ON SOME PROPERTIES OF A GENERAL CLASS OF TWO-PIECE SKEW NORMAL DISTRIBUTION

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A new class of generalized two-piece skew normal distribution is introduced here as a two-piece version of the generalized skew normal distribution of Kumar and Anusree (2011). It is shown that the proposed class of distribution will be more suitable for modelling skewed, multimodal data sets. Several properties of the model are studied and the maximum likelihood estimation of the parameters of the distribution is discussed. Further, the practical usefulness of the model is illustrated with the help of certain real life data sets.

Key words and phrases: Asymmetric distributions, method of maximum likelihood, plurimodality, reliability measures, skew normal distribution.

1. Introduction

An important drawback of the normal distribution from a practical point of view is its symmetric behaviour. Azzalini (1985) provided a methodology to introduce skewness in a normal distribution and he defined the skew normal distribution as follows:

A random variable X is said to have the skew normal distribution with skewness parameter $\lambda \in R = (-\infty, \infty)$, denoted by $SND(\lambda)$, if its probability density function (p.d.f.) is of the following form, for $x \in R$.

$$(1.1) \quad g_1(x; \lambda) = 2f(x)F(\lambda x)$$

where $f(\cdot)$ and $F(\cdot)$ are respectively the p.d.f. and cumulative distribution function (c.d.f.) of a standard normal variate. The $SND(\lambda)$ has been studied by Azzalini (1985, 1986), Henze (1986) and several others. Further, various generalizations were studied by Azzalini and Dalla Valle (1996), Mudholkar and Hutson (2000), Branco and Dey (2001), Arnold and Beaver (2002), Arellano-Valle *et al.* (2004), Sharafi and Behboodian (2008), Jamalizadeh *et al.* (2008, 2009), and Kumar and Anusree (2011, 2013a, b, 2014a, b).

Kim (2005) introduced a two-piece version of the $SND(\lambda)$ through the following p.d.f.

$$(1.2) \quad g_K(x; \lambda) = \frac{2\pi f(x)F(\lambda|x|)}{(\pi + 2\tan^{-1}(\lambda))}.$$

The distribution of a random variable with p.d.f. (1.2) hereafter we denote as $TSND_K(\lambda)$. It is seen that there exists a symmetric behaviour in $TSND_K(\lambda)$

Received August 12, 2014. Revised December 5, 2014. Accepted December 31, 2014.

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as determined by the sign of λ on either side of the origin. In order to mitigate this limitation, Kumar and Anusree (2013a) introduced a wide class of the two-piece skew normal distribution denoted by $TSND_{KA}(\lambda, \rho)$, through the following p.d.f., in which $\rho \in [-1, 1]$ and $C = 2\pi[\pi - \tan^{-1}(\lambda) + \tan^{-1}(\rho\lambda)]^{-1}$.

$$(1.3) \quad g_{KA}(x; \lambda, \rho) = \begin{cases} Cf(x)F(\lambda x), & x < 0 \\ Cf(x)F(\rho\lambda x), & x \geq 0. \end{cases}$$

Note that $TSND_{KA}(-\lambda, -1)$ is $TSND_K(\lambda)$. In order to accomodate plurimodality, Kumar and Anusree (2011) developed a generalized version of the $SND(\lambda)$ by considering a mixture of the standard normal and $SND(\lambda)$ through the following p.d.f., in which $x \in R$, $\lambda \in R$ and $\alpha \geq -1$.

$$(1.4) \quad g_2(x; \lambda, \alpha) = \frac{2}{(\alpha + 2)} f(x)[1 + \alpha F(\lambda x)].$$

A distribution with p.d.f. (1.4) they notated as the $GMNSND(\lambda, \alpha)$. In order to develop a more flexible class of asymmetric, plurimodal normal type distributions, through this paper we propose a two-piece version of the $GMNSND(\lambda, \alpha)$ and named it “the extended two-piece skew normal distribution” or in short “the ETPSND”. The ETPSND reduces to the standard normal distribution, the $SND(\lambda)$ of Azzalini (1985), the $GMNSND(\lambda, \alpha)$, the two-piece skew normal distribution of Kim (2005) and the generalized two-piece skew normal distribution of Kumar and Anusree (2013a) for particular choices of its shape parameters as its special cases. The rest of the paper is organized as follows: The definition of ETPSND and some of its important properties are given in Section 2. In Section 3, we obtain expressions for certain reliability measures and discuss also some concepts regarding the mode of the distribution. In Section 4, a location-scale extension of the ETPSND is suggested and some of its important structural properties are studied. Finally, the maximum likelihood estimation of the parameters is discussed in Section 5 and its practical usefulness is illustrated with the help of certain real life data sets.

We need the following notation in the sequel. For any $a \in R$ and $b > 0$, define

$$(1.5) \quad \xi(a, b) = \int_a^\infty f(x) \int_0^{bx} f(y) dy dx,$$

so that

$$(1.6) \quad \xi(0, \lambda) = \frac{\tan^{-1}(\lambda)}{2\pi}$$

and for any reals a , b and s such that $bx + s > 0$,

$$(1.7) \quad \xi_s(a, b) = \int_a^\infty f(x) \int_0^{bx+s} f(y) dy dx.$$

2. Definition and properties

First we present the definition of the extended two-piece skew normal distribution and discuss some of its properties.

DEFINITION 2.1. *A random variable Z is said to follow the extended two-piece skew normal distribution with parameters $\lambda_1, \lambda_2 \in R$ and $\alpha \geq -1$ if its p.d.f. $h(z; \lambda_1, \lambda_2, \alpha)$ is of the following form. For $z \in R$,*

$$(2.1) \quad h(z; \lambda_1, \lambda_2, \alpha) = \begin{cases} \frac{2}{(\alpha + 2)} f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_1 z)], & z < 0 \\ \frac{2}{(\alpha + 2)} f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_2 z)], & z \geq 0 \end{cases}$$

where $D(\lambda_1, \lambda_2, \alpha) = \alpha \pi [\pi + \tan^{-1}(\lambda_2) - \tan^{-1}(\lambda_1)]^{-1}$.

