SOME REMARKS ON BOUNDED INSURANCE CONTRACTS

Yoshinobu Teraoka*

The purpose of this paper is to show the "optimal" insurance contracts from the viewpoint of each of the two participants, i.e., buyer and seller, under different assumptions discussed in the author's previous paper. It is shown that optimal insurance contracts are "minimum truncated" for the seller and "boundedly proportional" for the buyer. These results suggest that the latter contract is the common one for two participants.

1. Introduction

Insurance contact can be formulated as a two-person quasi-game which consists of the buyer and the seller of the insurance. Arrow [1] and Teraoka [3] discussed the problem of choosing the "optimal" insurance contract from the viewpoint of the buyer, and Miller [2] and Teraoka [3] presented it for the seller, under some plausible conditions. In this paper, we shall derive two optimal insurance contracts from the viewpoint of each of the two participants, buyer and seller, under the different plausible conditions from the previous works. It will be shown that the optimal insurance contracts are "minimum truncated" for the seller and "boundedly proportional" for the buyer, and the latter contract is the common one for the two under disadvantage conditions for each other.

2. Assumptions

When a buyer faces a positive-valued monetary risk with a distribution function $F(\cdot)$, he buys a contract $T(\cdot)$ from the seller in such a way that if the loss $x$ is incurred by the buyer, the seller will pay the buyer an amount $T(x)$, $0 \leq T(x) \leq \min(x, K)$, where $K$ is a preassigned positive constant. Let $\pi$ be the premium which is usually equal to $\pi = \int_0^\infty T(x) dF(x)$. We also assume that the buyer and the seller have utility-of-money functions $u(\cdot)$ and $v(\cdot)$ respectively, and both of $u(\cdot)$ and $v(\cdot)$ are monotone increasing, twice differentiable, and concave, that is, $u'(x) \geq 0$, $u''(x) \leq 0$, $v'(x) \geq 0$, and $v''(x) \leq 0$ for all $x$. Then the expected utilities for each are

$$\int_0^\infty u[\pi - x + T(x)] dF(x) \quad \text{and} \quad \int_0^\infty v[\pi - T(x)] dF(x)$$

by making the contract.

Let $S_F(z) = \int_z^\infty (x-z) dF(x) = \int_z^\infty (1-F(x)) dx$, where the expected value

$$E(X) = \int_0^\infty xdF(x)$$

is assumed to exist. For any distribution function $F$ with finite mean $E(X)$, $S_F(z)$

Received Sept. 9, 1975, Revised Sept. 30, 1976

* Department of Electrical Engineering, Himeji Institute of Technology
is non-negative, convex, and strictly decreasing on the set where it is positive. Furthermore $S_F(z) \geq E(X) - z$, $(0 \leq z < \infty)$, $S_F(0) = E(X)$, and $\lim_{z \to \infty} S_F(z) = 0$. We denote the inverse function of $S_F(z)$ by $S_F^{-1}(-c)$ for $0 < c \leq E(X)$.

Now we shall define three sets of all insurance agreements $T(\cdot)$ for fixed positive number $\pi$ as follows:

(i) $\tau_K$ is the set of all insurance agreements $T(\cdot)$ such that $\int_0^x T(x)dF(x) = \pi$ and $0 \leq T(x) \leq \min(x, K)$, where $K = S_F^{-1}(E(X) - \pi)$.

(ii) $\tau_{K_0}$ is the set of all insurance agreements $T(\cdot)$ such that $\int_0^x T(x)dF(x) = \pi$, $0 \leq T(x) \leq \min(x, K)$, where $K = S_F^{-1}(E(X) - \pi)$, and $T(x)/x$ is a non-decreasing function of $x$ if $T(x) < K$.

(iii) $\tau_{K'}$ is the set of all insurance agreements $T(\cdot)$ such that $\int_0^x T(x)dF(x) = \pi$, $0 \leq T(x) \leq \min(x, K)$, where $K = S_F^{-1}(E(X) - \pi)$, and $T(x)/x$ is a non-increasing function of $x$ if $T(x) < K$.

We note that the requirement that $T(x)/x$ be nondecreasing is a disadvantage one to impose on the seller, but very natural, and the requirement that $T(x)/x$ be nonincreasing is a disadvantage one to impose on the buyer, but very popular.

