{q_3} - \frac{adg}{2} \alpha_{q_3}^t \mathbf{X}_{q_3} d_{q_3}^* d_{q_1}^{*t} \alpha_{q_1}.
\end{aligned}$$

where $\mathbf{X}_{q_1} = [\bar{X}_i^{-1}]_{(q_1 \times q_1)}$; $i = 1, \dots, q_1$, $\mathbf{X}_{q_2} = [\bar{X}_i^{-1}]_{(q_2 \times q_2)}$; $i = q_1 + 1, q_1 + 2, \dots, q_2$ and $\mathbf{X}_{q_3} = [\bar{X}_i^{-1}]_{(q_3 \times q_3)}$; $i = q_2 + 1, q_2 + 2, \dots, q_3$ are the diagonal matrices and $q = q_1 + q_2 + q_3$. α_{q_l} 's are vectors of unknown coefficient where $l = 1, 2, 3$ and the vector $d_{q_l \times 1}^{*t} = [d_i^*]_{q_l \times 1}$, with $d_i^* = (\bar{e}_{x_{(2)i}^*} - \bar{e}_{x_{(1)i}^*})$.

Again ignoring the second and higher order terms for obtaining the MSE to the terms of $o(1/n)$, we have

$$t_q - \bar{Y} \approx \bar{e}_{y_2^*} - a\alpha_{q_1}^t d_{q_1}^* - \bar{Y}bc\alpha_{q_2}^t \mathbf{X}_{q_2} d_{q_2}^* + \frac{\bar{Y}}{2} dg\alpha_{q_3}^t \mathbf{X}_{q_3} d_{q_3}^*.$$

Squaring and taking expectation, we have

$$(4.3) \quad \text{MSE}(t_q) = E[\bar{e}_{y_2^*} - \mathbf{h}_q^t \mathbf{m}_q]^2,$$

where $\mathbf{h}_q^t = [\alpha_{q_1}^t \quad \alpha_{q_2}^t \quad \alpha_{q_3}^t]_{1 \times q}$ and $\mathbf{m}_q^t = [a d_{q_1}^{*t} \quad \bar{Y}bc\mathbf{X}_{q_2} d_{q_2}^{*t} \quad -\frac{\bar{Y}dg}{2} \mathbf{X}_{q_3} d_{q_3}^{*t}]_{1 \times q}$.

To find the optimum value of $\mathbf{h}$, differentiating the equation (4.3) w.r.t. $\mathbf{h}$ and equating to zero, we get:

$$2\mathbf{m}_q E[\bar{e}_{y_2^*} - \mathbf{m}_q^t \mathbf{h}_q] = 0$$

or

$$E(\mathbf{m}_q \bar{e}_{y_2^*}) - E(\mathbf{m}_q \mathbf{m}_q^t) \mathbf{h}_q = 0$$

or

$$(4.4) \quad E(\mathbf{m}_q) - E(\mathbf{\Lambda}_q)\mathbf{h}_q = 0$$

where

$$(4.5) \quad E(\boldsymbol{\omega}) = \boldsymbol{\Phi}_{yx_q} = e \begin{bmatrix} a\mathbf{d}_{q_1}^* \bar{e}_{y_2}^* \\ \bar{Y}bc\mathbf{X}_{q_2}\mathbf{d}_{q_2}^* \bar{e}_{y_2}^* \\ -\frac{\bar{Y}dg}{2}\mathbf{X}_{q_3}\mathbf{d}_{q_3}^* \bar{e}_{y_2}^* \end{bmatrix}_{q \times 1} = \begin{bmatrix} a\boldsymbol{\Phi}_{yx_{q_1}} \\ \bar{Y}bc\mathbf{X}_{q_2}\boldsymbol{\Phi}_{yx_{q_2}} \\ -\frac{\bar{Y}dg}{2}\mathbf{X}_{q_3}\boldsymbol{\Phi}_{yx_{q_3}} \end{bmatrix}_{q \times 1}$$

and

$$\begin{aligned} E(\mathbf{\Lambda}_q) &= \mathbf{T}_q \\ &= E \begin{bmatrix} a^2\mathbf{d}_{q_1}^* \mathbf{d}_{q_1}^{*t} & \bar{Y}abc\mathbf{d}_{q_1}^* \mathbf{d}_{q_2}^{*t} \mathbf{X}_{q_2} & -\frac{\bar{Y}adg}{2}\mathbf{d}_{q_1}^* \mathbf{d}_{q_3}^{*t} \mathbf{X}_{q_3} \\ & \bar{Y}^2b^2c^2\mathbf{X}_{q_2}\mathbf{d}_{q_2}^* \mathbf{d}_{q_2}^{*t} \mathbf{X}_{q_2} & -\frac{\bar{Y}^2bcdg}{2}\mathbf{X}_{q_2}\mathbf{d}_{q_2}^* \mathbf{d}_{q_3}^{*t} \mathbf{X}_{q_3} \\ & & \frac{\bar{Y}^2d^2g^2}{4}\mathbf{X}_{q_3}\mathbf{d}_{q_3}^* \mathbf{d}_{q_3}^{*t} \mathbf{X}_{q_3} \end{bmatrix}_{q \times q} \end{aligned}$$

or

$$(4.6) \quad \mathbf{T}_q = \begin{bmatrix} a^2\mathbf{\Lambda}_{q_1} & \bar{Y}abc\mathbf{\Lambda}_{q_{12}}\mathbf{X}_{q_2} & -\frac{\bar{Y}adg}{2}\mathbf{\Lambda}_{q_{13}}\mathbf{X}_{q_3} \\ & \bar{Y}^2b^2c^2\mathbf{X}_{q_2}\mathbf{\Lambda}_{q_2}\mathbf{X}_{q_2} & -\frac{\bar{Y}^2bcdg}{2}\mathbf{X}_{q_2}\mathbf{\Lambda}_{q_{23}}\mathbf{X}_{q_3} \\ & & \frac{\bar{Y}^2d^2g^2}{4}\mathbf{X}_{q_3}\mathbf{\Lambda}_{q_3}\mathbf{X}_{q_3} \end{bmatrix},$$

where

$$E(\mathbf{d}_{q_l}e_{\bar{y}_2}^*) = [\lambda_3 S_{yx_i} + \theta_3 S_{yx_{i2}}]_{q_l \times 1} = \boldsymbol{\Phi}_{yx_{q_l}},$$

and

$$E(\mathbf{d}_{q_l}\mathbf{d}_{q_l}^t) = [\lambda_3 S_{x_i x_j} + \theta S_{x_i x_{j(2)}}]_{q_l \times q_l} = \mathbf{\Lambda}_{q_l}; \quad i = 1, 2, \dots, q_l, \quad l = 1, 2, 3.$$

Now (4.4) can be written as

$$\boldsymbol{\Phi}_{yx_q} - \mathbf{T}_q \mathbf{h}_q = 0$$

or

$$(4.7) \quad \mathbf{h}_q = \mathbf{T}_q^{-1} \boldsymbol{\Phi}_{yx_q}, \text{ provided } \mathbf{T}_q^{-1} \text{ exist.}$$

To use the members of the suggested class practically, we require the elements of $\mathbf{h}$. In almost all practical situations, the values of unknown parameters required for $\mathbf{h}$ are not available, hence for practical purposes we have to estimate

the elements of $\mathbf{h}$ from sample observations. Without knowing or estimating the elements, we cannot use the members of the proposed class.