Here $D(\lambda_1, \lambda_2, \alpha)$ is obtained by using Lemmas 2.1 and 2.2 as given in Kumar and Anusree (2013a). The distribution of a random variable Z with p.d.f. (2.1) we denoted as $ETPSND(\lambda_1, \lambda_2, \alpha)$.

For some particular choices of λ_1, λ_2 and α the p.d.f. given in (2.1) of $ETPSND(\lambda_1, \lambda_2, \alpha)$ is plotted and is shown in Fig. 1.

Some important special cases of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ are

(i) $ETPSND(\lambda_1, \lambda_2, 0)$, $ETPSND(0, 0, \alpha)$ or the limiting case of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ when $\lambda_1 \rightarrow -\infty, \lambda_2 \rightarrow \infty$, is the standard normal distribution,

(ii) the limiting case of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ when $\alpha \rightarrow \infty$ is the $TSND_{KA}(\lambda, \rho)$,

(iii) the limiting case of the $ETPSND(\lambda, \lambda, \alpha)$ when $\alpha \rightarrow \infty$ is the $TSND_K(\lambda)$,

(iv) the $ETPSND(\lambda, \lambda, \alpha)$ is the $GMNSND(\lambda, \alpha)$, which reduces to $SND(\lambda)$ when $\alpha = -1$ and

Figure 1. Probability plots of $ETPSND(\lambda_1, \lambda_2, \alpha)$ for different choices of α .

(v) the limiting case of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ as $\lambda_1 \rightarrow \infty$, $\lambda_2 \rightarrow \infty$ and $\alpha \rightarrow \infty$ or $\lambda_1 \rightarrow -\infty$, $\lambda_2 \rightarrow -\infty$ and $\alpha \rightarrow \infty$ is the standard half normal distribution.

We obtain certain structural properties of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ through the following results.

RESULT 2.1. If Z follows $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. $h(z; \lambda_1, \lambda_2, \alpha)$, then $Y_1 = -Z$ follows $ETPSND(-\lambda_2, -\lambda_1, \alpha)$.

PROOF. For any $y_1 \in R$, the p.d.f. $h_1(y_1; \lambda_1, \lambda_2, \alpha)$ of Y_1 is given by

$$\begin{aligned} h_1(y_1; \lambda_1, \lambda_2, \alpha) &= h(-y_1; \lambda_1, \lambda_2, \alpha) \left| \frac{dz}{dy_1} \right| \\ &= \begin{cases} \frac{2}{(\alpha+2)} f(y_1) [1 + D(\lambda_1, \lambda_2, \alpha) F(-\lambda_2 y_1)], & y_1 < 0 \\ \frac{2}{(\alpha+2)} f(y_1) [1 + D(\lambda_1, \lambda_2, \alpha) F(-\lambda_1 y_1)], & y_1 \geq 0, \end{cases} \end{aligned}$$

which shows that $Y_1 = -Z$ follows $ETPSND(-\lambda_2, -\lambda_1, \alpha)$.

RESULT 2.2. If Z follows $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. $h(z; \lambda_1, \lambda_2, \alpha)$, then $Y_2 = Z^2$ has p.d.f. (2.2).

PROOF. The p.d.f. $h_2(y_2; \lambda_1, \lambda_2, \alpha)$ of Y_2 is given by

$$\begin{aligned} (2.2) \quad h_2(y_2; \lambda_1, \lambda_2, \alpha) &= h(z; \lambda_1, \lambda_2, \alpha) \left| \frac{dz}{dy_2} \right| \\ &= h(-\sqrt{y_2}; \lambda_1, \lambda_2, \alpha) \left| \frac{dz}{dy_2} \right| + h(\sqrt{y_2}; \lambda_1, \lambda_2, \alpha) \left| \frac{dz}{dy_2} \right| \\ &= \frac{1}{(\alpha+2)} \left(\frac{f(\sqrt{y_2})}{2\sqrt{y_2}} \right) \\ &\quad \times [2 + D(\lambda_1, \lambda_2, \alpha)(F(-\lambda_1\sqrt{y_2}) + F(\lambda_2\sqrt{y_2}))]. \end{aligned}$$

Remark 2.1. When $\lambda_1 = \lambda_2$ and $\alpha = -1$, the p.d.f. given in (2.2) reduces to the p.d.f. of a Chi-square variate with one degree of freedom.

In order to find the distribution function of $ETPSND(\lambda_1, \lambda_2, \alpha)$, we need the following result.

RESULT 2.3. If Z is a $ETPSND(\lambda_1, \lambda_2, \alpha)$, then for any real d_1, d_2 such that $d_1 \leq d_2$, where $\xi(a, b)$ is as given in (1.5).

$$(2.3) \quad P(d_1 \leq Z \leq d_2)$$

$$= \begin{cases} \frac{2}{(\alpha+2)}[F(d_2) - F(d_1)] + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)}[F(d_2) - F(d_1)] \\ \quad + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)}[2\xi(d_1, \lambda_1) - 2\xi(d_2, \lambda_1)], & d_1 \leq d_2 < 0 \\ \frac{2}{(\alpha+2)}[F(d_2) - F(d_1)] + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)}[F(d_2) - F(d_1)] \\ \quad + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)}[2\xi(d_1, \lambda_2) - 2\xi(d_2, \lambda_2)], & 0 \leq d_1 \leq d_2. \end{cases}$$

PROOF. For any $d_1 \leq d_2 < 0$, by definition,

$$\begin{aligned} (2.4) \quad P(d_1 \leq Z \leq d_2) &= \int_{d_1}^{d_2} h(z; \lambda_1, \lambda_2, \alpha) dz \\ &= \int_{d_1}^{d_2} \left[\frac{2}{(\alpha+2)} f(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} 2f(z)F(\lambda_1 z) \right] dz \\ &= \frac{2}{(\alpha+2)} [F(d_2) - F(d_1)] \\ &\quad + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} [G(d_2, \lambda_1) - G(d_1, \lambda_1)] \end{aligned}$$

where $G(\cdot, \lambda)$ is the distribution function of the $SND(\lambda)$.

