THE MAXIMUM LIKELIHOOD ESTIMATES BASED ON THE INCOMPLETE QUANTAL RESPONSE DATA

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In this paper we present statistical techniques appropriate to one exceedingly important type of response, that known as all-or-nothing or quantal.

With incomplete quantal response data, the estimation of parameters has usually been solved approximately as the estimation problems based on grouped data, which is not always practicable or adequate [1]. We therefore propose a new method of maximum likelihood for estimating parameters based on 'individual incomplete quantal response data' without grouping. The existence theorems of its maximum likelihood estimates (its optimal solutions) are proved in the appendix.

1. Introduction

If the characteristic response is quantal, occurrence or nonoccurrence will depend upon the intensity of the stimulus. For any one subject under controlled conditions, there will be a certain level of intensity below which the response does not occur and above which the response occurs; such a value is called a tolerance value. This tolerance value will vary from one subject to another of the population under study. If the concept of repetition of tests on an individual is meaningful, tolerance is likely also to vary from one occasion to another as a result of uncontrolled internal and external conditions. Therefore, discussion of quantal response data requires recognition of the frequency distribution of tolerances over the population studied.

Let us consider the concrete example as follows. Suppose that we have held successive group milk teeth examinations (s times) to babies from p months old through q months old in some district. Then a set of babies who underwent all examinations cannot be a random sample. Some babies might take only the first examination and others might be examined twice and so on. We will set up a sample by all babies who underwent the examination at least one time. Such a sample will be most similar to a random sample from the population studied. Thus the information of individuals in the sample is not necessarily uniform, and we are obliged to deal with various types of quantal response data.

From the aspect of eruption age data of the specified deciduous tooth, the sample may be divided into the following three classes:
(a) babies who were examined at least twice and during the period of examination the eruption occurred,
(b) babies whose deciduous tooth studied has already erupted at his first examination,

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Babies whose deciduous tooth studied has not erupted at his last examination.

When we have finished the group examination which consists of $s$ times examinations, we may obtain the record of values $x_1^{(r)}, x_2^{(r)}, \ldots, x_r^{(r)}$ observed on the days of examinations of the $k$-th individual. Here $r$ is the number of times which the $k$-th individual underwent examinations actually. Among his tolerance value $X_k^*$ and his present values $x_j^{(r)}$ ($j=1, 2, \ldots, r$), one of the relations

$$(a') \ x_j^{(r)} < X_k^* < x_i^{(r)}, \quad (b') \ X_k^* < x_i^{(r)} \quad \text{or} \quad (c') \ x_i^{(r)} < X_k^*$$

holds according as he belongs to the class (a), (b) or (c) mentioned above. ($x$ is considered as a continuous variate, and the probability of equality is zero in the relations $(a'), (b')$ and $(c')$).

Practically we add the lower limit ($-\infty$) and upper limit ($\infty$) of the distribution of $X^*$ to the present values, and the statements $(b')$ and $(c')$ become the same type as $(a')$.

2. Formulation

We direct our attention to the variable $X$ of individuals; for example, age, dose, or concentration of a toxin. Let $X_k^*$ be the value of $X$ of the $k$-th individual at that time when he manifests the specified quantal response, and let this tolerance value $X_k^*$ be unobservable. We assume that the value of $X$ is continuous and the tolerance value $X_k^*$ is a realized value of the random variate $X^*$. We say that $\alpha$ and $\beta$ are a lower limit and an upper limit of the distribution of $X^*$ respectively if the probabilities of $\{X^*<\alpha\}$ and $\{\beta<X^*\}$ are zero. Note that $-\infty \leq \alpha < \beta \leq \infty$. We consider a random sample of size $n$. Let $X_k^*$ be the tolerance value of the $k$-th individual and let $x_1^{(r)}, x_2^{(r)}, \ldots, x_r^{(r)}$ be the observed values of $X$ of the $k$-th individual, where $t_k$ is the number of times which he was observed. For simplicity, let $x_1^{(r)} < x_2^{(r)} < \ldots < x_r^{(r)}$. Then we assume that the tolerance value $X_k^*$ is unobservable but it is known that $X_k^* \in (\alpha, x_1^{(r)})$ or $X_k^* \in (x_r^{(r)}, \beta)$ or $X_k^* \in (x_i^{(r)}, x_{i+1}^{(r)})$ for some $1 \leq i \leq t_k-1$. The interval, just stated above, including $X_k^*$ will be called the tolerance interval of the $k$-th individual. We rearrange end points of $n$ tolerance intervals in ascending order of magnitude;

$$(2.1) \quad x_{(1)} < x_{(2)} < \cdots < x_{(m)} < x_{(m+1)} ,$$

where $x_{(1)} = \alpha$, $x_{(m+1)} = \beta$. For each pair $\{x_{(j)}, x_{(i)}\}$ ($0 \leq j < i \leq m+1$), we define $c_{ij}$ by

$$(2.2) \quad c_{ij} = \text{number of } X_k^* \text{ having } (x_{(j)}, x_{(i)}) \text{ as its tolerance interval} .$$

Then $\{c_{ij}\}_{0 \leq j < i \leq m+1}$ satisfies the following relations;

$$(2.3) \quad \begin{aligned} c_{ij} = 0 \text{ or } 1 \quad \text{and} \quad c_{m+1,j} = 0 , \\
\sum_{i = k+1}^{m+1} c_{ik} + \sum_{j = 0}^{k-1} c_{kj} = 1 , \\
\sum_{i = 1}^{m+1} \sum_{j = 0}^{i-1} c_{ij} = n . \end{aligned}$$

As the value of $X$ is continuous, all end points of $n$ tolerance intervals are theoretically different. Then $m \leq 2n$, and $c_{ij} = 0$ or 1 with probability 1.