Now, from (4.2), ignoring the third and higher order terms for each expansion of the product, we get

$$\begin{aligned}
 E(t_1^m - \bar{Y}) \approx E \bigg(& \bar{e}_{y_2^*} - a\alpha_{q_1}^t \mathbf{d}_{q_1}^* - \bar{Y}bc\alpha_{q_2}^t \mathbf{X}_{q_2} \mathbf{d}_{q_2}^* + \frac{\bar{Y}}{2}dg\alpha_{q_3}^t \mathbf{X}_{q_3} \mathbf{d}_{q_3}^* \\
 & + \frac{\bar{Y}}{2}bc^2\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \alpha_{q_2} \mathbf{1}_{q_2}^t \bar{e}_{x_{2q_2}^*}^2 + \frac{\bar{Y}}{2}bc\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \bar{e}_{x_{2q_2}^*}^2 \\
 & + \frac{\bar{Y}}{2}bc^2\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \alpha_{q_2} \mathbf{1}_{q_2}^t \bar{e}_{x_{1q_2}^*}^2 - \bar{Y}bc^2\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \bar{e}_{x_{1q_2}^*} \bar{e}_{x_{2q_2}^*}^t \alpha_{q_2} \\
 & - \frac{\bar{Y}}{2}bc\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \bar{e}_{x_{1q_2}^*}^2 - bc\alpha_{q_2}^t \mathbf{X}_{q_2} \mathbf{d}_{q_2}^* \bar{e}_{y_2^*} \\
 & + abc\alpha_{q_2}^t \mathbf{X}_{q_2} \mathbf{d}_{q_2}^* \mathbf{d}_{q_1}^{*'} \alpha_{q_1} - \frac{\bar{Y}}{4}dg\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \bar{e}_{x_{2q_3}^*}^2 \\
 & + \frac{\bar{Y}}{4}dg\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \bar{e}_{x_{1q_3}^*}^2 + \frac{\bar{Y}}{8}dg^2\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} \mathbf{1}_{q_3}^t \bar{e}_{x_{2q_3}^*}^2 \\
 & + \frac{\bar{Y}}{8}dg^2\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} \mathbf{1}_{q_3}^t \bar{e}_{x_{1q_3}^*}^2 - \frac{\bar{Y}}{4}dg^2\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \bar{e}_{x_{1q_3}^*} \bar{e}_{x_{2q_3}^*}^t \alpha_{q_3} \\
 & - \frac{adg}{2}\alpha_{q_3}^t \mathbf{X}_{q_3} \mathbf{d}_{q_3}^* \mathbf{d}_{q_1}^{*'} \alpha_{q_1} + \frac{dg}{2}\alpha_{q_3}^t \mathbf{X}_{q_3} \mathbf{d}_{q_3}^* \bar{e}_{y_2^*} \bigg).
 \end{aligned}$$

Now taking expectation, bias can be written as:

(4.8)

$$\begin{aligned}
 \text{Bias}(t_q) \approx bc \bigg\{ & a\alpha_{q_2}^t \mathbf{X}_{q_2} \Lambda_{q_{21}} \alpha_{q_1} - \alpha_{q_2}^t \mathbf{X}_{q_2} \Phi_{yx_{q_2}} + \frac{\bar{Y}}{2}\alpha_{q_2}^t \mathbf{X}_{q_2}^2 (s_{2q_2}^* - s_{1q_2}^*) \\
 & + c \left(\frac{\bar{Y}}{2}\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \alpha_{q_2} \mathbf{1}_{q_2}^t (s_{1q_2}^* + s_{2q_2}^*) - \bar{Y}\alpha_{q_2}^t \mathbf{X}_{q_2}^2 \Sigma_{q_2} \alpha_{q_2} \right) \bigg\} \\
 & + dg \bigg\{ -\frac{a}{2}\alpha_{q_3}^t \mathbf{X}_{q_3} \Lambda_{3q_{31}} \alpha_{q_1} + \frac{\bar{Y}}{4}\alpha_{q_3}^t \mathbf{X}_{q_3}^2 (s_{1q_3}^* - s_{2q_3}^*) \\
 & + \frac{1}{2}\alpha_{q_3}^t \mathbf{X}_{q_3} \Phi_{yx_{q_3}} + g \left(\frac{\bar{Y}}{8}\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} \mathbf{1}_{q_3}^t (s_{1q_3}^* + s_{2q_3}^*) \right. \\
 & \left. - \frac{\bar{Y}}{4}\alpha_{q_3}^t \mathbf{X}_{q_3}^2 \Sigma_{q_3} \alpha_{q_3} \right) \bigg\}
 \end{aligned}$$

Bias(t_q) = Bias due to regression-cum-ratio

+ Bias due to regression-cum-exponential,

where the values of α_{q_i} 's can be used from $\mathbf{h}$ given in (4.7) and

$$\begin{aligned}
 E(\bar{e}_{x_{1q_l}^*} \bar{e}_{x_{2q_l}^*}^t) &= E(\bar{e}_{x_{1q_l}^*} \bar{e}_{x_{1q_l}^*}^t) = [\lambda_1 S_{x_i x_{j(2)}}]_{q_l \times q_l} = \Sigma_{q_l}, \\
 E(\bar{e}_{x_{1q_l}^*}^2) &= [\lambda_1 S_{x_i}^2 + \theta_1 S_{x_{i(2)}}^2]_{q_l \times 1} = \mathbf{s}_{1q_l}^*
 \end{aligned}$$

and

$$E(\bar{\mathbf{e}}_{x_{2q_l}}^2) = [\lambda_2 S_{x_i}^2 + \theta_2 S_{x_i(2)}^2]_{q_l \times 1} = \mathbf{s}_{2q_l}^*.$$

By using the normal equations that are used to find the optimum values given in equation (4.7), equation (4.3) can be written as:

$$\text{MSE}(t_q) = E(\bar{e}_{y_2^*}(\bar{e}_{y_2^*} - \mathbf{h}_q^t \mathbf{m}_q))$$

or

$$\text{MSE}(t_q) = E(\bar{e}_{y_2^*}^2) - \mathbf{h}_q^t E(\mathbf{m}_q \bar{e}_{y_2^*})$$

or

$$\text{MSE}(t_q) = \lambda_2 S_y^2 + \theta S_{y(2)}^2 - \mathbf{h}_q^t \Phi_{yx_q}.$$

Now, using equation (4.5), we get:

$$(4.9) \quad \text{MSE}(t_q) = \lambda_2 S_y^2 + \theta S_{y(2)}^2 - \Phi_{yx_q}^t \mathbf{T}_q^{-1} \Phi_{yx_q}.$$

A number of special cases of proposed class (4.1) may be deduced. These members of the class with their expressions of bias and MSE's are given in the following remarks.