Now, for the case $0 \leq d_1 \leq d_2$,

$$\begin{aligned} (2.5) \quad P(d_1 \leq Z \leq d_2) &= \int_{d_1}^{d_2} h(z; \lambda_1, \lambda_2, \alpha) dz \\ &= \int_{d_1}^{d_2} \left[\frac{2}{(\alpha+2)} f(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} 2f(z)F(\lambda_2 z) \right] dz \\ &= \frac{2}{(\alpha+2)} [F(d_2) - F(d_1)] \\ &\quad + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} [G(d_2, \lambda_2) - G(d_1, \lambda_2)]. \end{aligned}$$

Thus (2.4) and (2.5) implies (2.3).

RESULT 2.4. The c.d.f. $H(z)$ of a random variable Z following $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. (2.1) is the following, in which $\xi(a, b)$ is as given in (1.5).

$$(2.6) \quad H(z) = \frac{2}{(\alpha+2)} \begin{cases} F(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} [F(z) - 2\xi(z, \lambda_1)], & z < 0 \\ F(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \left[F(z) - 2\xi(z, \lambda_2) \right. \\ \quad \left. - \frac{\tan^{-1}(\lambda_1)}{\pi} + \frac{\tan^{-1}(\lambda_2)}{\pi} \right], & z \geq 0. \end{cases}$$

PROOF. Let Z be a random variable with p.d.f. (2.1) and c.d.f. $H(z)$. Then by definition,

$$(2.7) \quad H(z) = \begin{cases} L_1, & z < 0 \\ L_2, & z \geq 0, \end{cases}$$

where

$$(2.8) \quad \begin{aligned} L_1 &= \frac{2}{(\alpha+2)} \int_{-\infty}^z f(t)dt + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} \int_{-\infty}^z 2f(t)F(\lambda_1 t)dt \\ &= \frac{2}{(\alpha+2)} F(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} G(z, \lambda_1) \\ &= \frac{2}{(\alpha+2)} F(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{(\alpha+2)} [F(z) - 2\xi(z, \lambda_1)] \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} L_2 &= \int_{-\infty}^0 \frac{2}{(\alpha+2)} f(t)[1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_1 t)]dt \\ &\quad + \int_0^z \frac{2}{(\alpha+2)} f(t)[1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_2 t)]dt \\ &= \frac{2}{(\alpha+2)} \left[F(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} G(0, \lambda_1) \right. \\ &\quad \left. + \frac{D(\lambda, \alpha, \rho)}{2} (G(z, \lambda_2) - G(0, \lambda_2)) \right] \\ &= \frac{2}{(\alpha+2)} \left[F(z) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \left(\frac{1}{2} - 2\xi(0, \lambda_1) \right) \right. \\ &\quad \left. + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \left(F(z) - 2\xi(z, \lambda_2) - \frac{1}{2} - 2\xi(0, \lambda_2) \right) \right]. \end{aligned}$$

Since $\xi(0, \cdot)$ is as defined in (1.6), on substituting (2.8) and (2.9) in (2.7) we get (2.6).

RESULT 2.5. The characteristic function, $\phi_Z(t)$ of a random variable Z following $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. (2.1) is the following, for any $t \in R$ and $i^2 = -1$.

$$(2.10) \quad \begin{aligned} \phi_Z(t) &= \frac{2}{(\alpha+2)} e^{-t^2/2} [1 + D(\lambda_1, \lambda_2, \alpha)F(i\delta_1 t)] \\ &\quad - \frac{2}{(\alpha+2)} D(\lambda_1, \lambda_2, \alpha) e^{-t^2/2} [\xi_{k_1}(-it, \lambda_1) - \xi_{k_2}(-it, \lambda_2)] \end{aligned}$$

where, $\delta_1 = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}$ for $j = 1, 2$ and $\xi_s(a, b)$ is defined in (1.7).

PROOF. Let Z follow $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. (2.1). By the definition of the characteristic function, for any $t \in R$ we have

$$(2.11) \quad \begin{aligned} \phi_Z(t) &= E(e^{itZ}) \\ &= \int_{-\infty}^{\infty} e^{itz} h(z; \lambda_1, \lambda_2, \alpha) dz. \end{aligned}$$

On substituting (2.1) in (2.11) and rearranging the terms in light of the expression for the characteristic function of a standard normal variate, we obtain the following,

$$(2.12) \quad \begin{aligned} \phi_Z(t) &= \frac{2}{(\alpha + 2)} e^{-t^2/2} \left\{ 1 + D(\lambda_1, \lambda_2, \alpha) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-it)^2/2} F(\lambda_1 z) dz \right. \\ &\quad - D(\lambda_1, \lambda_2, \alpha) \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-it)^2/2} F(\lambda_1 z) dz \\ &\quad \left. + D(\lambda_1, \lambda_2, \alpha) \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-it)^2/2} F(\lambda_2 z) dz \right\}. \end{aligned}$$

