IDENTIFIABILITY OF COUNTABLE MIXTURES

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The identifiability of a countable mixture of Weibull distributions and that of a countable mixture of log-normal distributions are studied provided that the supports of mixing distributions are well-ordered sets for a total ordering of the parameter space. Both classes of all finite mixtures of Weibull distributions and all finite mixtures of log-normal distributions are shown to be identifiable as special cases.

1. Introduction

Let \( \mathcal{F}_1 = \{F(x, \theta) : \theta \in \mathbb{R}^k\} \) be a family of one-dimensional cumulative distribution functions (cdf's) and \( R_1 \) a Borel subset of \( k \)-dimensional Euclidean space \( \mathbb{R}^k \). Let \( \mathcal{G}_k = \{G(\theta)\} \) be a class of cdf's such that \( P_\theta(R_1) = 1 \), where \( P_\theta \) is the probability measure induced by \( G \). For any \( G \in \mathcal{G}_k \),

\[
H_\theta(x) = \int_{R_1} F(x; \theta) dG(\theta)
\]

will be called a \( G \)-mixture (or a \( P_\theta \)-mixture) of \( \mathcal{F}_1 \) and \( G(P_\theta) \) a mixing cdf (a mixing probability measure). In particular, \( H_\theta \) will be called a countably infinite mixture (a finite mixture) in a case the support of \( P_\theta \) is a countably infinite subset of \( R_1 \) (a finite subset of \( R_1 \)). We here regard also \( F(x, \theta) \) as a finite mixture for a \( G \) such as \( P_\theta(\theta) = 1 \). The class of all \( H_\theta \), that is,

\[
\mathcal{H} = \left\{ H_\theta : H_\theta(x) = \int_{R_1} F(x; \theta) dG(\theta), G \in \mathcal{G}_k \right\}
\]

will be called a \( \mathcal{G}_k \)-mixture of \( \mathcal{F}_1 \). The following is given by Teicher [4].

**Definition.** A \( G^* \)-mixture \( H_{G^*} \) of \( \mathcal{F}_1 \) will be called "identifiable" if the relationship \( H_{G^*}(x) = \int F(x, \theta) dG^*(\theta) = \int F(x, \theta) dG(\theta) \) implies that \( G^* = G \) for any \( G (\in \mathcal{G}_k) \). If every member of \( \mathcal{H} \) is identifiable, then \( \mathcal{H} \) itself will be called identifiable.

The identifiability is a necessary condition when we estimate the mixing cdf \( G \) on the basis of an independent random sample from the distribution \( H_\theta \). Teicher [4] has given a sufficient condition for the identifiability of finite mixtures to hold and shown that the class of all finite mixtures of normal distributions is identifiable. Chandra [2] has given a modification of the condition of Teicher. Henna [3] has given a generalization of the condition of Teicher [4] and Chandra [2] which can be applied to a countably infinite mixture. Ahmad [1] has shown that both classes of all finite mixtures of Weibull distributions and all finite mixtures of log-normal distributions are identifiable by using a result of Chandra [2]. We will give a generalization of the results of Ahmad [1] to some

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countable mixtures by using a result of Henna [3].

In Section 2, we introduce a sufficient condition given by Henna [3] for the identifiability of a countable mixture to hold. In Sections 3 and 4, we consider the identifiability of a countable mixture of Weibull distributions and that of a countable mixture of log-normal distributions, respectively. In both sections, the supports of mixing cdf's are assumed to be well-ordered subsets of $R_k$ for a given total ordering. We lastly give an Appendix where a convergence problem of a ratio of gamma functions, which is needed to achieve our object, is studied.

2. A sufficient condition for the identifiability to hold

In order to give a generalization of the results of Ahmad [1], we need to introduce a notation and a result for identifiability. Suppose that there exists a total ordering $(\prec)$ of $R_k$ such that $\theta_1 \prec \theta_2$ and that, for any $\theta_0 \in R_k$, the set $\{\theta : \theta_0 \prec \theta, \theta \in R_k\}$ is a Borel subset of $R_k$. For a subset $\Theta$ of $R_k$, $(\Theta, \prec)$ will be called a well-ordered set if every non-empty subset $\Theta_1$ of $\Theta$ has the least element, that is, the element $\theta_0 \in \Theta_1$ satisfying $\theta_0 \prec \theta$ for every $\theta$ in $\Theta_1$ other than $\theta_0$. We denote it by $\min \Theta_1$. Let

$$\mathcal{A} = \{ (\Theta, \prec) : (\Theta, \prec) \text{ is a well-ordered set and } \Theta \text{ a finite or a countably infinite subset of } R_k \} .$$

We define a class of cdf's by

$$G^* = \bigcup_{(\Theta, \prec) \in \mathcal{A}} \left\{ G(\theta) : \sum_{\theta \in \Theta} g_r = 1, 0 < g_r \right\} ,$$

where $G$ is a discrete cdf such as $P_G(\theta_0) = g_r$ for $\theta_0 \in \Theta$. Then each member of the $G^*$-mixture of $\mathcal{F}$ is a finite or a countably infinite mixture and the $G^*$-mixture of $\mathcal{F}$ includes the class of all finite mixtures of $\mathcal{F}$.

