ON BALANCED ARRAYS OF 2 SYMBOLS, STRENGTH 2l, m CONSTRAINTS AND INDEX SET
\[ \{ \mu_0, \mu_1, \cdots, \mu_{2l} \} \] WITH \( \mu_1 = 0 \)

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Consider a balanced array ("partially balanced" array, in the terminology of Chakravarti [1]) of 2 symbols, strength 2l, m constraints and index set \( \{ \mu_0, \mu_1, \cdots, \mu_{2l} \} \). Then it is shown that balanced arrays with \( \mu_1 = 0 \) are simple. A necessary and sufficient condition for the existence of such balanced arrays is given.

1. Introduction

Let \( T \) be a \((0,1)\) matrix of size \( m \times N \). We shall define a balanced array (B-array) with 2 symbols. \( T \) is said to be a B-array of strength \( t \), size \( N \), \( m \) constraints and index set \( \{ \mu_0, \mu_1, \cdots, \mu_t \} \) (or indices \( \mu_i \) \((i=0,1,\cdots,t)\)), if for every \( t \)-rowed submatrix (subarray) \( T^{(t)} \) of \( T \), every vector with weight (or number of non-zero elements) \( i \) occurs exactly \( \mu_i \) times \((i=0,1,\cdots,t)\) as a column of \( T^{(t)} \). Then it is clear that \( N = \sum_{i=0}^{t} \binom{t}{i} \mu_i \). A B-array with \( s (>2) \) symbols can be similarly defined but we shall not consider it here. Thus the term "2 symbols" will be omitted.

We observe that B-arrays reduce to orthogonal arrays or \( t \)-designs. The B-array defined above becomes an orthogonal array with parameters \((N, m, 2, t)\) of index \( \mu \) when \( \mu_0 = \mu_1 = \cdots = \mu_t \) (say). On the other hand, it becomes a \( t-(m, k, \lambda) \) design when all the weights of its column vectors are equal to \( k \) and \( \mu_\lambda = \lambda > 0 \).

Srivastava [5] has constructively given necessary and sufficient conditions for the existence of a B-array with strength \( t \) and \( m (\leq t+2) \) constraints. Furthermore it is known (see, for example, Srivastava [4], Yamamoto, Shirakura and Kuwada [6]) that B-arrays of strength 2l, size \( N \) and \( m \) constraints have close relationships with balanced fractional \( 2^m \) factorial designs of resolution 2l+1 with \( N \) assemblies. In this paper, special B-arrays of strength 2l with \( \mu_1 = 0 \) are investigated. We may stress that such B-arrays are urgently needed as balanced designs of even resolution. That is, Shirakura [2] has shown that under some conditions B-arrays of strength 2l, size \( N \), \( m \) constraints and index set \( \{ \mu_0, \mu_1, \cdots, \mu_t \} \) with \( \mu_1 = 0 \) yield balanced fractional \( 2^m \) factorial designs of resolution 2l with \( N \) assemblies such that the mean of the effects of \( l \)-factor interactions and \( \left\{ \binom{m}{l-1} - 1 \right\} \) independent contrasts between these effects are estimable. Note that such B-arrays do not correspond to designs of resolution 2l+1 at all, since a necessary condition for a general B-array of strength 2l to be a balanced design of resolution 2l+1 is \( \mu_1 \neq 0 \) (see Shirakura and Kuwada [3]).

Received Oct. 12, 1974, revised Feb. 8, 1975

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2. Simple arrays

Let \( \Omega(k; m) \) \((0 \leq k \leq m)\) be the \((0, 1)\) matrix of size \( m \times \binom{m}{k} \) whose columns are all distinct vectors of weight \( k \).

**Definition.** An array obtained by juxtaposing each \( \Omega(k; m) \ \alpha_k \ (\geq 0) \) times, \( k = 0, 1, \ldots, m \), i.e.,

\[
\left[ \underbrace{\Omega(0; m)}_{\alpha_0} : \ldots : \underbrace{\Omega(0; m)}_{\alpha_1} : \ldots : \underbrace{\Omega(1; m)}_{\alpha_1} : \ldots : \underbrace{\Omega(m; m)}_{\alpha_m} \right]
\]

is said to be a simple array with parameters \((m, \alpha_0, \alpha_1, \ldots, \alpha_m)\). Each \( \Omega(k; m) \), of course, is a simple array with \( \alpha_k = 1 \).

Suppose that \( (a, b) = 0 \) if and only if \( 0 \leq a < b \) or \( b < 0 \), throughout this paper. Then it can be easily checked that \( \Omega(k; m) \) is a B-array of strength \( t \) with indices \( (m-tk-i) \ (i=0, 1, \ldots, t) \). Thus we have

\[\Omega(k; m) \] is a simple array with parameters \((m, \alpha_0, \alpha_1, \ldots, \alpha_m)\). Each \( \Omega(k; m) \), of course, is a simple array with \( \alpha_k = 1 \).

**Lemma.** A simple array with parameters \((m, \alpha_0, \alpha_1, \ldots, \alpha_m)\) is a B-array of strength \( t \) and \( m \) constraints with indices \( \sum_{i=0}^{m} \binom{m-t}{k-i} \alpha_k \ (i=0, 1, \ldots, t) \).