If we put $z - it = x$ in (2.12), we get

$$\begin{aligned} \phi_Z(t) &= \frac{2}{(\alpha + 2)} e^{-t^2/2} \left[1 + D(\lambda_1, \lambda_2, \alpha) \int_{-\infty}^{\infty} \frac{e^{-x^2/2} F(\lambda_1(x + it)) dx}{\sqrt{2\pi}} \right] \\ &\quad - D(\lambda_1, \lambda_2, \alpha) \int_{-it}^{\infty} \frac{e^{-x^2/2} F(\lambda_1(x + it)) dx}{\sqrt{2\pi}} \\ &\quad + D(\lambda_1, \lambda_2, \alpha) \int_{-it}^{\infty} \frac{e^{-x^2/2} F(\lambda_2(x + it)) dx}{\sqrt{2\pi}} \\ &= \frac{2}{(\alpha + 2)} e^{-t^2/2} \left\{ 1 + D(\lambda_1, \lambda_2, \alpha) F(i\delta_1 t) \right. \\ &\quad - D(\lambda_1, \lambda_2, \alpha) \int_{-it}^{\infty} f(x) F(\lambda_1(x + it)) dx \\ &\quad \left. + D(\lambda_1, \lambda_2, \alpha) \int_{-it}^{\infty} f(x) F(\lambda_2(x + it)) dx \right\}, \end{aligned}$$

in light of lemma 2.1 of Kumar and Anusree (2013a). Now, on rearranging the terms and putting $k = \lambda_1 t$ to arrive at the following.

$$\begin{aligned} \phi_Z(t) &= \frac{2}{(\alpha + 2)} e^{-t^2/2} [1 + D(\lambda_1, \lambda_2, \alpha) F(i\delta_1 t)] \\ &\quad - \frac{D(2\lambda_1, \lambda_2, \alpha)}{(\alpha + 2)} e^{-t^2/2} \int_{-it}^{\infty} f(x) \left[\int_{-\infty}^0 f(u) du + \int_0^{\lambda_1 x + k_1} f(u) du \right] dx \\ &\quad + \frac{2D(\lambda_1, \lambda_2, \alpha)}{(\alpha + 2)} e^{-t^2/2} \int_{-it}^{\infty} f(x) \left[\int_{-\infty}^0 f(u) du + \int_0^{\lambda_2 x + \rho k_2} f(u) du \right] dx, \end{aligned}$$

which implies (2.10), in light of (1.7).

3. Reliability measures and mode

Here we present some reliability aspects of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. (2.1) and discuss some concepts regarding its mode.

RESULT 3.1. The reliability function $R(t)$ of a random variable Z having the $ETPSND(\lambda_1, \lambda_2, \alpha)$ is the following, in which $\xi(t, \cdot)$ is as defined in (1.5).

$$(3.1) \quad R(t) = 1 - \frac{2}{(\alpha + 2)} \begin{cases} F(t) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} [F(t) - 2\xi(t, \lambda_1)], & t < 0 \\ F(t) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} [F(t) - 2\xi(t, \lambda_2)] \\ + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \\ \times \left[\frac{-1}{\pi} \tan^{-1}(\lambda_1) + \frac{1}{\pi} \tan^{-1}(\lambda_2) \right], & t \geq 0. \end{cases}$$

Proof follows from the definition of the reliability function $R(t) = 1 - H(t)$, where $H(\cdot)$ is the c.d.f. of Z as obtained in Result 2.4.

RESULT 3.2. The failure rate $r(t)$ of a random variable Z following the $ETPSND(\lambda_1, \lambda_2, \alpha)$ with p.d.f. $h(z; \lambda_1, \lambda_2, \alpha)$ is

$$(3.2) \quad r(t) = \begin{cases} \frac{2f(t)[1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_1 t)]}{(\alpha + 2) - 2(F(t) + D(\lambda_1, \lambda_2, \alpha)[F(t) - 2\xi(t, \lambda_1)])}, & t < 0 \\ \frac{2f(t)[1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_2 t)]}{(\alpha + 2) - 2\left(F(t) + D(\lambda_1, \lambda_2, \alpha)\left[F(t) - 2\xi(t, \lambda_2) - \frac{1}{\pi} \tan^{-1}(\lambda_1) + \frac{1}{\pi} \tan^{-1}(\lambda_2)\right]\right)}, & t \geq 0. \end{cases}$$

Proof follows from the definition of failure rate, $r(t) = \frac{h(t; \lambda_1, \lambda_2, \alpha)}{R(t)}$, where $R(t)$ is the reliability function as given in (3.1).

RESULT 3.3. The mean residual life function (MRLF) $\mu(t)$ of $ETPSND(\lambda_1, \lambda_2, \alpha)$ is the following, in which $\Delta = \frac{2}{(\alpha+2)R(t)}$, $\delta_j = \frac{\lambda_j}{\sqrt{1+\lambda_j^2}}$ for $j = 1, 2$.

$$(3.3) \quad \mu(t) = \Delta \begin{cases} f(t) + D(\lambda_1, \lambda_2, \alpha)f(t)F(\lambda_1 t) + \frac{\delta_1}{2\sqrt{2\pi}}D(\lambda_1, \lambda_2, \alpha) \\ - \frac{\delta_1}{\sqrt{2\pi}}D(\lambda_1, \lambda_2, \alpha)F(t\sqrt{1+\lambda_1^2}) \\ + \frac{\delta_2}{2\sqrt{2\pi}}D(\lambda_1, \lambda_2, \alpha) - t, & t \leq 0 \\ f(t) + D(\lambda_1, \lambda_2, \alpha)f(t)F(\lambda_2 t) + \frac{\delta_2}{2\sqrt{2\pi}}D(\lambda_1, \lambda_2, \alpha) \\ - \frac{\delta_2}{\sqrt{2\pi}}D(\lambda_1, \lambda_2, \alpha)F(t\sqrt{1+\lambda_2^2}) - t, & t > 0 \end{cases}$$