Let $G = \{G(\theta)\}$ be a class of cdfs such that $G^* \subseteq G$ and $P_G(R_k) = 1$ for any $G \in G$, and $\mathcal{G}$ the $G$-mixture of $\mathcal{F}$. Suppose that a $T$ transforms $H_\theta(\in \mathcal{G})$ to a function $\phi_{H_\theta}(t)$ defined on a subset $S_{H_\theta}$ of $[-\infty, +\infty]$. In particular, if $F(x, \theta) = H_\theta$, then we denote $\phi_{H_\theta}(t) = \phi_{H_\theta}(t)$ and $S_{H_\theta} = S_{H_\theta}$, respectively. Assume that the $T$ satisfies the following conditions:

(C-2.1) There exists a non-empty set $S$ such as $S \subseteq S_{H_\theta}$ for all $H_\theta \in \mathcal{G}$.

(C-2.2) $\phi_{H_\theta}(t) \neq 0$ for any $t \in S$ and for any $\theta \in R_k$.

(C-2.3) $\phi_{H_\theta}(t) = \int_{R_k} G(\theta) \phi_{H_\theta}(t) dG(\theta)$, for any $t \in S$ and for any $H_\theta \in \mathcal{G}$.

(C-2.4) For any $\theta_1$, there exists a $\bar{t} \in \bar{S}$ such that, if $\theta_1 \prec \theta$, then $\phi_{H_\theta}(t) / \phi_{H_{\bar{\theta}}}(t) \to 0$ as $t \to \bar{t}$, $\phi_{H_\theta}(t) / \phi_{H_{\bar{\theta}}}(t)$ is uniformly bounded and has the same sign in $\theta$ on a neighbourhood of $\bar{t}$, where $\bar{S}$ is the closure of $S$ and $\bar{t}$ is independent of $\theta_1$.

Under these conditions, Henna [3] has shown that, if the equality

$$\int_{R_k} F(x, \theta) dG(\theta) = \int_{R_k} F(x, \theta) dG^*(\theta), \quad (G, G^* \in G) ,$$

holds and either $G$ or $G^*$ is a member of $G^*$, then $G = G^*$.
3. Identifiability of a mixture of Weibull distributions

We consider here a family $\mathcal{F}_1$ of Weibull cdf's as follows:

\begin{equation}
\mathcal{F}_1 = \left\{ F : F(x; \alpha, \beta) = \int_{\alpha}^{x} \frac{y^{\beta-1}}{\alpha^\beta} \exp \left[ -\left( \frac{y}{\alpha} \right)^\beta \right] dy, \ (x, \beta) \in R_1^* \right\},
\end{equation}

where $R_1^* = [\alpha, \alpha_0] \times [\beta, \beta_0]$ with $0 < \alpha_0 < \alpha$ and $0 < \beta_0 < \beta$.

We first define two total orderings ($\preceq$) and ($\prec$) of $[\beta_0, \beta_0^*]$ and $R_1^*$, respectively, as follows:

\begin{equation}
\beta_1 \prec \beta_2 \iff \beta_1 < \beta_2.
\end{equation}

\begin{equation}
(\alpha_1, \beta_1) \prec (\alpha_2, \beta_2) \iff \beta_1 < \beta_2 \text{ or } \beta_1 = \beta_2 \text{ but } \alpha_1 < \alpha_2.
\end{equation}

We next consider two classes of well-ordered sets as follows:

$\mathcal{B} = \{(B, \preceq) : B \subset [\beta_0, \beta_0^*] \text{ and } (B, \preceq) \text{ is a well-ordered set}\}$,

$\mathcal{A} = \{(\Theta, \prec) : \Theta \subset R_1^* \text{ and } (\Theta, \prec) \text{ is a well-ordered set}\}$.

