k-MULTIPLE CHART AND ITS APPLICATION TO THE TEST FOR HOMOGENEITY AGAINST ORDERED ALTERNATIVES

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The paper explains how a k-multiple chart in two dimensions can be constructed as a descriptive tool for the k-samples. This chart is regarded as an extension to the k-sample problem of the pair chart discussed by Quade for the two-sample problem. The area Sk lying below the path on the k-multiple chart is applied to the non-parametric test for homogeneity against ordered alternatives. The Pitman efficiency of Sk with respect to the well-known test statistic Wk of Terpstra and Jonckheere is shown, and its value is illustrated for k=4, 5, ..., 15. As the results, though Sk and Wk are nearly equivalent test statistics, Sk may be superior to Wk in terms of the test statistic with some graphical characteristics.

1. Introduction

Let \( X_{i1}, X_{i2}, \ldots, X_{in_i} \) be a random sample of size \( n_i \) from a population with unknown distribution function \( F_i(x) \) (i=1, 2, ..., k). Let \( N = \sum_{i=1}^{k} n_i \) be the size of the total observations and suppose that there is no tie among \( N \) observations. Then we propose a k-multiple chart which is constructed by the following steps:

Step 1: Draw a horizontal line (the x-axis) and a vertical line (the y-axis), intersecting at the point O (the origin). Starting from the origin O, draw a line \( OP_1 \) of length \( n_1 \) in the direction of the x-axis. Next, starting from the end of this line, draw another line \( P_1P_2 \) of length \( n_2 \) in the direction of \( \pi/(2(k-1)) \) radian from x-axis. Continue in the same manner, draw another line \( P_{i-1}P_i \) of length \( n_i \) (i=3, 4, ..., k) in the direction of \( (i-1)\pi/(2(k-1)) \) radian from x-axis, and construct a polygone with 2k sides as shown in Fig. 1, where \( P_i \) (i=1, 2, ..., k-1) is a symmetric point of \( P_i \) with respect to the line \( OP_k \).

Step 2: Let us combine all the observations of size \( N \). Starting from the origin O, draw a line \( P_1 \) of unit length in the direction of \( (i-1)\pi/(2(k-1)) \) radian from the x-axis, if the smallest observation in the combined samples is one from the population with \( F_i(x) \) (i=1, 2, ..., k). Next, starting from the end of this line, draw another line in the direction of \( (j-1)\pi/(2(k-1)) \) radian from x-axis if the second smallest observation is a sample from the population having \( F_j(x) \) (j=1, 2, ..., k).

Step 3: Continue in the same manner for all \( N \) observations. The \( N \) line segments then form a path from the origin O to the point

\[
P_i \left( \sum_{i=1}^{k} n_i \cos \left( \frac{(i-1)\pi}{2(k-1)} \right), \sum_{i=1}^{k} n_i \sin \left( \frac{(i-1)\pi}{2(k-1)} \right) \right).
\]

We call the chart which is constructed by the above steps “k-multiple chart” (see

Received Nov. 15, 1980, Revised Feb. 23, 1981.

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Fig. 1 A multiple chart

Note: The 2-multiple chart is called "pair chart", which seems to have been introduced by Drion [1] whose typical effects were illustrated by Quade [3].

2. An area $S_k$ and its application to the non-parametric test

Let us denote the area of a polygone which is enclosed by the path with $N$ line segments from the origin $O$ to the point $P_k$, and the $k$ sidelines $OP_1, P_1P_2, \ldots, P_{k-1}P_k$, by $S_k$ (the shadowed portion in Fig. 1). Put

\begin{equation}
S_{ij} = \sin \frac{(i-j)\pi}{2(k-1)} \text{ for } i<j
\end{equation}

and

\begin{equation}
U_{ij} = \sum_{s=1}^{n_i} \sum_{v=1}^{n_j} \phi(X_{is}, X_{jv}) \text{ for } i<j,
\end{equation}

where $\phi(a, b)=1$ if $a<b$, 0 otherwise. Thus we obtain the following simple equation:

\begin{equation}
S_k = \sum_{i<j} S_{ij} U_{ij},
\end{equation}

where $0 \leq S_k \leq \sum_{i<j} n_i n_j s_{ij}$.

Let us consider the testing problem of the null hypothesis

$H_0: F_1(x) = F_2(x) = \cdots = F_k(x)$ \text{ for all } x

against the alternative of the form

$H_1: F_1(x) \leq F_2(x) \leq \cdots \leq F_k(x)$ \text{ for all } x,
where at least one strict inequality holds.