PROOF. By definition, the mean residual life function of Z following the $ETPSND(\lambda_1, \lambda_2, \alpha)$ is given by

$$(3.4) \quad \begin{aligned} \mu(t) &= E(Z \mid Z > t) - t \\ &= \frac{2}{R(t)(\alpha + 2)} \\ &\quad \times \begin{cases} \int_t^0 z f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_1 z)] dz \\ \quad + \int_0^\infty z f(z) dz \\ \quad + D(\lambda_1, \lambda_2, \alpha) \int_0^\infty z f(z) F(\lambda_2 z) dz - t, & t \leq 0 \\ \int_t^\infty z f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_2 z)] dz - t, & t > 0. \end{cases} \end{aligned}$$

Now for any $t < 0$,

$$(3.5) \quad \begin{aligned} &\int_t^0 z f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_1 z)] dz \\ &= - \int_t^0 f'(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_1 z)] dz \\ &= f(t) - \frac{1}{\sqrt{2\pi}} - D(\lambda_1, \lambda_2, \alpha) \left[\frac{1}{2\sqrt{(2\pi)}} - F(\lambda_1 t) f(t) \right] \\ &\quad - D(\lambda_1, \lambda_2, \alpha) \lambda_1 \int_t^0 f(\lambda_1 z) f(z) dz \\ &= f(t) - \frac{1}{\sqrt{2\pi}} - \frac{D(\lambda_1, \lambda_2, \alpha)}{2\sqrt{2\pi}} + D(\lambda_1, \lambda_2, \alpha) F(\lambda_1 t) f(t) \\ &\quad + \frac{\lambda_1 D(\lambda_1, \lambda_2, \alpha) \left[\frac{1}{2} - F(t\sqrt{1 + \lambda_1^2}) \right]}{\sqrt{2\pi}\sqrt{1 + \lambda_1^2}} \end{aligned}$$

and for any $t > 0$,

$$(3.6) \quad \begin{aligned} &\int_t^\infty z f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_2 z)] dz \\ &= - \int_t^\infty f'(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_2 z)] dz \\ &= f(t) + D(\lambda_1, \lambda_2, \alpha) F(\lambda_2 t) f(t) + \frac{\lambda_2 D(\lambda_1, \lambda_2, \alpha)}{\sqrt{2\pi}\sqrt{1 + \lambda_2^2}} \\ &\quad - D(\lambda_1, \lambda_2, \alpha) \frac{\lambda_2}{\sqrt{2\pi}\sqrt{1 + \lambda_2^2}} F(t\sqrt{1 + \lambda_2^2}). \end{aligned}$$

In particular, when $t = 0$ in (3.6) we have

$$(3.7) \quad \begin{aligned} \int_0^\infty z f(z) [1 + D(\lambda_1, \lambda_2, \alpha) F(\lambda_2 z)] dz &= \frac{1}{\sqrt{(2\pi)}} + \frac{D(\lambda_1, \lambda_2, \alpha)}{2\sqrt{2\pi}} \\ &\quad + \frac{\lambda_2 D(\lambda_1, \lambda_2, \alpha)}{2\sqrt{2\pi}\sqrt{1 + \lambda_2^2}}. \end{aligned}$$

Now on substituting (3.5), (3.6) and (3.7) in (3.4), we get (3.3).

RESULT 3.4. The p.d.f. of $ETPSND(\lambda_1, \lambda_2, \alpha)$ is bimodal with unimodes in the regions: $z \in (-\infty, 0)$ and $z \in [0, \infty)$ subject to the conditions given below, in which for $i = 0, 1$ and $j = 1, 2$,

$$a_{ij} = \frac{D^i(\lambda_1, \lambda_2, \alpha)(\lambda_j z)^{1-i} f(\lambda_j^i z)}{[1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_j z)]^{i+1}}.$$

Region $z \in (-\infty, 0)$: (i) For all $\alpha \geq 0$ for which either $\lambda_1 \leq 0$ or $\lambda_1 > 0$ with $|a_{01}| < a_{11}$,

(ii) For all $\alpha < 0$ for which either $\lambda_1 \leq 0$ with $|a_{01}| < a_{11}$ or $\lambda_1 > 0$ with $\lambda_1^2 f(\lambda_1 z)(a_{01} + a_{11}) > -1$.

Region $z \in [0, \infty)$: (i) For all $\alpha \geq 0$ for which either $\lambda_2 > 0$ or $\lambda_2 \leq 0$ with $|a_{02}| < a_{12}$,

(ii) For all $\alpha < 0$ for which either $\lambda_2 > 0$ with $|a_{12}| < a_{02}$ or $\lambda_2 \leq 0$ with $\lambda_2^2 f(\lambda_2 z)(a_{02} + a_{12}) > -1$.

PROOF. In order to show that there exists unimodes in regions of $z \in (-\infty, 0)$ and $z \in [0, \infty)$, it is enough to show that the second derivative of $h(z; \lambda_1, \lambda_2, \alpha)$ is negative for all α, λ_1 and λ_2 in the respective regions.

For $z \in (-\infty, 0)$, we have

$$(3.8) \quad \frac{d^2}{dz^2} \{\log[h(z; \lambda_1, \lambda_2, \alpha)]\} = -1 - \lambda_1^2 f(\lambda_1 z)[a_{00} + a_{10}]$$

and for $z \in [0, \infty)$, we have

$$(3.9) \quad \frac{d^2}{dz^2} \{\log[h(z; \lambda_1, \lambda_2, \alpha)]\} = -1 - \lambda_2^2 f(\lambda_2 z)[a_{01} + a_{11}].$$

Note that for all α , $1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_j z)$ is positive for each $j = 1, 2$. Now for the region $z \in (-\infty, 0)$, a_{01} is positive or negative according to whether the value of λ_1 is negative or positive and a_{11} is positive or negative according to whether the value of α is positive or negative. If $\alpha \geq 0$, then (3.8) is negative either for $\lambda_1 \leq 0$ or for $\lambda_1 > 0$ if $a_{01} + a_{11} > 0$ and for $\alpha < 0$, (3.8) is negative either for $\lambda_1 < 0$ provided $a_{01} + a_{11} > 0$ or for $\lambda_1 > 0$ with $\lambda_1^2 f(\lambda_1 z)(a_{01} + a_{11}) > -1$. Hence the p.d.f. given in (2.1) is log-concave and thus unimodal under these cases.