Remark 1. We can obtain a regression estimator using multi-auxiliary variables when non-response occurs at both phases by substituting $a = 1$, $b = c = d = g = 0$ in (4.1) i.e.

$$(4.10) \quad t_{q_{\text{reg}}} = \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{1i} (\bar{x}_{1i}^* - \bar{x}_{2i}^*).$$

The bias of (4.10) can be obtained by substituting $a = 1$, $b = c = d = g = 0$ in (4.8) as

$$\text{Bias}(t_{q_{\text{reg}}}) = 0.$$

The optimum value $\alpha_{1q_1} = (\Lambda_{q_1})^{-1} \Phi_{yx_{q_1}}$, the mean square error of (4.10) can be obtained by substituting $a = 1$, $b = c = d = g = 0$ in (4.5) and (4.6) and using (4.9) as

$$(4.11) \quad \text{MSE}(t_{q_{\text{reg}}}) = \lambda_2 S_y^2 + \theta_2 S_{y(2)}^2 - \Phi_{yx_{q_1}} (\Lambda_{q_1})^{-1} \Phi_{yx_{q_1}}.$$

For $a = -1$, the difference estimator can be obtained as

$$t_{q_{\text{reg}}} = \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{1i} (\bar{x}_{2i}^* - \bar{x}_{1i}^*).$$

Now for the optimum value $\alpha_{2q_1} = -\alpha_{1q_1}$, the MSE will remain the same as in (4.11) and bias will be zero.

Remark 2. We can obtain a ratio estimator by substituting $b = c = 1$, $a = d = g = 0$ in (4.1) i.e.

$$(4.12) \quad t_{q_{\text{ratio}}} = \bar{y}_2^* \prod_{i=1}^{q_2} \left(\frac{\bar{x}_{1i}^*}{\bar{x}_{2i}^*} \right)^{\alpha_{1i}}.$$

The bias of (4.12) can be obtained by substituting $b = c = 1$, $a = d = g = 0$ in (4.8) as

$$\begin{aligned} \text{Bias}(t_{q_{\text{ratio}}}) &\approx -\alpha_{1_{q_2}}^t X_{q_2} \Phi_{yx_{q_2}} + \frac{\bar{Y}}{2} \alpha_{1_{q_2}}^t X_{q_2}^2 (s_{2_{q_2}}^* - s_{1_{q_2}}^*) \\ &\quad + \frac{\bar{Y}}{2} \alpha_{1_{q_2}}^t X_{q_2}^2 \alpha_{1_{q_2}} 1_{q_2} (s_{1_{q_2}}^* + s_{2_{q_2}}^*) - \bar{Y} \alpha_{1_{q_2}}^t X_{q_2}^2 \Sigma_{q_2} \alpha_{1_{q_2}}. \end{aligned}$$

For the optimum value $\alpha_{1_{q_2}} = \bar{Y}^{-1} \mathbf{X}_{q_2}^{-1} (\Lambda_{q_2})^{-1} \Phi_{yx_{q_2}}$, the mean square error of (4.12) can be obtained by substituting $b = c = 1$, $a = d = g = 0$ in (4.5) and (4.6) and using (4.9) as:

$$(4.13) \quad \text{MSE}(t_{q_{\text{ratio}}}) \approx \lambda_2 S_y^2 + \theta_2 S_{y_2}^2 - \Phi_{yx_{q_2}} (\Lambda_{q_2})^{-1} \Phi_{yx_{q_2}}.$$

We can obtain a product estimator when $b = -1$ as

$$t_{q_{\text{prod}}} = \bar{y}_2^* \prod_{i=1}^{q_2} \left(\frac{\bar{x}_{2i}^*}{\bar{x}_{1i}^*} \right)^{\alpha_{2i}}.$$

For the optimum value $\alpha_{2_{q_2}} = -\alpha_{1_{q_2}}$, the bias and MSE will remain the same as for the estimator given (4.12). If $q_1 = q_2$, (4.13) will be the same as (4.11).

Remark 3. We can obtain an exponential-ratio estimator by substituting $d = g = -1$, $a = b = c = 0$ in (4.1) i.e.

$$(4.14) \quad t_{q_{\text{er}}} = \bar{y}_2^* \exp \sum_{i=q_2+1}^{q_3} \alpha_{1i} \left(\frac{(\bar{x}_{1i}^* - \bar{x}_{2i}^*)}{(\bar{x}_{2i}^* + \bar{x}_{1i}^*)} \right).$$

The bias of (4.14) can be obtained by substituting $d = g = -1$, $a = b = c = 0$ in (4.8) as

$$\begin{aligned} \text{Bias}(t_{q_{\text{er}}}) &\approx \frac{\bar{Y}}{4} \alpha_{q_3}^t \mathbf{X}_{q_3}^2 (s_{1_{q_3}}^* - s_{2_{q_3}}^*) + \frac{1}{2} \alpha_{q_3}^t \mathbf{X}_{q_3} \Phi_{yx_{q_3}} \\ &\quad + \frac{\bar{Y}}{8} \alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} 1_{q_3} (s_{1_{q_3}}^* + s_{2_{q_3}}^*) - \frac{\bar{Y}}{4} \alpha_{q_3}^t \mathbf{X}_{q_3}^2 \Sigma_{q_3} \alpha_{q_3} \end{aligned}$$

The optimum value $\alpha_{1_{q_3}} = 2\bar{Y}^{-1} \mathbf{X}_{q_3}^{-1} (\Delta_{q_3})^{-1} \Phi_{yx_{q_3}}$. The mean square error of (4.14) can be obtained by substituting $d = g = -1$, $a = b = c = 0$ in (4.5) and (4.6) and using (4.9) as

$$(4.15) \quad \text{MSE}(t_{q_{\text{er}}}) = \lambda_2 S_y^2 + \theta_2 S_{y_2}^2 - \Phi_{yx_{q_2}} (\Delta_{q_2})^{-1} \Phi_{yx_{q_2}}.$$

For $g = 1$ we can obtain the exponential-product estimator

$$t_{q_{er}} = \bar{y}_2^* \exp \sum_{i=q_2+1}^{q_3} \alpha_{2i} \left(\frac{(\bar{x}_{2i}^* - \bar{x}_{1i}^*)}{(\bar{x}_{2i}^* + \bar{x}_{1i}^*)} \right).$$

The optimum value $\alpha_{2_{q_3}} = -\alpha_{1_{q_3}}$, The bias MSE will remain the same as for the estimator given in (4.14).