We use \( S_k \) to test the null hypothesis \( H_0 \) against \( H_1 \) with the following test procedure:

At the \( \alpha \) level of significance,

\[
\begin{align*}
&\text{reject } H_0 \text{ if } S_k \geq A_2(\alpha, k, (n_1, n_2, \ldots, n_k)), \\
&\text{accept } H_0 \text{ if } S_k < A_2(\alpha, k, (n_1, n_2, \ldots, n_k)),
\end{align*}
\]

where the constant \( A_2(\alpha, k, (n_1, n_2, \ldots, n_k)) \) satisfies the equation \( P_0\{S_k \geq A_2(\alpha, k, (n_1, n_2, \ldots, n_k))\} = \alpha \).

Then we call \( S_k \) "Area test statistic."

Let us denote the mean value and the variance of a random variable \( X \) by \( E(X) \) and \( V(X) \), respectively. Then we obtain the following equations under the null hypothesis \( H_0 \).

\[
E(S_k) = \frac{1}{2} \sum_{i<j} n_in_j s_{ij},
\]
\[
V(S_k) = E[(S_k - E(S_k))^2] = E\left[\left(\sum_{i<j} (U_{ij} - E(U_{ij})) s_{ij}\right)^2\right] = \frac{1}{12} \left[ \sum_{i<j} n_i n_j (n_i + n_j + 1)s_{ij}^2 + 2 \sum_{i=1}^{k-2} \sum_{i<j}^{k} n_i n_j s_{ij} s_{il} \right].
\]

Put

\[
(2.4) \quad S_k^* = \frac{S_k - E(S_k)}{\sqrt{V(S_k)}}.
\]

When \( H_0 \) is true, the statistic \( S_k^* \) has an asymptotic (min \( (n_1, \ldots, n_k) \) tend to infinity) \( N(0, 1) \) distribution. Thus we have the following approximate \( \alpha \)-level test procedure:

\[
(2.5) \quad \text{reject } H_0 \text{ if } S_k^* \geq z_{\alpha}, \quad \text{accept } H_0 \text{ if } S_k^* < z_{\alpha},
\]

where \( z_{\alpha} \) is the lower \( \alpha \)-percentile of the \( N(0, 1) \) distribution.

To test \( H_0 \) against \( H_1 \), we usually use the well-known test statistic \( W_k \) proposed by Terpstra [4] and Jonckheere [2] as follows:

\[
(2.6) \quad W_k = \sum_{i<j} U_{ij}.
\]

In the next section, we discuss some comparisons between \( S_k \) and \( W_k \) by using the Pitman efficiency.
(3.2) \[ W_k^* = \sum_{p=1}^{k-1} b_{p} U_{p,p+1} \], where \( b_{p} = p(k-p) \).

(3.3) \[ T_k^* = \sum_{p=1}^{k-1} a_{p} U_{p,p+1} \], where \( a_{p} = \sum_{i=1}^{p} \sum_{j=p+1}^{k} a_{ij} \), \( p=1,2,\ldots,k-1 \).

Let us suppose \( F_{i}(x) = F(x + \theta_{i}) \) \( (i=1,2,\ldots,k) \) and denote the Pitman efficiency of a test statistic \( X \) with respect to another test statistic \( Y \) by \( e_{p}(X, Y) \). Then by using the results of Tryon and Hettmansperger [5]; \( e_{p}(W_{k}, W_{k}^*) = 1 \) and the formula of \( e_{p}(T_{k}^*, W_{k}^*) \), we have the following equation:

(3.4) \[ e_{p}(T_{k}, W_{k}) = \frac{a' \delta' a}{a' \hat{H} a} \frac{b' \delta' b}{b' \hat{H} b} \]

where \( a' = (a_{1}, a_{2}, \ldots, a_{k-1}) \), \( b' = (b_{1}, b_{2}, \ldots, b_{k-1}) \), \( \delta' = (1, a_{1}/a_{2}, a_{3}/a_{2}, \ldots, a_{k-1}/a_{2}) \), \( a_{p} = (\theta_{p+1} - \theta_{p})/(\theta_{k} - \theta_{1}) \) and

\[
\hat{H}_{(a-1)\times(a-1)} = \begin{pmatrix}
1 + \frac{n_{2}}{n_{1}} & -\frac{n_{4}}{n_{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{n_{3}}{n_{2}} & 1 + \frac{n_{4}}{n_{3}} & -\frac{n_{4}}{n_{3}} & 0 & \cdots & 0 & 0 & 0 \\
0 & -\frac{n_{3}}{n_{2}} & 1 + \frac{n_{4}}{n_{3}} & -\frac{n_{5}}{n_{4}} & 0 & \cdots & 0 & 0 \\
0 & 0 & -\frac{n_{3}}{n_{2}} & 1 + \frac{n_{4}}{n_{3}} - \frac{n_{5}}{n_{4}} & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{n_{k-1}}{n_{k-2}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 + \frac{n_{k-1}}{n_{k-2}} - \frac{n_{k}}{n_{k-1}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{n_{k}}{n_{k-1}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 + \frac{n_{k}}{n_{k-1}} \end{pmatrix}
\]

Furthermore they proved that \( a_{ij} = 1 \) for all \( i, j \) attain \( \max \frac{e_{p}(T_{k}, W_{k})}{e_{p}(S_{k}, W_{k})} \) under the condition:

(3.5) \[ n_{1} = n_{2} = \cdots = n_{k} \quad \text{and} \quad \delta' = (1, 1, \ldots, 1) \quad \text{(equal spacings)}. \]

This result suggests that the test statistic \( W_{k} \) is the best of all the linear combinations with \( a_{ij} \) given by (3.1) under the condition (3.5).