As will be seen from definition, for any index set \( \{ \mu_0, \mu_1, \ldots, \mu_l \} \), we can always construct a B-array of strength \( t \) and \( t \) constraints. Moreover, it is a simple array with parameters \( (t, \alpha_0=\mu_0, \alpha_1=\mu_1, \ldots, \alpha_t=\mu_t) \).

3. B-arrays of strength \( 2l \) with \( \mu_l=0 \)

**Definition.** A B-array of strength \( 2l \), \( m \) constraints and index set \( \{ \mu_0, \mu_1, \ldots, \mu_l \} \) is called a coreless B-array if \( \mu_l=0 \).

**Theorem 1.** Let \( T \) be a coreless B-array of strength \( 2l \), \( m \) constraints and index set \( \{ \mu_0, \mu_1, \ldots, \mu_l \} \). Then the weight \( k \) of a column of \( T \) must satisfy \( l > k \) or \( m - l < k \).

**Proof.** Assume that there exists a column vector of \( T \) with weight \( k \) satisfying \( 1 \leq k \leq m-l \). Then we can obtain a \( 2l \)-rowed subarray \( T^{(2l)} \) of \( T \) such that a column vector with weight \( l \) occurs in \( T^{(2l)} \). This implies \( \mu_l \neq 0 \), a contradiction.

In view of the result of Theorem 1 a coreless B-array \( T \) of strength \( 2l \) and \( m \) constraints can be expressed without loss of generality as

\[ T = [T_0: T_1: \ldots: T_{l-1}: T_{m-l+1}: \ldots: T_m], \]

where each \( T_k \) is a subarray of \( T \) whose columns are of weight \( k \) only.

**Theorem 2.** Let \( T \) be a coreless B-array of strength \( 2l \), \( m \) constraints and index set \( \{ \mu_0, \mu_1, \ldots, \mu_{l-1}, 0, \mu_{l+1}, \ldots, \mu_l \} \). Then the subarrays \([T_0: T_1: \ldots: T_{l-1}]\) and \([T_{m-l+1}: T_{m-l+2}: \ldots: T_m]\) are B-arrays of strength \( 2l \) with index sets \( \{ \mu_0, \ldots, \mu_{l-1}, 0, \ldots, 0 \} \) and \( \{0, \ldots, 0, \mu_{l+1}, \ldots, \mu_l\} \) respectively.

**Proof.** The number of times any column vector of weight \( i \) \((0 \leq i \leq l-1)\) occurs in any \( 2l \)-rowed subarray of \( T \) does not depend on \( T_{m-l+1}, \ldots, T_m \). Hence \([T_0: T_1: \ldots: T_{l-1}]\) is a B-array of strength \( 2l \) with \( \{\mu_1, \ldots, \mu_{l-1}, 0, \ldots, 0\} \). Similarly, it can be shown that \([T_{m-l+1}: \ldots: T_m]\) is a B-array with the indicated parameters.

**Theorem 3.** Consider the coreless B-array \( T \) of Theorem 2. Then it is a simple array.
**Proof.** Now we shall prove by induction that each $T_i$ ($i=0,1,\ldots, l-1$) is a simple array. From Theorem 2, the index set of the $B$-array $[T_0:T_1: \cdots : T_{l-1}]$ is given as $\{\mu_0, \mu_1, \cdots, \mu_{l-1}, 0, 0, \cdots, 0\}$. On the other hand, the number of times a vector with weight $l-1$ occurs as a column of this array depends on $T_{l-1}$ only. Let $\nu(i_1, i_2, \cdots, i_{l-1})$ denote the number of times the vector $x(m \times 1)$ occurs as a column of $T_{l-1}$ where $x$ contains 1 exactly at the $i_1$-th, $i_2$-th, \ldots, $i_{l-1}$-th positions and 0 elsewhere. Then in a 2l-rowed subarray of $T_{l-1}$ which includes the $i_1$-th, $i_2$-th, \ldots, $i_{l-1}$-th rows, the column vector corresponding to $x$ must occur exactly $\nu(i_1, i_2, \cdots, i_{l-1})$ times. From the definition of $B$-arrays, it follows that $\nu(i_1, \cdots, i_{l-1})$ is equal to $\mu_{l-1}$. Thus the number $\nu(i_1, i_2, \cdots, i_{l-1})$ does not depend on the $i_1$-th, $i_2$-th, \ldots, $i_{l-1}$-th positions of $x$. This means that $T_{l-1}$ is a simple array with $\alpha_{l-1} = \mu_{l-1}$. Assume that $[T_{i+1}: T_{i+2}: \cdots : T_{l-1}]$ is a simple array. Then, since it is a $B$-array of strength 2l from Lemma, $[T_0:T_1: \cdots : T_i]$ is also a $B$-array of strength 2l and its index set takes the form of $\{\mu_0, \mu_1, \cdots, \mu_i, 0, 0, \cdots, 0\}$. From an argument similar to the above, it follows that $T_i$ is a simple array with $\alpha_i = \mu'_i$. This proves that $[T_0:T_1: \cdots : T_{l-1}]$ is a simple array. In the same way, it can be shown that the $B$-array $[T_{m-l+1}: T_{m-l+2}: \cdots : T_m]$ is also a simple array. Thus $T$ is a simple array.