### 3. Preliminary Results

We begin by starting two theorems due to Teraoka [3] which generalize the works of Arrow [1] and Miller [2].

**Result 1.** For any utility function $u(\cdot)$

\[
\max_{T \in \tau_K} \int_0^\infty u[-\pi - x + T(x)]dF(x) = \int_0^\infty u[-\pi - x + T_\pi(x)]dF(x)
\]

where

\[
T_\pi(x) = \begin{cases} 0 & \text{if } 0 \leq x < a_K \\ \min(x - a_K, K) & \text{if } x \geq a_K, \end{cases}
\]

and $a_K$ is a unique root of the equation

\[
S_F(a_K) - S_F(a_K + K) = \pi,
\]

and satisfies

\[
0 \leq a_K \leq S_F^{-1}(\pi).
\]

**Result 2.** For any utility function $v(\cdot)$

\[
\max_{T \in \tau_{K_0}} \int_0^\infty v[\pi - T(x)]dF(x) = \int_0^\infty v[\pi - T_\pi(x)]dF(x),
\]

where

\[
T_\pi(x) = \begin{cases} q_K & \text{if } 0 \leq x < K/q_K \\ K & \text{if } x \geq K/q_K, \end{cases}
\]

and $q_K$ is a unique root of the equation

\[
S_F(K/q_K) = E(X) - (\pi/q_K)\]
and satisfies
\begin{equation}
\pi/E(X) \leq q_\kappa \leq \min (1, K/E(X)).
\end{equation}

We shall note that Result 1 says that the buyer wants to maximize his own expected utility of net loss under generous conditions, he should buy a boundedly stop-loss contract, and Result 2 says that the seller wishes to maximize his own expected utility of net gain under disadvantage conditions, he should sell a boundedly proportional contract. Finally note that \( a_\kappa \) and \( q_\kappa \) depend on preassigned positive constant \( K \).

4. The Main Results

Theorem 1 shows an optimal bounded insurance contract from the viewpoint of the seller under generous conditions, and Theorem 2 suggests it for the buyer under disadvantage conditions. It is found that the optimal contract for the seller is “minimum truncated” and one for the buyer is “boundedly proportional” which is the very same contract as one for the seller under disadvantage conditions given by Result 2.

**Theorem 1.** For any utility function \( v(\cdot) \)
\begin{equation}
\max_{T \in \tau_\kappa} \int_0^\infty v[\pi - T(x)]dF(x) = \int_0^\infty v[\pi - T_\kappa(x)]dF(x),
\end{equation}
where
\begin{equation}
T_\kappa(x) = \begin{cases} x & \text{if } 0 \leq x < b \\ b & \text{if } x \geq b \end{cases}
\end{equation}
and \( b \) is a unique root of the equation \( S_\kappa(b) = E(X) - \pi \), that is,
\begin{equation}
b = S_\kappa'(E(X) - \pi).
\end{equation}
Proof. Since \( v''(\cdot) \leq 0 \), we have
\begin{equation}
v[\pi - T(x)] - v[\pi - T_\kappa(x)] \leq \{ T_\kappa(x) - T(x) \} v'[\pi - T_\kappa(x)].
\end{equation}
It is found that
\begin{equation}
v'[\pi - T_\kappa(x)] = v'(\pi - x) \leq v'(\pi - b) \quad \text{for } 0 \leq x < b,
\end{equation}
and
\begin{equation}
T_\kappa(x) - T(x) = x - T(x) \geq 0 \quad \text{for } 0 \leq x < b.
\end{equation}
By considering (12), (13), and (14), it follows that for any \( T(\cdot) \in \tau_\kappa \)
\begin{align*}
\int_0^\infty v[\pi - T(x)]dF(x) - \int_0^\infty v[\pi - T_\kappa(x)]dF(x) & \leq \int_0^b \{ T_\kappa(x) - T(x) \} v'(\pi - b)dF(x) + \int_b^\infty \{ T_\kappa(x) - T(x) \} v'(\pi - b)dF(x) \\
& = v'(\pi - b) \int_0^\infty \{ T_\kappa(x) - T(x) \}dF(x) = 0
\end{align*}
yields the equation (9).