In the formula (3.1), put

\[ a_{ij} = s_{ij} = \sin \frac{(j-i)r}{2(k-1)} \quad \text{for} \quad i<j. \]

### Table 1.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( e_{p}(S_{k}, W_{k}) )</th>
<th>( k )</th>
<th>( e_{p}(S_{k}, W_{k}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.999152</td>
<td>10</td>
<td>0.998965</td>
</tr>
<tr>
<td>5</td>
<td>0.998850</td>
<td>11</td>
<td>0.998983</td>
</tr>
<tr>
<td>6</td>
<td>0.998907</td>
<td>12</td>
<td>0.999000</td>
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<td>7</td>
<td>0.998909</td>
<td>13</td>
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<tr>
<td>8</td>
<td>0.998825</td>
<td>14</td>
<td>0.999028</td>
</tr>
<tr>
<td>9</td>
<td>0.998945</td>
<td>15</td>
<td>0.999040</td>
</tr>
</tbody>
</table>
Then we have $T_k = S_k$, and (3.4) gives the Pitman efficiency of $S_k$ with respect to $W_k$. Under the condition (3.5), we will illustrate in Table 1 the values of $e_p(S_k, W_k)$ for $k=4, 5, \cdots, 15$. As a result, we conclude that in terms of Pitman efficiency the test statistics $S_k$ and $W_k$ are nearly equivalent. We also find that $e_p(S_k, W_k)$ is increasing for $k \leq 6$, and it is decreasing for $k > 7$.

4. Descriptive examples

Let $(X_{i1}, X_{i2}, X_{i3}, X_{i4}, X_{i5})$ be a random sample from the population having a distribution function $F_i(x)$ ($i=1, 2, 3, 4$). Combine these samples and arrange in ascending order of magnitude from the smallest observation to the largest one. If an observation in the combined samples is one from the population with $F_i(x)$ ($i=1, 2, 3, 4$), assign the number $i$ to it. Thus, suppose that the following three sequences are obtained.

(i) 1 1 1 2 1 2 1 3 2 2 3 2 3 4 4 4 4 3 4
(ii) 2 4 1 3 2 2 3 1 4 4 3 2 1 2 3 1 4 1 3 4
(iii) 1 4 2 4 1 1 4 4 1 4 1 2 3 3 3 3 3 2 2 2

Then 4-multiple charts for the above number sequences are shown in Fig. 2, Fig. 3 and Fig. 4. The values of $S_4$ in (2.3) and $S_4^*$ in (2.4) for the number sequences (i), (ii) and (iii) are obtained in Table 2, where $E(S_4)=52.9$, $V(S_4)=121.0$ and $0 \leq S_4 \leq 105.8$. The value $z_a$ in (2.5) is 1.96 for $\alpha=0.05$. Therefore the null hypothesis $H_0$ is rejected at the 0.05 level of significance for the sequence (i), and $H_0$ is accepted at the same level of significance for the sequences (ii) and (iii). Though two values of $S_4$ for the sequences (ii) and (iii) are nearly equal, we can see from the Fig. 3 and Fig. 4 that the two paths corresponding to the two sequences are different in pattern at the following points:

![Fig. 2 Multiple chart for the sequence (i)](image-url)
Table 2.

<table>
<thead>
<tr>
<th>Number sequence</th>
<th>$S_4$</th>
<th>$S_4^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>99.8</td>
<td>4.26</td>
</tr>
<tr>
<td>(ii)</td>
<td>56.6</td>
<td>0.34</td>
</tr>
<tr>
<td>(iii)</td>
<td>52.6</td>
<td>-0.03</td>
</tr>
</tbody>
</table>

(1) The number of the run for the sequence (ii) is larger than that for the sequence (iii).

(2) The path for the sequence (iii) has two long parallel lines to $P_1P_2$ and $P_1P_3$, respectively.
Acknowledgements

The author wishes to thank Dr. T. Yanagawa, Kyushu University, Dr. S. Shirahata, Osaka University, Dr. N. I. Fisher, CSIRO of Australia, and the referees for their valuable suggestions and comments. His thanks are extended to the members of Okayama statistical research group for their helpful advice and discussions.

REFERENCES