As an ultimate theorem, we have

**Theorem 4.** $T$ is a coreless $B$-array of strength 2l, m constraints and index set $(\mu_0, \mu_1, \cdots, \mu_{l-1}, 0, \mu_{l+1}, \cdots, \mu_m)$ if and only if $T$ is a simple array with parameters $(m, \alpha_0, \alpha_1, \cdots, \alpha_{l-1}, 0, 0, \alpha_{m-l+1}, \cdots, \alpha_m)$. The connection between the indices $\mu_0$ and the parameters $\alpha_i$ is given as follows: For $i=0, 1, \cdots, l-1$,

\begin{align}
(1a) \quad \mu_i &= \sum_{k=i}^{l-1} \binom{m-2l}{k-i} \alpha_k, \\
(1b) \quad \mu_{i+1+k} &= -\sum_{k=0}^{i} \binom{m-2l}{i-k} \alpha_{m-l+1+k},
\end{align}

or for $k=0, 1, \cdots, l-1$,

\begin{align}
(2a) \quad \alpha_k &= \sum_{i=0}^{l-1} (-1)^{i+k} \binom{m-2l-1+i-k}{i-k} \mu_i, \\
(2b) \quad \alpha_{m-l+1+k} &= \sum_{i=0}^{l-1} (-1)^{i+k} \binom{m-2l-1+k-i}{k-i} \mu_{i+1+k}.
\end{align}

**Proof.** The proof of the first part of the theorem follows from Lemma and Theorems 1-3. Now, (1a, b) are obvious from Lemma. Next, we show that (2a, b) hold. From (1a), we have $(\mu_0, \mu_1, \cdots, \mu_{l-1})' = H(\alpha_0, \alpha_1, \cdots, \alpha_{l-1})'$, where $H$ is the $l \times l$ matrix whose $(i, j)$ elements are given by $\binom{m-2l}{j-i}$. Clearly, $H$ is a non-singular upper triangular matrix. From Lemma of appendix, it follows that $(i, j)$ elements of the inverse matrix $H^{-1}$ are given by $(-1)^{i+j} \binom{m-2l-1+j-i}{j-i}$. This leads to (2a).

Since from (1b) $(\mu_{m-l}, \mu_{m-l-1}, \cdots, \mu_{l-1})' = H(\alpha_{m-l}, \alpha_{m-l-1}, \cdots, \alpha_{m-1})'$ holds, we have similarly $\alpha_{m-k} = \sum_{i=0}^{l-1} (-1)^{i+k} \binom{m-2l-1+i-k}{i-k} \mu_{m-k}$, for $k=0, 1, \cdots, l-1$. This leads to (2b).

As an easy consequence to the above theorem, we have

**Corollary.** A necessary and sufficient condition for the existence of a coreless $B$-array of strength 2l, m constraints and index set $(\mu_0, \mu_1, \cdots, \mu_{l-1}, 0, \mu_{l+1}, \cdots, \mu_m)$ is that
the following inequalities hold:

\[(3a) \sum_{i,k} (-1)^{i+k} \binom{m-2l-1+i-k}{i-k} \mu_i \geq 0 \]

and

\[(3b) \sum_{i,k} (-1)^{i+k} \binom{m-2l-1+k-i}{k-i} \mu_{i+1} \geq 0, \]

for all \(k=0,1, \ldots, l-1\).

4. Appendix

**Lemma.** For non-negative integers \(i, j\) and \(n\), we have the following combinatorial identity:

\[(4) \sum_k (-1)^{i+j} \binom{n}{i-k} \binom{n-1+k-j}{k-j} = \delta_{i,j}, \]

where \(\delta_{i,j}\) denotes the Kronecker delta \((\delta_{ii}=1, \delta_{ij}=0, i \neq j)\) and \(\sum\) denotes the summation over all possible \(k \geq 0\).

**Proof.** If \(i < j\), then the left hand side of (4) is always zero. Hence the identity (4) holds for \(i < j\). For \(i \geq j\), consider the following generating function:

\[
(1-x)^n(1-x)^{-n} = \sum_{r=0}^{n} \sum_{q=0}^{m} (-1)^r \binom{n}{r} \binom{n-1+q}{q} x^r.
\]

Since this function is identically equal to 1, we have the identity

\[
\sum_k (-1)^{i+j} \binom{n}{i-k} \binom{n-1+k-j}{k-j} = \delta_{i,j}.
\]

By replacing \(r\) and \(q\) with \(i-j\) and \(k-j\) respectively, and multiplying its both sides by \((-1)^{i-j}\), we obtain

\[
\sum_k (-1)^{i+j} \binom{n}{i-k} \binom{n-1+k-j}{k-j} = (-1)^{i-j} \delta_{i,j} = \delta_{i,j}.
\]

Acknowledgment

The author wishes to thank the referees for their valuable comments.

**References**