Next we shall prove the equation (11). Since

$$\pi = \int_0^\infty T_k(x) dF(x) = \int_0^b x dF(x) + \int_0^\infty b dF(x) = E(X) - S(b),$$

the root $b$ exists uniquely, giving (11).

This completes the proof of theorem 1.

Note that $b$ is the minimum constant that $K$ can be taken, that is, $b$ is independent of $K$.

**THEOREM 2.** For any utility function $u(\cdot)$

$$\max_{r \in \tau_K} \int_0^\infty u[-\pi - x + T(x)] dF(x) = \int_0^\infty u[-\pi - x + T_K(x)] dF(x)$$

where $T_K(x)$ is the very contract given by (6), (7), and (8) in Result 2.

**PROOF.** Since $u''(\cdot) \leq 0$, we clearly have

$$u[-\pi - x + T(x)] - u[-\pi - x + T_K(x)] \leq (T(x) - T_K(x)) u'[-\pi - x + T_K(x)].$$

The definition of $T_K$ gives

$$T(x) - T_K(x) = x[T(x)/x - q_K] \quad \text{for} \quad 0 < x \leq K/q_K$$
$$= T(x) - K \leq 0 \quad \text{for} \quad x > K/q_K.$$

We first consider the case where $F(K/q_K) = P_r(X \leq K/q_K) = 0$. As

$$\int_0^\infty T(x) dF(x) = \int_{K/q_K}^\infty T(x) dF(x) = \pi, \quad T(x) \leq K,$$

and

$$\int_0^\infty T_K(x) dF(x) = \int_{K/q_K}^\infty T_K(x) dF(x) = \int_{K/q_K}^\infty K dF(x) = \pi,$$

we obtain

$$T(x) = K, \quad \text{almost everywhere}.$$

Hence we conclude that

$$\int_0^\infty \{T_K(x) - T(x)\} u'[-\pi - x + T_K(x)] dF(x) = \int_{K/q_K}^\infty (K - T(x)) u'[-\pi - x + K] dF(x) = 0,$$

which gives (16) in conjunction with (17).

We shall therefore prove the case where $F(K/q_K) = P_r(X \leq K/q_K) > 0$. If we suppose

$$T(x)/x - q_K > 0 \quad \text{for all} \quad x \in (0, K/q_K]$$

then $T(K/q_K) > K$, contradicting to $T(x) \in \tau_K$.

We also assume that

$$T(x)/x - q_K \leq 0 \quad \text{for all} \quad x \in (0, K/q_K]$$
$$< 0 \quad \text{for some} \quad x \in (0, K/q_K].$$
then, since $T(x) \leq K$,

$$\int_0^\infty \{T(x) - T^*_K(x)\}dF(x) < 0$$

which contradicts to $T(x) \in \tau^*_K$. This implies that there exists $r \in (0, K/q_K]$ such that

$$T^*_K(x) = q_K \begin{cases} \frac{x}{q_K} & \text{if } 0 < x \leq r \\ \frac{K}{q_K} & \text{if } r \leq x \leq K \\ \end{cases} T(x),$$

as $T(x)/x$ is a non-increasing function of $x$. And the condition $0 < q_K \leq 1$ leads us to

$$u'[\pi - x + T^*_K(x)] = u'[\pi - (1 - q_K)x] \geq u'[\pi - (1 - q_K)r] \text{ if } 0 \leq x \leq r$$

$$= u'[\pi - (1 - q_K)x] \geq u'[\pi - (1 - q_K)r] \text{ if } r \leq x \leq K/q_K$$

$$= u'[\pi - x + K] \geq u'[\pi - (1 - q_K)r] \text{ if } x \geq K/q_K.$$
Table 1. The data of automobile physical damage insurance for private passenger automobile (small-size) from April 1974 to March 1975 in Japan.