Then $B (e \in \mathcal{B})$ and $\Theta (e \in \mathcal{A})$ are finite or countably infinite subsets. Furthermore we construct two classes of cdf's on $R_1^*$ as follows.

\begin{equation}
\mathcal{G}_2 = \bigcup_{(B, \preceq) \in \mathcal{B}} \left\{ G(\alpha, \beta) : \sum_{\beta_r \in B} q_r = 1, \ 0 < q_r \right\},
\end{equation}

where $P_\Theta(\alpha, \beta_r) : x \in [\alpha_0, \alpha_0^*] = g_r$ for each $\beta_r \in B$, and

\begin{equation}
\mathcal{G}_2^* = \bigcup_{(\Theta, \prec) \in \mathcal{A}} \left\{ G(\alpha, \beta) : \sum_{(\alpha_r, \beta_r) \in \Theta} g_r = 1, \ 0 < g_r \right\},
\end{equation}

where $P_\Theta(\alpha_r, \beta_r) = g_r$ for each $\Theta \in \Theta$.

By the construction, it can be easily seen that $\mathcal{G}_2 \subset \mathcal{G}_2^*$ and $\mathcal{G}_2^*$-mixture of $\mathcal{F}_1$ is composed of some finite and some countably infinite mixtures of $\mathcal{F}_1$. In order to apply a result of Henna \cite{3}, to a mixture of $\mathcal{F}_1$ above, it is appropriate to consider a $T$ which transforms a cdf $H_\Theta$ to the moment generating function $\phi_{x, \Theta}$ of log $X$, where $X$ is a random variable distributed according to $H_\Theta$. Then, for $F(x; \alpha, \beta)$, we have $\phi_{x, \beta}(t) = x^\beta \Gamma(t/\beta + 1)$ with $S_{x, \beta} = (-\beta, \infty)$. It can be easily seen that (C-2.1)-(C-2.3) in Section 2 are satisfied with $S = [0, \infty)$ and $T = +\infty$. Unfortunately the uniform boundedness in (C-2.4) is unsatisfied. However the difficulty can be conquered by a modification of the proof of Theorem 3.1 of Henna \cite{3}.

PROPOSITION 3.1. Assume that

\begin{equation}
\int_{R_1^*} F(x; \alpha, \beta) dG(\alpha, \beta) = \int_{R_1^*} F(x; \alpha, \beta) dG^*(\alpha, \beta), \quad (G, G^* \in \mathcal{G}_1),
\end{equation}

and either $G$ or $G^*$ is a member of $\mathcal{G}_2^*$. Then $G = G^*$.

PROOF. We give the first half of the proof. Let $G^*$ be any member of $\mathcal{G}_2^*$ such as $P_\Theta(\alpha^*, \beta^*) = g^*$ for $(\alpha^*, \beta^*) \in \Theta^{\circ \circ}$, where $\Theta^{\circ \circ}$ is the support of $G^*$. Let $A^* = \{ \alpha^* : (\alpha^*, \beta^*) \in \Theta^{\circ \circ}\}$ and $B^* = \{ \beta^* : (\alpha^*, \beta^*) \in \Theta^{\circ \circ}\}$. Then $A^*$ is a finite or a countably infinite
subset of \([a_0, a_{00}]\) and \(B^* \in \mathcal{B}\). Let the support of \(G\) be \(\Theta_0\), \(A = \{a: (a, \beta_0) \in \Theta_0\}\) and \(B = \{\beta_0: (a, \beta_0) \in \Theta_0\}\), then \(A \subset [a_0, a_{00}]\) and \(B \in \mathcal{B}\). By the construction of \(G^*\) and \(G\), there exist \(\min \Theta_0 = (\alpha_0^*, \beta_0^*)\) and \(\min B = \beta_1\). In a similar way to (3.4) of Henna [3], by (C-2.3), we have

\[
(3.5) \quad \int_{(a, \alpha) < (\alpha_1^*, \beta_1^*)} \frac{\phi_{a, \alpha}(t)}{\phi_{0, 0}(t)} dG(a, \beta) + \int_{(\alpha_1^*, \beta_1^*) \leq (a, \alpha)} \frac{\phi_{a, \alpha}(t)}{\phi_{0, 0}(t)} dG(a, \beta) = g_1^* + \int_{(\alpha_1^*, \beta_1^*) \leq (a, \alpha)} \frac{\phi_{a, \alpha}(t)}{\phi_{0, 0}(t)} dG_*(a, \beta),
\]

where \((\alpha_1^*, \beta_1^*) = \min(\Theta_0, (\alpha_1^*, \beta_1^*))\).