Remark 4. We can obtain a regression-cum-ratio estimator by substituting $a = b = c = 1$, $d = g = 0$ in (4.1) i.e.

$$(4.16) \quad t_{q_{rcr}} = \left\{ \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{3i} (\bar{x}_{1i}^* - \bar{x}_{2i}^*) \right\} \left\{ \prod_{i=1}^{q_2} \left(\frac{\bar{x}_{1i}^*}{\bar{x}_{2i}^*} \right)^{\alpha_{3i}} \right\}.$$

The bias of (4.16) can be obtained by substituting $a = b = c = 1$, $d = g = 0$ in (4.8) as

$$\begin{aligned} \text{Bias}(t_{q_{rcr}}) &\approx \alpha_{3_{q_2}}^t X_{q_2} \Delta_{q_{21}} \alpha_{3_{q_1}} - \alpha_{3_{q_2}}^t X_{q_2} \Phi_{yx_{q_2}} + \frac{\bar{Y}}{2} \alpha_{3_{q_2}}^t X_{q_2}^2 (s_{2_{q_2}}^* - s_{1_{q_2}}^*) \\ &\quad + \frac{\bar{Y}}{2} \alpha_{3_{q_2}}^t X_{q_2}^2 \alpha_{3_{q_2}} 1_{q_2}^t (s_{1_{q_2}}^* + s_{2_{q_2}}^*) - \bar{Y} \alpha_{3_{q_2}}^t X_{q_2}^2 \sum_{q_2} \alpha_{3_{q_2}}. \end{aligned}$$

The optimum values $\alpha_{3_{q_1}}$ and $\alpha_{3_{q_2}}$ are $\alpha_{3_{q_1}} = \Delta_{q_1}^{-1} \Phi_{yx_{q_3}} + \Delta_{q_1}^{-1} \Delta_{q_{12}} \Delta_{q_2}^{*-1} \Delta_{q_{21}}^{-1} \cdot \Delta_{q_1}^{-1} \Phi_{yx_{q_1}} - \Delta_{q_1}^{-1} \Delta_{q_{12}} \Delta_{q_2}^{*-1} \Phi_{yx_{q_1}}$ and $\alpha_{3_{q_2}} = \bar{Y}^{-1} X_{q_2}^{-1} \Delta_{q_2}^{*-1} (\Phi_{yx_{q_2}} - \Delta_{q_{21}} \Delta_{q_1}^{-1} \cdot \Phi_{yx_{q_1}})$, where $\Delta_{q_2}^* = (\Delta_{q_2} - \Delta_{q_{21}} \Delta_{q_1}^{-1} \Delta_{q_{12}})$.

The mean square error of (4.16) can be obtained by substituting $a = b = c = 1$, $d = g = 0$ in (4.5) and (4.6) and using (4.9) as

$$(4.17) \quad \begin{aligned} \text{MSE}(t_{q_{rcr}}) &\approx \lambda_2 S_y^2 + \theta_2 S_{y_2}^2 - \Phi_{yx_{q_1}}^t \Delta_{q_1}^{-1} \Phi_{yx_{q_1}} \\ &\quad + (2\Phi_{yx_{q_2}}^t - \Phi_{yx_{q_1}}^t B_{q_{12}}^*) \Delta_{q_2}^{*-1} B_{q_{21}}^* \Phi_{yx_{q_1}}^t - \Phi_{yx_{q_2}}^t \Delta_{q_2}^{*-1} \Phi_{yx_{q_2}}^t \end{aligned}$$

where $B_{(q_{12})}^* = \Delta_{q_1}^{-1} \Delta_{q_{12}}$.

For $(a, b, c) = \{(1, 1, -1), (-1, 1, 1), (-1, 1, -1)\}$, the regression-cum-product, difference-cum-ratio and differencecum-product estimator are

$$\begin{aligned} t_{q_{rcp}} &= \left\{ \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{r_i} (\bar{x}_{1i}^* - \bar{x}_{2i}^*) \right\} \left\{ \prod_{i=1}^{q_2} \left(\frac{\bar{x}_{2i}^*}{\bar{x}_{1i}^*} \right)^{\alpha_{4i}} \right\}, \\ t_{q_{dcr}} &= \left\{ \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{5i} (\bar{x}_{2i}^* - \bar{x}_{1i}^*) \right\} \left\{ \prod_{i=1}^{q_2} \left(\frac{\bar{x}_{1i}^*}{\bar{x}_{2i}^*} \right)^{\alpha_{5i}} \right\}. \end{aligned}$$

and $t_{q_{dcp}} = \{\bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{6i} (\bar{x}_{2i}^* - \bar{x}_{1i}^*)\} \{\prod_{i=1}^{q_2} (\frac{\bar{x}_{2i}^*}{\bar{x}_{1i}^*})^{\alpha_{6i}}\}$ respectively. The optimum values for $t_{q_{rcp}}$ are $\alpha_{4_{q_1}} = \alpha_{q_{q_1}}$ and $\alpha_{4_{q_2}} = -\alpha_{3_{q_2}}$, for $t_{m_{dcr}} \alpha_{5_{q_1}} = -\alpha_{3_{q_1}}$ and $\alpha_{5_{q_2}} = \alpha_{3_{q_2}}$ and for $t_{q_{dcp}} \alpha_{6_{q_1}} = -\alpha_{3_{q_1}}$ and $\alpha_{6_{q_2}} = -\alpha_{3_{q_2}}$.

The bias and MSE of regression-cum-product, difference-cum-ratio and difference-cum-product estimators are the same as for the estimator given in (4.16).