Now for the region $z \in [0, \infty)$, a_{02} is positive or negative according to whether the value of λ_2 is positive or negative and a_{12} is positive or negative according to whether the value of α is positive or negative. If $\alpha \geq 0$, then (3.9) is negative either for $\lambda_2 > 0$ or for $\lambda_2 < 0$ if $a_{02} + a_{12} > 0$. Now for $\alpha < 0$, (3.9) is negative either for $\lambda_2 > 0$ if $a_{02} + a_{12} > 0$ or $\lambda_2 < 0$ if $\lambda_2^2 f(\lambda_2 z)(a_{02} + a_{12}) > -1$. Thus the p.d.f. given in (2.1) is log-concave and hence unimodal under these conditions.

As a consequence of Result 3.4 we obtain the following result.

RESULT 3.5. The p.d.f. of $ETPSND(\lambda_1, \lambda_2, \alpha)$ is plurimodal in the regions: $z \in (-\infty, 0)$ and $z \in [0, \infty)$ is as given below:

Region $z \in (-\infty, 0)$: (i) For all $\alpha \geq 0$ and $\lambda_1 > 0$ such that $|a_{01}| > a_{11}$.

(ii) For all $\alpha < 0$, $\lambda_1 \leq 0$ such that $|a_{11}| > a_{01}$ or $\lambda_1 > 0$ and $\lambda^2 f(\lambda_1 z)(a_{01} + a_{11}) < -1$.

Region $z \in [0, \infty)$: (i) For all $\alpha \geq 0$, $\lambda_2 \leq 0$ such that $|a_{02}| > a_{12}$,

(ii) For all $\alpha < 0$ for which either $\lambda_2 > 0$ with $|a_{12}| > a_{02}$ or $\lambda_2 \leq 0$ with $\lambda_2^2 f(\lambda_2 z)(a_{02} + a_{12}) < -1$.

4. Location-scale extension

In this section we consider the location-scale extension of the $ETPSND(\lambda_1, \lambda_2, \alpha)$ and discuss some of its important properties. Let Z follow $ETPSND(\lambda_1, \lambda_2, \alpha)$, then $X = \mu + \sigma Z$ is said to have a location scale extension of the two-piece skew normal distribution with location parameter μ , scale parameter σ and shape parameters λ_1 , λ_2 and α , which is denoted as $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$. The p.d.f. of X is given by

$$(4.1) \quad h(x; \mu, \sigma, \lambda_1, \lambda_2, \alpha) = \begin{cases} \frac{2}{(\alpha + 2)\sigma} f\left(\frac{x - \mu}{\sigma}\right) \\ \quad \times \left[1 + D(\lambda_1, \lambda_2, \alpha) F\left(\lambda_1 \frac{x - \mu}{\sigma}\right)\right], & x < \mu \\ \frac{2}{(\alpha + 2)\sigma} f\left(\frac{x - \mu}{\sigma}\right) \\ \quad \times \left[1 + D(\lambda_1, \lambda_2, \alpha) F\left(\lambda_2 \frac{x - \mu}{\sigma}\right)\right], & x \geq \mu, \end{cases}$$

in which $\mu, \lambda_1, \lambda_2 \in R$, $\sigma > 0$ and $\alpha \geq -1$.

Clearly we have the following special cases.

(i) $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, 0)$, $ETPSND^{LS}(\mu, \sigma; 0, 0, \alpha)$ or the limiting case of the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ when $\lambda_1 \rightarrow -\infty$, $\lambda_2 \rightarrow \infty$, is the normal distribution with parameters μ and σ^2 ,

(ii) the limiting case of the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ when $\alpha \rightarrow \infty$ is the location scale extension of $TSND_{KA}(\lambda, \rho)$, denoted by $ETSND_{KA}(\mu, \sigma; \lambda, \rho)$

(iii) the limiting case of the $ETPSND^{LS}(\mu, \sigma; \lambda, \lambda, \alpha)$ when $\alpha \rightarrow \infty$ is the location scale extension of $TSND_K(\lambda)$, denoted by $ETSND_K(\mu, \sigma; \lambda)$

(iv) the $ETPSND^{LS}(\mu, \sigma; \lambda, \lambda, \alpha)$ is the location scale extension of $GMNSND(\lambda, \alpha)$, denoted by $EGMNSND(\mu, \sigma; \lambda, \alpha)$ which reduces to the location scale extension of $SND(\lambda)$, denoted by $ESND(\mu, \sigma; \lambda)$ when $\alpha = -1$ and

(v) the limiting case of the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ as $\lambda_1 \rightarrow \infty$, $\lambda_2 \rightarrow \infty$ and $\alpha \rightarrow \infty$ or $\lambda_1 \rightarrow -\infty$, $\lambda_2 \rightarrow -\infty$ and $\alpha \rightarrow \infty$ is the half normal distribution.

We obtain the following properties of $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ with p.d.f. (4.1) in a similar approach as we did in earlier sections.