<table>
<thead>
<tr>
<th>Monetary Risk</th>
<th>Percentage</th>
<th>$F(x)$</th>
<th>$\int_x^\infty x dF(x)$</th>
<th>$\int_x^\infty zdF(x)$</th>
<th>$S_F(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Just 0 Yen</td>
<td>78.40</td>
<td>0.7840</td>
<td>16,858 Yen</td>
<td>0 Yen</td>
<td>16,858 Yen</td>
</tr>
<tr>
<td>Up to 20,000 Yen</td>
<td>4.01</td>
<td>0.8241</td>
<td>16,858 Yen</td>
<td>0 Yen</td>
<td>16,858 Yen</td>
</tr>
<tr>
<td>40,000</td>
<td>5.71</td>
<td>0.8812</td>
<td>16,457</td>
<td>3,518</td>
<td>12,939</td>
</tr>
<tr>
<td>60,000</td>
<td>3.86</td>
<td>0.9198</td>
<td>14,744</td>
<td>4,752</td>
<td>9,992</td>
</tr>
<tr>
<td>80,000</td>
<td>2.39</td>
<td>0.9437</td>
<td>12,814</td>
<td>4,812</td>
<td>8,002</td>
</tr>
<tr>
<td>100,000</td>
<td>1.49</td>
<td>0.9586</td>
<td>11,141</td>
<td>4,504</td>
<td>6,637</td>
</tr>
<tr>
<td>200,000</td>
<td>2.62</td>
<td>0.9848</td>
<td>9,800</td>
<td>4,140</td>
<td>5,660</td>
</tr>
<tr>
<td>300,000</td>
<td>0.78</td>
<td>0.9926</td>
<td>5,870</td>
<td>3,040</td>
<td>2,830</td>
</tr>
<tr>
<td>400,000</td>
<td>0.33</td>
<td>0.9959</td>
<td>3,920</td>
<td>2,220</td>
<td>1,700</td>
</tr>
<tr>
<td>500,000</td>
<td>0.17</td>
<td>0.9976</td>
<td>2,765</td>
<td>1,640</td>
<td>1,125</td>
</tr>
<tr>
<td>1,000,000</td>
<td>0.20</td>
<td>0.9996</td>
<td>2,000</td>
<td>1,200</td>
<td>800</td>
</tr>
<tr>
<td>1,500,000</td>
<td>0.04</td>
<td>1.0000</td>
<td>500</td>
<td>400</td>
<td>100</td>
</tr>
</tbody>
</table>

"boundedly stop-loss" with stop-loss point 39,000 Yen. On the other hand, the optimal contract under generous conditions on the seller in the above case is "minimum truncated" with truncated point 75,000 Yen. Furthermore the com-
Fig. 1. The graph of $S_F(z)$ in Table 1.

Table 2. The numerical examples of $a_K$, $q_K$, and $b$ for the data in Table 1.

<table>
<thead>
<tr>
<th>$K$ (Yen)</th>
<th>$\pi$ (Yen)</th>
<th>$a_K$ (Yen)</th>
<th>$q_K$ (rate)</th>
<th>$b$ (Yen)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500,000</td>
<td>500,000</td>
<td>1,000,000</td>
<td>1,500,000</td>
<td></td>
</tr>
<tr>
<td>10,000</td>
<td>10,000</td>
<td>10,000</td>
<td>10,000</td>
<td>9,000</td>
</tr>
<tr>
<td>15,000</td>
<td>15,000</td>
<td>15,000</td>
<td>15,000</td>
<td>15,000</td>
</tr>
<tr>
<td>16,858</td>
<td>16,858</td>
<td>16,858</td>
<td>16,858</td>
<td>16,858</td>
</tr>
</tbody>
</table>

For all $K$ ($b \leq K$)

Fig. 2. The patterns of $T_K^*(x)$, $T_K^*(x)$, and $T^*(x)$ for $\pi=10,000$ and $K=1,500,000$ in Table 2.
promized contract for the two, i.e., optimal contract under disadvantage conditions for each other, is "boundedly proportional" with rate 59%. Fig. 2 shows the patterns of $T^*(x)$, $T^*_k(x)$, and $T^*_w(x)$ for the case of $x=10,000$ (Yen) and $K=1,500,000$ (Yen). Finally we note that Teraoka [3] is applicable to the case where the monetary risk faced by the buyer has an exponential distribution in this paper.

Acknowledgement

The author is very grateful to Professor M. Sakaguchi, Osaka University and Professor K. Sugahara, Himeji Institute of Technology for their encouragement. Furthermore he heartily thanks to the benefactors who helped him to obtain the data of insurance.

References