Remark 5. We can obtain a regression-cum-exponential-ratio estimator by substituting $a = d = g = -1$, $b = c = 0$ in (4.1) i.e.

$$(4.18) \quad t_{q_{rcer}} = \left\{ \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{7_i} (\bar{x}_{1i}^* - \bar{x}_{2i}^*) \right\} \left\{ \exp \sum_{i=q_2+1}^{q_3} \alpha_{7_i} \left(\frac{(\bar{x}_{1i}^* - \bar{x}_{2i}^*)}{(\bar{x}_{2i}^* + \bar{x}_{1i}^*)} \right) \right\}.$$

The bias of (4.18) can be obtained by substituting $a = d = g = -1$, $b = c = 0$ in (4.8) as

$$\begin{aligned} \text{Bias}(t_{q_{rcer}}) \approx & -\frac{1}{2} \alpha_{q_3}^t \mathbf{X}_{q_3} \Delta_{3q_{31}} \alpha_{q_1} + \frac{\bar{Y}}{4} \alpha_{q_3}^t \mathbf{X}_{q_3}^2 (s_{1q_3}^* - s_{2q_3}^*) + \frac{1}{2} \alpha_{q_3}^t \mathbf{X}_{q_3} \Phi_{yx_{q_3}} \\ & + \frac{\bar{Y}}{8} \alpha_{q_3}^t \mathbf{X}_{q_3}^2 \alpha_{q_3} \mathbf{1}_{q_3} (s_{1q_3}^* + s_{2q_3}^*) - \frac{\bar{Y}}{4} \alpha_{q_3}^t \mathbf{X}_{q_3}^2 \Sigma_{q_3} \alpha_{q_3}. \end{aligned}$$

The optimum values $\alpha_{7_{q_1}}$ and $\alpha_{7_{q_3}}$ are $\alpha_{7_{q_1}} = \Delta_{q_1}^{-1} \Phi_{yx_{q_3}} + \Delta_{q_1}^{-1} \Delta_{q_{12}} \Delta_{q_2}^{*-1} \Delta_{q_{21}}^{-1} \cdot \Delta_{q_1}^{-1} \Phi_{yx_{q_1}} - \Delta_{q_1}^{-1} \Delta_{q_{12}} \Delta_{q_2}^{*-1} \Phi_{yx_{q_1}}$ and $\alpha_{7_{q_3}} = 2\bar{Y}^{-1} \mathbf{X}_{q_2}^{-1} \Delta_{q_2}^{*-1} (\Phi_{yx_{q_2}} - \Delta_{q_{21}} \Delta_{q_1}^{-1} \cdot \Phi_{yx_{q_1}})$.

The mean square error of (4.16) can be obtained by substituting $a = d = g = 1$, $b = c = 0$ in (4.5) and (4.6) and using (4.9) as

$$(4.19) \quad \text{MSE}(t_{q_{rcer}}) \approx \lambda_2 S_y^2 + \theta_y^2 - \Phi_{yx_{q_1}}^t \Delta_{q_1}^{-1} \Phi_{yx_{q_1}} + (2\Phi_{yx_{q_2}}^t - \Phi_{yx_{q_1}}^t B_{q_{12}}^*) \Delta_{q_2}^{*-1} B_{q_{21}}^* \Phi_{yx_{q_1}}^t - \Phi_{yx_{q_2}}^t \Delta_{q_2}^{*-1} \Phi_{yx_{q_2}}^t.$$

For $(a, d, b) = \{(1, 1, 1), (-1, 1, -1), (-1, 1, 1)\}$ the regression-cum-exponential-product, difference-cum-exponential ratio and difference-cum-exponential-product estimator can be obtained as

$$\begin{aligned} t_{q_{rcer}} &= \left\{ \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{8_i} (\bar{x}_{1i}^* - \bar{x}_{2i}^*) \right\} \left\{ \exp \sum_{i=q_2+1}^{q_3} \alpha_{8_i} \left(\frac{(\bar{x}_{2i}^* - \bar{x}_{1i}^*)}{(\bar{x}_{2i}^* + \bar{x}_{1i}^*)} \right) \right\}, \\ t_{q_{dcer}} &= \left\{ \bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{9_i} (\bar{x}_{2i}^* - \bar{x}_{1i}^*) \right\} \left\{ \exp \sum_{i=q_2+1}^{q_3} \alpha_{9_i} \left(\frac{(\bar{x}_{1i}^* - \bar{x}_{2i}^*)}{(\bar{x}_{2i}^* + \bar{x}_{1i}^*)} \right) \right\} \end{aligned}$$

and $t_{q_{dcep}} = \{\bar{y}_2^* + \sum_{i=1}^{q_1} \alpha_{10_i} (\bar{x}_{2i}^* - \bar{x}_{1i}^*)\} \{\exp \sum_{i=q_2+1}^{q_3} \alpha_{10_i} (\frac{\bar{x}_{2i}^* - \bar{x}_{1i}^*}{\bar{x}_{2i}^* + \bar{x}_{1i}^*})\}$ respectively. The optimum values for $t_{m_{rcp}}$ are $\alpha_{8_{q_1}} = \alpha_{7_{q_1}}$ and $\alpha_{8_{q_3}} = -\alpha_{7_{q_3}}$, for $t_{m_{dcer}}$ $\alpha_{9_{q_1}} = -\alpha_{7_{q_1}}$ and $\alpha_{9_{q_3 \times 1}} = \alpha_{7_{q_3 \times 1}}$ and for $t_{m_{dcp}}$ $\alpha_{10_{q_1}} = -\alpha_{7_{q_1}}$ and $\alpha_{10_{q_3}} = -\alpha_{7_{q_3}}$.

The bias and MSE of regression-cum-exponential-product, difference-cum-exponential-ratio and difference-cum-exponential-product estimators are the same as for the estimators given in (4.19).

The expressions of bias and MSE for the above special cases can be obtained from (2.9) and (2.10) by using a suitable combination of (a, b, c, d, g) . The following table contains some existing estimators which are modified according to the situation discussed in this paper and are special cases of the proposed class.

Table 1. Modified form of some existing estimators as special cases of the proposed class.

Sr. #	$(a, b, c, d, g, q_1, q_2, q_3)$	Estimator	Reference
1	$(0, 0, 0, 1, 0, 0, 0, 0)$	$\bar{y}_2^*$	Hansen and Hurwitz (1946)
2	$(0, 1, 1, 0, 0, 0, 1, 0)$	$\bar{y}_2^* \left(\frac{\bar{x}_1^*}{\bar{x}_2^*} \right)$	Khare and Srivastava (1993, 1995)
3	$(0, 1, -1, 0, 0, 0, 1, 0)$	$\bar{y}_2^* \left(\frac{\bar{x}_2^*}{\bar{x}_1^*} \right)$	
4	$(1, 0, 0, 1, 0, 1, 0, 0)$	$\bar{y}_2^* + \alpha(\bar{x}_1^* + \bar{x}_2^*)$	Singh <i>et al.</i> (2010)
5	$(0, 0, 0, 1, -1, 0, 0, 1)$	$\bar{y}_2^* \exp \left(\frac{\bar{x}_1^* - \bar{x}_2^*}{\bar{x}_1^* + \bar{x}_2^*} \right)$	
6	$(0, 0, 0, 1, 1, 0, 0, 1)$	$\bar{y}_2^* \exp \left(\frac{\bar{x}_2^* - \bar{x}_1^*}{\bar{x}_2^* + \bar{x}_1^*} \right)$	

4.1. Theoretical comparison for the proposed class

In this section we compare our proposed class of estimators with Hansen and Hurwitz (1946). The comparison of the proposed class of estimators with Hansen and Hurwitz (1946) is:

$$\text{MSE}(\bar{y}_2^*) - \text{MSE}(t_q) = \Phi_{yx_q}^t \mathbf{T}_q^{-1} \Phi_{yx_q} > 0.$$

We can observe that our proposed class of estimators is more efficient than Hansen and Hurwitz (1946).