RESULT 4.1. The characteristic function $\phi_X(t)$ of a random variable X following $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ is the following, in which $k_j = \lambda_j it$, $\delta_j = \frac{\lambda_j}{\sqrt{1+\lambda_j^2}}$ for $j = 1, 2$. For $i^2 = -1$ and $t \in R$,

$$\begin{aligned} \phi_X(t) = & \frac{2}{(\alpha+2)} e^{i\mu t - t^2 \sigma^2/2} [1 + D(\lambda_1, \lambda_2, \alpha) F(i\delta_1 \sigma t)] \\ & - \frac{2}{(\alpha+2)} D(\lambda_1, \lambda_2, \alpha) e^{i\mu t - t^2 \sigma^2/2} [\xi_{k_1 \sigma}(-it\sigma, \lambda_1) - \xi_{k_2 \sigma}(-it\sigma, \lambda_2)]. \end{aligned}$$

RESULT 4.2. The distribution function $H_1^*(t)$ of a random variable X following $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ is the following,

$$H_1^*(t) = \frac{2}{(\alpha+2)} \begin{cases} F\left(\frac{t-\mu}{\sigma}\right) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \\ \quad \times \left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\left(\frac{t-\mu}{\sigma}\right), \lambda_1\right) \right], & t < \mu \\ F\left(\frac{t-\mu}{\sigma}\right) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \\ \quad \times \left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\left(\frac{t-\mu}{\sigma}\right), \lambda_2\right) \right] \\ \quad - \frac{\tan^{-1}(\lambda_1)}{\pi} + \frac{\tan^{-1}(\lambda_2)}{\pi} \Big], & t \geq \mu. \end{cases}$$

RESULT 4.3. The reliability function $R_1^*(t)$ of a random variable X following $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ is the following,

$$R_1^*(t) = 1 - \frac{2}{(\alpha+2)} \begin{cases} F\left(\frac{t-\mu}{\sigma}\right) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \\ \quad \times \left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\frac{t-\mu}{\sigma}, \lambda_1\right) \right], & t < \mu \\ F\left(\frac{t-\mu}{\sigma}\right) + \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \\ \quad \times \left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\frac{t-\mu}{\sigma}, \lambda_2\right) \right] \\ \quad - \frac{D(\lambda_1, \lambda_2, \alpha)}{2} \\ \quad \times \left[\frac{-1}{\pi} \tan^{-1}(\lambda_1) + \frac{1}{\pi} \tan^{-1}(\lambda_2) \right], & t \geq \mu. \end{cases}$$

RESULT 4.4. The failure rate $r_1(t)$ of a random variable X following $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ is the following,

$$r_1(t) = \begin{cases} \frac{2f\left(\frac{t-\mu}{\sigma}\right) [1 + D(\lambda_1, \lambda_2, \alpha)F\left(\lambda_1 \frac{t-\mu}{\sigma}\right)]}{\sigma(\alpha+2) - 2\sigma\left(F\left(\frac{t-\mu}{\sigma}\right) + D(\lambda_1, \lambda_2, \alpha)\left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\frac{t-\mu}{\sigma}, \lambda_1\right)\right]\right)}, & t < \mu \\ \frac{2f\left(\frac{t-\mu}{\sigma}\right) [1 + D(\lambda_1, \lambda_2, \alpha)F\left(\lambda_2 \frac{t-\mu}{\sigma}\right)]}{\sigma(\alpha+2) - 2\sigma\left(F\left(\frac{t-\mu}{\sigma}\right) + D(\lambda_1, \lambda_2, \alpha)\left[F\left(\frac{t-\mu}{\sigma}\right) - 2\xi\left(\frac{t-\mu}{\sigma}, \lambda_2\right) - \frac{1}{\pi}\tan^{-1}(\lambda_1) + \frac{1}{\pi}\tan^{-1}(\lambda_2)\right]\right)}, & t \geq \mu. \end{cases}$$

5. Estimation

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ with p.d.f. (4.1). Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the ordered sample. Assume $X_{(r)} < \mu < X_{(r+1)}$, for a particular $r = 1, 2, \dots, n$. Then the log-likelihood function $l(\underline{\theta}) = l(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ of the sample is the following, in which Σ_{I_j} , denotes the summation over the set I_j such that $I_1 = \{i : X_{(i)} < \mu, \text{ for } i = 1, 2, \dots, r\}$ and $I_2 = \{i : X_{(i)} \geq \mu, \text{ for } i = r+1, \dots, n\}$.

$$(5.1) \quad l(\underline{\theta}) = n \ln \left(\frac{2}{\sigma(\alpha+2)} \right) + \sum_{I_1} \ln f \left(\frac{x_i - \mu}{\sigma} \right) \left[1 + D(\lambda_1, \lambda_2, \alpha) F \left(\frac{\lambda_1(x_i - \mu)}{\sigma} \right) \right] + \sum_{I_2} \ln f \left(\frac{x_i - \mu}{\sigma} \right) \left[1 + D(\lambda_1, \lambda_2, \alpha) F \left(\frac{\lambda_2(x_i - \mu)}{\sigma} \right) \right].$$

On differentiating (5.1) with respect to the parameters $\mu, \sigma, \lambda_1, \lambda_2$ and α and equating to zero, we obtain the following likelihood equations, in which $u_i = \frac{x_i - \mu}{\sigma}$ for $i = 1, 2, \dots, n$, $D'_j(\lambda_1, \lambda_2, \alpha) = \frac{d}{d\lambda_k} D(\lambda_1, \lambda_2, \alpha)$ and for $j = 1, 2$ and $k = 0, 1$ $\beta_j(x_i; \lambda_1, \lambda_2, \alpha) = 1 + D(\lambda_1, \lambda_2, \alpha)F(\lambda_j \frac{x_i - \mu}{\sigma})$, $\gamma_{jk}(x_i) = \frac{f^k(\lambda_j)F^{1-k}(\lambda_j u_i)}{\beta_j(x_i; \lambda_1, \lambda_2, \alpha)}$.