Assume that \(P_\theta(\alpha, \beta) < (\alpha_1^*, \beta_1^*) > 0\). Then, by the definition of ordering, we have \(\beta_1 < \beta_1^*\) or \(\beta_1 = \beta_1^*\) but \(\alpha > \alpha_1^*\). In the case of \(\beta_1 < \beta_1^*\), by Lemma 5.2 in Appendix, the left hand side of (3.5) diverges as \(t \to +\infty\). In the case of \(\beta_1 = \beta_1^*\) but \(\alpha > \alpha_1^*\), the left hand side of (3.5) also diverges because \(\phi_{a, \alpha}(t)/\phi_{0, 0}(t) = (\alpha/\alpha_1^*)^{t-\alpha_1^*} \to \infty\) as \(t \to +\infty\). On the other hand the right hand side converges to \(g_1^*\) by the Lebesgue convergence theorem and Lemmas 5.1 and 5.2 in both cases. This is a contradiction. Accordingly \(P_\theta(\alpha, \beta) < (\alpha_1^*, \beta_1^*) = 0\).

Assume that \(\beta_1^* < \beta_1\), then we have

\[
(3.6) \quad \int_{(a, \beta) < (\alpha_1^*, \beta_1^*)} \frac{\phi_{a, \beta}(t)}{\phi_{0, 0}(t)} dG(a, \beta) = g_1^* + \int_{(\alpha_1^*, \beta_1^*) \leq (a, \beta)} \frac{\phi_{a, \beta}(t)}{\phi_{0, 0}(t)} dG(a, \beta).
\]

Again letting \(t \to +\infty\), by Lemmas 5.1 and 5.2, the left and the right hand sides of (3.6) converge to 0 and \(g_1^*\), respectively. This is a contradiction. Therefore \(\beta_1^* = \beta_1\). Hence we have

\[
(3.7) \quad \int_{(a, \beta) = (\alpha_1^*, \beta_1^*)} dG(a, \beta) + \int_{(a, \beta) > (\alpha_1^*, \beta_1^*)} \frac{\phi_{a, \beta}(t)}{\phi_{0, 0}(t)} dG(a, \beta) + \int_{(a, \beta) < (\alpha_1^*, \beta_1^*)} \frac{\phi_{a, \beta}(t)}{\phi_{0, 0}(t)} dG(a, \beta) = g_1^* + \int_{(\alpha_1^*, \beta_1^*) \leq (a, \beta)} \frac{\phi_{a, \beta}(t)}{\phi_{0, 0}(t)} dG_*(a, \beta),
\]

where \(\beta_1 = \min(B - \beta_1^*)\). Again letting \(t \to +\infty\), by the Lebesgue convergence theorem and Lemmas 5.1 and 5.2, we have

\[
(3.8) \quad \int_{(a, \beta) = (\alpha_1^*, \beta_1^*)} dG(a, \beta) = g_1^*.
\]

Repeating the argument similar to the above, we can show the latter half by the same way to the proof of Theorem 3.1 of Henna [3].

The following is an immediate consequence of the above lemma by taking a \(R^1\) such as \(\Theta_0 \subset R^1\) and \(\Theta_0 \subset R^1\) for any two finite mixtures \(H_0\) and \(H_0\) holding \(H_0 = H_0\).

**Corollary 3.1.** (Ahmad [1]). The class of all finite mixtures of Weibull distributions is identifiable.

4. **Identifiability of a mixture of log-normal distributions**

In this section we consider a family \(\mathcal{E}_1\) of log-normal cdf's as follows:
\begin{equation}
\mathcal{C}_1 = \left\{ E : E(x; \mu, \sigma^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma y} \exp \left[ -\frac{(\log y - \mu)^2}{2\sigma^2} \right] dy, \quad (\mu, \sigma^2) \in R_i^1 \right\},
\end{equation}

where $R_i^1 = [\mu_0, \mu_{00}] \times [\sigma_0^2, \sigma_{00}^2]$ with $\mu_0 < \mu_{00}$ and $0 < \sigma_0 < \sigma_{00}$.

We now define two total orderings $(\preceq)$ and $(\prec)$ of $[\sigma_0^2, \sigma_{00}^2]$ and $R_i^1$, respectively, as follows:

\begin{equation}
\sigma_i^2 \preceq \sigma_j^2 \iff \sigma_i^2 > \sigma_j^2
\end{equation}

In the same way to the last section, we construct two classes $\mathcal{B}$ and $\mathcal{A}$ of well-ordered sets and two families $\mathcal{Q}_2$ and $\mathcal{Q}_4^*$ of cdf's on $R_i^1$. We consider also a $T$ which transforms a cdf $H_0$ to the moment generating function $\phi_{H_0}$ of $\log X$ as the last section. Then, for $E(x; \mu, \sigma^2)$, we have $\phi_{G*}(t) = \exp \left\{ \sigma^2 t^2 / 2 + \mu t \right\}$ with $S_{\sigma^2, \mu} = (-\infty, \infty)$. Then we can show the following proposition in the same way to the proof of Proposition 3.1 by $t \to +\infty$.