5. Empirical comparison

For empirical comparison, the data set of the Census Report of the Faisalabad District (1998) Pakistan has been used, which was also used by Ahmad *et al.* (2009b). For calculation of Bias and MSE of the estimators, it is assumed from 283 population units that 28% are non-respondents i.e. $N_2 = 80$ and remaining $N_1 = 203$ are respondents. The details of the study and auxiliary variables are given in Table A1. Each variable is taken from Rural Locality. The necessary results required for the calculation of bias and MSE are given in Tables A2, A3 and A4. The values of bias and percent relative efficiency of the proposed and existing estimators for different values of n_1 , n_2 , k_1 and k_2 are given in Tables A5 and A6.

For different combinations of a , b , c , d and g fourteen members of the proposed class have been considered for empirical comparison. As in the literature, the only case discussed was where non response was considered in the second phase. Therefore, for comparison purpose, estimators of the literature are modified according to the situation where non response is considered in both phases and empirical results are given in Section 5.

From Table A5, the MSE of six estimators (ratio, product, regression, difference, exponential, exponential ratio and exponential product) is the same and similarly, for eight remaining estimators is the same. The group of eight estimators is more efficient because the Percent Relative Efficiency (PRE) of

Table A1. Description of variables (Each variable is taken from Rural Locality).

Description of variables	
Y_2	Population of currently married
X_1	Population of both sexes
X_2	Population of primary but below matric
X_3	Population of matric and above
X_4	Population of 18 years old and above
X_5	Population of women 15-49 years old

Table A2. Average of all the Variables w.r.t. N (283).

Variables	Y_2	X_1	X_2	X_3	X_4	X_5
Average	1511.261	10931.44	1969.286	754.3604	6173.163	2457.682

Table A3. Variance-covariance matrix w.r.t. N .

	Y_2	X_1	X_2	X_3	X_4	X_5
Y_2	621542.7328	3963614.8	792130.418	303104.0047	2481864.876	823818.0161
X_1	3963614.842	29441849	5836183.44	2230311.546	17507723.7	6588651.038
X_2	792130.4178	5836183.4	2815406.61	624113.7475	3788276.829	1390088.258
X_3	303104.0047	2230311.5	624113.748	276331.9476	1795351.395	533963.3136
X_4	2481864.876	17507724	3788276.83	1795351.395	39560446.26	4060606.428
X_5	823818.0161	6588651	1390088.26	533963.3136	4060606.428	2196836.707

Table A4. Variance-covariance matrix w.r.t. N_2 .

	Y_2	X_1	X_2	X_3	X_4	X_5
Y_2	915368.5722	5470158.3	1207503.21	429915.8	3939985.41	1236879.163
X_1	5470158.306	43396500	8701020.14	3059560.347	27495437.31	8863944.5
X_2	1207503.209	8701020.1	2605859.46	788441.1218	6822705.087	1983612.909
X_3	429915.8	3059560.3	788441.122	382683.9487	3553687.451	716956.6554
X_4	3939985.41	27495437	6822705.09	3553687.451	120333017.3	6217267.278
X_5	1236879.163	8863944.5	1983612.91	716956.6554	6217267.278	2000402.069

this group is greater than the group consisting of six estimators. To check the performance within these two groups, there is a need to observe the bias of these from Table A6. This group of eight estimators having the same MSE can be further divided into two groups on the basis of biases, each consisting of four estimators. The first group of four estimators (Regression-cum-exponential-ratio, Regression-cum-exponential-product, difference-cum-exponential-ratio and difference-cum-exponential-product) is preferred because it has less bias as compared to the second group of estimators (Regression-cum-ratio, Regression-cum-product, difference-cum-ratio and difference-cum-product).

Table A5. Results of the percent relative efficiency w.r.t. $\bar{y}_2^*$ of existing and proposed estimators.

$k_1 = k_2$	n_1	n_2	T_{R1d}	T_{lr1d}	$t_{R(-1)d}^{(1)}$	$t_{lr1}^m, t_{d1}^m, t_{r1}^m,$ $t_{ep1}^m, t_{er1}^m, t_{p1}^m$	$t_{rcr1}^m, t_{rcp1}^m, t_{dcr1}^m, t_{dcp1}^m,$ $t_{rcer1}^m, t_{rcep1}^m, t_{dcer1}^m, t_{dcep1}^m$
2	130	90	107.68	112.10	126.54	158.86	160.34
2	130	100	105.85	108.99	119.41	138.68	139.53
2	130	110	103.96	105.93	112.63	122.95	123.39
2	130	120	102.01	102.94	106.16	110.34	110.52
2	140	90	109.03	114.32	132.18	178.35	180.58
2	140	100	107.35	111.37	125.19	154.71	156.06
2	140	110	105.61	108.46	118.50	136.41	137.19
2	140	120	103.81	105.60	112.08	121.81	122.23
2	150	90	110.22	116.32	137.48	200.55	203.78
2	150	100	108.68	113.52	130.68	172.82	174.83
2	150	110	107.08	110.75	124.12	151.51	152.75
2	150	120	105.41	108.02	117.78	134.63	135.36
2	160	90	111.29	118.12	142.49	226.07	230.65
2	160	100	109.87	115.47	135.89	193.44	196.33
2	160	110	108.39	112.84	129.48	168.59	170.43
2	160	120	106.86	110.23	123.26	149.03	150.19
3	130	90	107.41	110.49	124.41	150.01	151.14
3	130	100	105.59	107.79	117.75	132.37	133.01
3	130	110	103.75	105.15	111.48	118.98	119.32
3	130	120	101.89	102.55	105.57	108.47	108.61
3	140	90	108.70	112.38	129.50	166.66	168.33
3	140	100	107.02	109.83	122.94	145.74	146.75
3	140	110	105.31	107.31	116.74	130.04	130.62
3	140	120	103.57	104.84	110.87	117.81	118.12
3	150	90	109.85	114.07	134.25	185.67	188.07
3	150	100	108.29	111.66	127.83	160.82	162.30
3	150	110	106.69	109.27	121.72	142.38	143.29
3	150	120	105.07	106.91	115.90	128.16	128.70
3	160	90	110.87	115.60	138.70	207.59	210.98
3	160	100	109.42	113.31	132.43	177.95	180.05
3	160	110	107.93	111.04	126.44	156.26	157.58
3	160	120	106.42	108.79	120.70	139.69	140.52

Now we want to check the rate of increase/decrease in Bias and MSE of the different estimators by increasing the values of n_1 , n_2 , k_1 and k_2 under the assumption that the trend will be linear. For this purpose, two values are considered for each n_1 , n_2 , k_1 and k_2 and slopes are calculated for Bias and MSE of all estimators. The results are given in Table A7.