$$(5.2) \quad \sum_{I_1} \frac{(x_i - \mu)}{\sigma} + \sum_{I_2} \frac{(x_i - \mu)}{\sigma} = D(\lambda_1, \lambda_2, \alpha) \left[\lambda_1 \sum_{I_1} \gamma_{11}(x_i) + \lambda_2 \sum_{I_2} \gamma_{21}(x_i) \right],$$

$$(5.3) \quad \frac{n}{2\sigma^2} = \frac{1}{2} \sum_{I_1} \frac{(x_i - \mu)^2}{\sigma^4} + \frac{1}{2} \sum_{I_2} \frac{(x_i - \mu)^2}{\sigma^4} - \frac{D(\lambda_1, \lambda_2, \alpha)}{2\sigma^3} \left(\lambda_1 \sum_{I_1} \gamma_{11}(x_i)(x_i - \mu) + \lambda_2 \sum_{I_2} \gamma_{21}(x_i)(x_i - \mu) \right),$$

$$(5.4) \quad \frac{D(\lambda_1, \lambda_2, \alpha)}{\sigma} \sum_{I_1} \gamma_{11}(x_i)(x_i - \mu) + \sum_{I_1} \gamma_{10}(x_i)D_1'(\lambda_1, \lambda_2, \alpha) \\ + \sum_{I_1} D_1'(\lambda_1, \lambda_2, \alpha)\beta_1(x_i; \lambda_1, \lambda_2, \alpha) = 0,$$

$$(5.5) \quad \frac{D(\lambda_1, \lambda_2, \alpha)}{\sigma} \sum_{I_1} \gamma_{21}(x_i)(x_i - \mu) + \sum_{I_1} \gamma_{20}(x_i)D_2'(\lambda_1, \lambda_2, \alpha) \\ + \sum_{I_1} D_2'(\lambda_1, \lambda_2, \alpha)\beta_2(x_i; \lambda_1, \lambda_2, \alpha) = 0$$

and

$$(5.6) \quad D(\lambda_1, \lambda_2, \alpha) \sum_{I_1} \gamma_{10} + D(\lambda_1, \lambda_2, \alpha) \sum_{I_2} \gamma_{20} = \frac{n\alpha}{(\alpha + 2)}.$$

Solving the non-linear system of equations (5.2) to (5.6) by simultaneous solution method using mathematical software such as *MATHCAD*, *MATLAB*, *MATHEMATICA* etc., the maximum likelihood estimates (MLE) of the parameters of $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ can be obtained.

6. Applications

For illustrating the usefulness of the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$, in this section we have considered the IQ data set for 52 non-white males and 87 white males hired by a large insurance company in 1971 given in Roberts (1988).

Data set 1: 91 102 100 117 122 115 97 109 108 104 108 118 103 123 123 103 106 102 118 100 103 107 108 107 97 95 119 102 108 103 102 112 99 116 114 102 111 104 122 103 111 101 91 99 121 97 109 106 102 104 107 95.

Data set 2: 85 94 94 97 98 100 100 101 102 102 103 103 103 103 104 104 106 106 106 106 106 107 107 108 108 108 108 108 108 109 109 111 111 112 112 112 112 112 112 112 112 113 113 113 113 113 113 113 113 114 114 115 116 116 116 116 117 117 117 118 118 118 119 120 120 120 121 121 121 122 122 122 122 122 124 124 125 129 131 132 135 136 140.

Here we have fitted the proposed model to these two data sets, non-white males and white males, and compared results with the existing models such as the normal distribution $N(\mu, \sigma)$, $ESND(\mu, \sigma, \lambda)$, $ETSND_K(\mu, \sigma; \lambda, \rho)$ and $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$. For model comparison, we have computed some well-known information criterion-such as Akaike's Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the corrected Akaike's Information Criterion (AICc). For numerical evaluation, we have used *MATHCAD* software and the results obtained are included in Table 1 for the case of Data set 1 and in Table 2 for the case of Data set 2 along with the log-likelihood(l).

In light of the computed values listed in Tables 1 and 2, it can be seen that the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ gives the best results compared to other existing models.

Table 1. MLE of the parameters of the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ and computed values of the l , AIC, BIC and AICc values in the case of various fitted models to data set 1.

Distribution:	N (μ, σ)	$ESND$ (μ, σ, λ)	$ETSND_K$ $(\mu, \sigma; \lambda, \rho)$	$ETPSND^{LS}$ $(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$
$\hat{\mu}$	106.653	98.79	104.64	102.45
$\hat{\sigma}$	8.229	11.38	8.9	9.8
$\hat{\lambda}$	—	1.71	0.8	2.54, -0.457
$\hat{\alpha}$	—	—	—	1.66
$\hat{\rho}$	—	—	-0.04	—
l	-183.374	-182.423	-182	-177
AIC	371	371	370	365
BIC	375	377	376	374
AICc	370.99	371.34	370.73	366

Table 2. MLE of the parameters of the $ETPSND^{LS}(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$ and computed values of the l , AIC, BIC and AICc values in the case of various fitted models to data set 2.

Distribution:	N (μ, σ)	$ESND$ (μ, σ, λ)	$ETSND_K$ $(\mu, \sigma; \lambda, \rho)$	$ETPSND^{LS}$ $(\mu, \sigma; \lambda_1, \lambda_2, \alpha)$
$\hat{\mu}$	112.86	105.78	107.1	108.3
$\hat{\sigma}$	9.58	11.94	10.9	11.5
$\hat{\lambda}$	—	1.14	0.899	0.49, -0.0098
$\hat{\alpha}$	—	—	—	-0.91
$\hat{\rho}$	—	—	0.9	—
l	-319.6	-319.29	-319	-314
AIC	643.2	644.57	647	638
BIC	648.14	651.97	657	647
AICc	643.35	644.86	647.32	639.01

Acknowledgements

The authors are grateful to the Editor, Associate Editor and the anonymous referee for carefully reading the paper and for the valuable comments and suggestions. The second author is thankful to the Department of Science and Technology, Ministry of Science and Technology, New Delhi for financial support (IF 110142).

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