**Proposition 4.1.** Assume that

and either $G$ or $G^*$ is a member of $\mathcal{Q}_4^*$. Then $G=G^*$.

Let $\mathcal{C}_1$ be that of (4.1), the ordering $(\preceq)$ that of (4.2) and the ordering $(\prec)$ as follows:

\begin{equation}
(\mu_1, \sigma_1^2) \prec (\mu_2, \sigma_2^2) \iff \sigma_1^2 > \sigma_2^2; \quad \text{or} \quad \sigma_1^2 = \sigma_2^2 \quad \text{but} \quad \mu_1 < \mu_2.
\end{equation}

Furthermore we construct $\mathcal{B}$, $\mathcal{A}$, $\mathcal{Q}_2$ and $\mathcal{Q}_4^*$ as the last section. Then we have the following proposition by $t \to +\infty$.

**Proposition 4.2.** Assume that

and either $G$ or $G^*$ is a member of $\mathcal{Q}_4^*$. Then $G=G^*$.

The following can be immediately obtained from Proposition 4.1 or 4.2 by taking a $R_i^1$ such that $\Theta_0 \subset R_i^1$ and $\Theta_* \subset R_i^1$ for any two finite mixtures $H_0$ and $H_0^*$, holding $H_0 = H_0^*$.

**Corollary 4.1.** (Ahmad [1]). The class of all finite mixtures of log-normal distributions is identifiable.

**Appendix**

**Lemma 5.1.** Let $0 < \beta_1 < \beta_2$ and $0 < \alpha_0$. Then

\[ \frac{\alpha^2 \Gamma(l/\beta_1 + 1)}{\Gamma(l/\beta_2 + 1)} \]

is bounded in $(\alpha, \beta, l)$ on $[0, \alpha_0] \times [\beta_1, \infty) \times [l_0, \infty)$ for a $l_0(>0)$ sufficiently large.

**Proof.** There exist positive integers $l(\geq 2)$, $m$ and $n$ such as $\beta_1 \leq m/l < n/l \leq \beta_2 \leq \beta$. 

\[ \frac{\alpha^2 \Gamma(l/\beta_1 + 1)}{\Gamma(l/\beta_2 + 1)} \]
Then, by the monotone increasing of $\Gamma(\cdot)$, we have
\[
\frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)} < \frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)}.
\]

Let $k$ be a positive integer such as $mnk < t \leq mn(k+1)$. Then
\[
\frac{\Gamma(m|n|t+1)}{\Gamma(n|\beta_1+1)} < \frac{\Gamma([m|n|t+1]}{\Gamma(n|\beta_1+1)}.
\]

So, for $x > 1$, we have
\[
x^t \frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)} < \alpha_{n0}^{mn(k+1)} \frac{\Gamma([m|n|t+1]}{\Gamma(n|\beta_1+1)}.
\]

Therefore, by $n-m \geq 1$, we have
\[
x^t \frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)} \leq \alpha_{n0}^{mn(k+1)} \frac{\Gamma([m|n|t+1]}{\Gamma(n|\beta_1+1)}
\]
for $k$ (accordingly for $t$) sufficiently large. Furthermore, by $l(n-m) \geq 2$, we have
\[
(5.1) \quad x^t \frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)} \leq \left( \frac{\alpha_{n0}^{mn}}{nlk} \right) \left( \frac{1}{nlk-k} \right) \left( \frac{1}{nlk-k-1} \right).
\]

Hence
\[
(5.2) \quad x^t \frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)} \leq 1
\]
if $k \geq k_0$ (accordingly if $t \geq t_0$) for a $k_0$ (for a $t_0$) independent of $(x, \beta)$ because of $\alpha_{n0}, l, m$ and $n$ being independent of $(x, \beta)$.

For $x \leq 1$, again we have the inequality (5.2) by substituting 1 for $\alpha_{n0}$ in the above inequalities.

The inequality (5.1) in the proof above shows also the following.

**Lemma 5.2.** If $0 < \beta_1 < \beta$ and $0 \leq x$, then we have
\[
\lim_{t \to +\infty} x^t \frac{\Gamma(n|\beta+1)}{\Gamma(n|\beta_1+1)} = 0.
\]

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**References**