From Table A7, by comparing our suggested and existing estimators, we can observe that when we increase the value of k_1 and k_2 , the value of PRE of estimators decreased as expected. The decrease in PRE of T_{R1d} is more when

Table A5. (continued).

$k_1 = k_2$	n_1	n_2	T_{R1d}	T_{lr1d}	$t_{R(-1)d}^{(1)}$	$t_{r1}^m, t_{p1}^m, t_{lr1}^m,$ $t_{d1}^m, t_{ep1}^m, t_{er1}^m$	$t_{rcr1}^m, t_{rcp1}^m, t_{dcr1}^m, t_{dcp1}^m,$ $t_{rcer1}^m, t_{rcep1}^m, t_{dcer1}^m, t_{dcep1}^m$
4	130	90	107.25	112.10	123.24	144.14	145.07
4	130	100	105.45	108.99	116.83	128.38	128.92
4	130	110	103.64	105.93	110.85	116.56	116.84
4	130	120	101.82	102.94	105.25	107.36	107.48
4	140	90	108.52	114.32	128.02	158.76	160.14
4	140	100	106.84	111.37	121.71	140.03	140.85
4	140	110	105.14	108.46	115.78	126.14	126.62
4	140	120	103.44	105.60	110.21	115.43	115.69
4	150	90	109.64	116.32	132.47	175.43	177.39
4	150	100	108.07	113.52	126.28	153.10	154.30
4	150	110	106.48	110.75	120.43	136.77	137.51
4	150	120	104.88	108.02	114.91	124.31	124.76
4	160	90	110.64	118.12	136.63	194.60	197.34
4	160	100	109.17	115.47	130.57	167.89	169.57
4	160	110	107.68	112.84	124.81	148.65	149.72
4	160	120	106.18	110.23	119.35	134.14	134.82
5	130	90	107.15	110.49	122.48	139.81	140.64
5	130	100	105.36	107.79	116.25	125.53	126.00
5	130	110	103.57	105.15	110.46	114.86	115.11
5	130	120	101.78	102.55	105.05	106.60	106.70
5	140	90	108.40	112.38	127.08	152.91	154.11
5	140	100	106.72	109.83	120.93	135.92	136.65
5	140	110	105.04	107.31	115.19	123.40	123.83
5	140	120	103.36	104.84	109.80	113.79	114.02
5	150	90	109.51	114.07	131.34	167.76	169.45
5	150	100	107.93	111.66	125.30	147.54	148.58
5	150	110	106.35	109.27	119.63	132.84	133.48
5	150	120	104.76	106.91	114.29	121.67	122.06
5	160	90	110.49	115.60	135.32	184.76	187.09
5	160	100	109.01	113.31	129.40	160.60	162.04
5	160	110	107.52	111.04	123.80	143.32	144.23
5	160	120	106.03	108.79	118.52	130.34	130.92

compared to the proposed estimators and is less for the group of eight estimators. Increasing the value of k_1 and k_2 , bias increases for all estimators as expected. The bias of $t_{R(-1)d}^{(1)}$ increases the most and the bias of the exponential product and exponential ratio increase the least.

Increases in the value of n_1 increases the PRE of all the estimators considered in the comparison. The increase is high in suggested estimators particularly in the group of eight estimators. In the case of bias, almost all of the suggested

Table A6. Results of bias of existing and proposed estimators.

$k_1 = k_2$	n_1	n_2	T_{R1d}	T_{lr1d}	$t_{R(-1)d}^{(1)}$	$t_{r_1}^m, t_{p_1}^m$	$t_{ep_1}^m, t_{er_1}^m$	$t_{rcr_1}^m, t_{rcp_1}^m$, $t_{dcr_1}^m, t_{dcp_1}^m$	$t_{rcer_1}^m, t_{rcep_1}^m$, $t_{dcer_1}^m, t_{dcep_1}^m$
2	130	90	1.56	0.00027	5.75	0.04	-1.68	0.0662	0.063
2	130	100	1.05	0.00019	3.88	0.02	-1.13	0.0456	0.043
2	130	110	0.64	0.00012	2.35	0.01	-0.69	0.0287	0.026
2	130	120	0.29	0.00006	1.08	0.01	-0.31	0.0146	0.012
2	140	90	1.81	0.00032	6.68	0.04	-1.95	0.0761	0.074
2	140	100	1.31	0.00023	4.81	0.03	-1.40	0.0555	0.053
2	140	110	0.89	0.00016	3.28	0.02	-0.96	0.0386	0.036
2	140	120	0.54	0.00011	2.00	0.01	-0.58	0.0246	0.022
2	150	90	2.03	0.00035	7.48	0.05	-2.18	0.0847	0.082
2	150	100	1.52	0.00027	5.61	0.03	-1.64	0.0641	0.062
2	150	110	1.11	0.00020	4.08	0.03	-1.19	0.0472	0.045
2	150	120	0.76	0.00014	2.80	0.02	-0.82	0.0332	0.031
2	160	90	2.22	0.00038	8.18	0.05	-2.39	0.0923	0.090
2	160	100	1.71	0.00030	6.31	0.04	-1.84	0.0717	0.070
2	160	110	1.30	0.00023	4.78	0.03	-1.39	0.0548	0.053
2	160	120	0.95	0.00017	3.50	0.02	-1.02	0.0407	0.039
3	130	90	1.99	0.00054	7.21	-0.02	-2.04	0.0791	0.076
3	130	100	1.34	0.00037	4.87	-0.02	-1.38	0.0544	0.051
3	130	110	0.81	0.00023	2.95	-0.01	-0.84	0.0342	0.031
3	130	120	0.37	0.00011	1.35	0.00	-0.38	0.0174	0.014
3	140	90	2.31	0.00062	8.37	-0.03	-2.37	0.0910	0.088
3	140	100	1.66	0.00045	6.03	-0.02	-1.71	0.066	0.063
3	140	110	1.13	0.00031	4.11	-0.01	-1.16	0.046	0.043
3	140	120	0.69	0.00020	2.51	-0.01	-0.71	0.029	0.026
3	150	90	2.59	0.00069	9.37	-0.03	-2.65	0.101	0.099
3	150	100	1.94	0.00052	7.03	-0.02	-1.99	0.077	0.074
3	150	110	1.41	0.00038	5.11	-0.02	-1.45	0.056	0.054
3	150	120	0.97	0.00027	3.52	-0.01	-1.00	0.040	0.037
3	160	90	2.83	0.00076	10.25	-0.03	-2.90	0.110	0.108
3	160	100	2.18	0.00059	7.91	-0.03	-2.24	0.086	0.083
3	160	110	1.65	0.00045	5.99	-0.02	-1.70	0.066	0.063
3	160	120	1.21	0.00033	4.39	-0.01	-1.24	0.049	0.046

estimators are stable except $t_{R(-1)d}^{(1)}$. Increases in n_2 decreases the PRE for all estimators unexpectedly. The increase is high in the suggested estimators, especially in the group of eight. In the case of bias all are stable except $t_{R(-1)d}^{(1)}$. Furthermore, stability is high in the suggested estimators.

Table A6. (continued).

$k_1 = k_2$	n_1	n_2	T_{R1d}	T_{lr1d}	$t_{R(-1)d}^{(1)}$	$t_{r_1}^m, t_{p_1}^m$	$t_{ep_1}^m, t_{er_1}^m$	$t_{rcr_1}^m, t_{rcp_1}^m$, $t_{dcr_1}^m, t_{dcp_1}^m$	$t_{rcer_1}^m, t_{rcep_1}^m$, $t_{dcer_1}^m, t_{dcep_1}^m$
4	130	90	2.42	0.00080	8.67	-0.09	-2.38	0.091	0.088
4	130	100	1.63	0.00054	5.85	-0.06	-1.60	0.063	0.059
4	130	110	0.99	0.00033	3.55	-0.04	-0.97	0.039	0.036
4	130	120	0.45	0.00016	1.63	-0.02	-0.45	0.020	0.016
4	140	90	2.81	0.00092	10.06	-0.10	-2.76	0.105	0.102
4	140	100	2.02	0.00067	7.25	-0.07	-1.99	0.077	0.073
4	140	110	1.38	0.00046	4.94	-0.05	-1.35	0.053	0.050
4	140	120	0.84	0.00029	3.02	-0.03	-0.83	0.034	0.031
4	150	90	3.14	0.00103	11.27	-0.11	-3.09	0.117	0.114
4	150	100	2.36	0.00078	8.45	-0.08	-2.32	0.088	0.086
4	150	110	1.71	0.00057	6.15	-0.06	-1.69	0.065	0.062
4	150	120	1.18	0.00040	4.23	-0.04	-1.16	0.046	0.043
4	160	90	3.44	0.00113	12.33	-0.12	-3.38	0.127	0.125
4	160	100	2.65	0.00087	9.51	-0.09	-2.61	0.099	0.096
4	160	110	2.01	0.00066	7.20	-0.07	-1.98	0.076	0.073
4	160	120	1.47	0.00049	5.28	-0.05	-1.45	0.056	0.053
5	130	90	2.84	0.00106	10.13	-0.15	-2.69	0.103	0.099
5	130	100	1.92	0.00072	6.84	-0.10	-1.81	0.071	0.067
5	130	110	1.16	0.00044	4.14	-0.06	-1.10	0.045	0.041
5	130	120	0.53	0.00021	1.90	-0.03	-0.50	0.023	0.019
5	140	90	3.30	0.00122	11.76	-0.17	-3.12	0.119	0.115
5	140	100	2.38	0.00089	8.46	-0.13	-2.25	0.087	0.083
5	140	110	1.62	0.00061	5.77	-0.09	-1.53	0.060	0.057
5	140	120	0.99	0.00038	3.53	-0.05	-0.94	0.038	0.035
5	150	90	3.70	0.00137	13.17	-0.20	-3.49	0.133	0.129
5	150	100	2.77	0.00103	9.88	-0.15	-2.62	0.100	0.097
5	150	110	2.02	0.00075	7.18	-0.11	-1.91	0.074	0.071
5	150	120	1.39	0.00052	4.94	-0.07	-1.31	0.052	0.048
5	160	90	4.04	0.00150	14.40	-0.21	-3.82	0.145	0.141
5	160	100	3.12	0.00116	11.11	-0.16	-2.95	0.112	0.109
5	160	110	2.36	0.00088	8.42	-0.12	-2.23	0.086	0.083
5	160	120	1.73	0.00065	6.17	-0.09	-1.64	0.064	0.061

6. Conclusions

We can conclude from the above discussion regarding bias, relative efficiency and slopes that four classes of estimators (Regression-cum-exponential-ratio, Regression-cum-exponential-product, difference-cum-exponential-ratio and difference-cum-exponential-product) have less Bias and MSE than all suggested and existing estimators, and are also less sensitive to increasing the values of n_1 , n_2 , k_1 and k_2 . Hence this group is preferred over all other considered estimators.

Table A7. Slopes of Proposed and Existing Estimators

Estimators		$k_1 = k_2$	n_1	n_2
PREs	T_{R1d}	4.97	0.12	-0.15
	T_{lr1d}	4.64	0.20	-0.25
	$t_{R(-1)d}^{(1)}$	4.80	0.43	-0.54
	$t_{r_1}^m, t_{p_1}^m, t_{lr_1}^m,$ $t_{d_1}^m, t_{ep_1}^m, t_{er_1}^m$	4.02	1.02	-1.28
	$t_{rcr_1}^m, t_{rcp_1}^m, t_{dcr_1}^m, t_{dcp_1}^m,$ $t_{rcer_1}^m, t_{rcep_1}^m, t_{dcer_1}^m, t_{dcep_1}^m$	3.99	1.05	-1.31
	T_{R1d}	5.07	0.03	-0.06
	T_{lr1d}	5.00	0.00	0.00
Bias	$t_{R(-1)d}^{(1)}$	5.24	0.11	-0.21
	$t_{r_1}^m, t_{p_1}^m$	4.99	0.00	0.00
	$t_{ep_1}^m, t_{er_1}^m$	4.95	-0.03	0.06
	$t_{rcr_1}^m, t_{rcp_1}^m, t_{dcr_1}^m, t_{dcp_1}^m$	5.00	0.00	0.00
	$t_{rcer_1}^m, t_{rcep_1}^m, t_{dcer_1}^m, t_{dcep_1}^m$	5.00	0.00	0.00
	T_{R1d}	5.07	0.03	-0.06

It is also found that on increase in first phase sample size improves the performance of estimators rather than increases in the size of the second phase sample. Increase in the re-contacted sample increases the performance of estimators.

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