In practice, observed values of $X^{(r)}$ are not always different even if $X$ is con-
tinuous. And then \( m \leq 2n \), \( c_{ij} = 0 \) or positive integer and \( \sum_{i=k+1}^{m+1} c_{ik} + \sum_{j=0}^{k-1} c_{j} = \text{positive integer} \). But these modifications do not disturb our following discussion.

Assume that \( X^{*} \) is a random variate with continuous distribution function \( F(x, \theta) \). Then the likelihood function \( L(\theta) \) assuming that \( (X^{*}, X^{*}, \cdots, X^{*}) \) is a random sample is of the form;

\[
(2.4) \quad L(\theta) = \prod_{i=1}^{m+1} \prod_{j=0}^{k-1} \left( F(x_{ij}, \theta) - F(x_{ij}, \theta) \right)^{c_{ij}}.
\]

Put \( M(\theta) = \ln L(\theta) \) and we obtain

\[
(2.5) \quad M(\theta) = \sum_{i=1}^{m+1} \sum_{j=0}^{k-1} c_{ij} \ln \left( F(x_{ij}, \theta) - F(x_{ij}, \theta) \right).
\]

Now we will prove the existence of parameter estimates which maximize \( M(\theta) \) over a parameter space.

3. The existence theorem of the maximum likelihood estimates

We will consider the problem of estimating parameter \( \theta \) for the normal distribution function \( F(x, \theta) = \Phi(x, \theta) \). In the appendix, the problem will be expressed in more general form and the existence theorem of its optimal solution will be proved. In the appendix, \( \{c_{ij}\}_{0 \leq j < i \leq m+1} \) is a set of nonnegative real numbers that are not all zero. When \( \{c_{ij}\}_{0 \leq j < i \leq m+1} \) is defined by (2.2), this \( \{c_{ij}\} \) satisfies the conditions, just mentioned above, and \( M(\theta) \) in the appendix coincides with the log-likelihood (2.5). Therefore we can easily derive the existence theorems of the maximum likelihood estimates from Theorem 1 and Theorem 2 in the appendix.

In the following theorems, the conditions \( (C_{0}) \) and \( (C_{k}) (1 \leq k \leq m) \) are as follows;

\begin{itemize}
    \item \( (C_{0}) \) \( c_{ij} \neq 0 \) for some pair \( (i, j) \) such that \( 1 \leq j < i \leq m \).
    \item \( (C_{k}) \) \( c_{ij} \neq 0 \) for some pair \( (i, j) \) such that \( 0 \leq j < i < k \) or \( k < j < i \leq m+1 \).
\end{itemize}

**Theorem A.** Assume that \( \{c_{ij}\} \) fulfills the condition \( (C_{0}) \). Then the maximum likelihood estimates exist if and only if \( \{c_{ij}\} \) satisfies all of the conditions \( (C_{1}), \cdots, (C_{m}) \).

**Theorem B.** Assume that \( \{c_{ij}\} \) does not satisfy the condition \( (C_{0}) \), and let \( \{c_{ij}\} \) satisfy all of the conditions \( (C_{1}), \cdots, (C_{m}) \). If \( a = \sum_{i=1}^{m} c_{i0} \neq 0 \), \( b = \sum_{j=1}^{m} c_{m+1,i} \neq 0 \) and the inequality

\[
\left( \sum_{i=1}^{m} x_{ij}(c_{i0} + c_{m+1,i}) \right) / (a+b) < \left( \sum_{i=1}^{m} x_{ij}c_{i0} \right) / a
\]

holds, the maximum likelihood estimates exist.

**Example.** We are interested in estimating menarche age distribution in adolescent girls or the eruption age distribution of deciduous teeth. It is very difficult to obtain reliable records of the age of menarche or the eruption age of specified deciduous tooth. Our available data are a set of the result of each observation which is not a sample value of the random variable being tested, but two numbers, together with the information that the sample value is between those num-
The analysis of these data and applications of Theorem B are shown in [2] and [3].

Appendix

Let $\Theta$ denote the upper-half plane of the 2-dimensional Euclidean space $\mathbb{R}^2$ and let $x_1, \cdots, x_m$ be real numbers such that $x_0 < x_1 < \cdots < x_m < x_{m+1}$, where $x_0 = -\infty$ and $x_{m+1} = \infty$. Denote by $\Phi$ and $\phi$ the distribution function and the density function of the standard normal distribution respectively. For $\theta = (\mu, \sigma) \in \Theta$ and $x \in \mathbb{R}$ (the real number system), we define functions $t(x, \theta)$, $\phi_i(\theta)$ $(i = 0, \cdots, m+1)$, $\Phi_i(\theta)$ $(i = 0, \cdots, m+1)$, $\Phi_{i,j}(\theta)$ $(0 \leq j < i \leq m+1)$ and $M(\theta)$ by

\[
t(x, \theta) = (x - \mu)\sigma \quad \text{if} \quad x \in \mathbb{R} \quad \text{and} \quad t(\pm \infty, \theta) = \pm \infty,
\]

\[
\phi_i(\theta) = \phi(t(x_i, \theta)), \quad \Phi_i(\theta) = \Phi(t(x_i, \theta)), \quad \Phi_{i,j}(\theta) = \ln \left[ \Phi_i(\theta) - \Phi_j(\theta) \right],
\]

and

\[
M(\theta) = \sum_{i=0}^{m+1} \sum_{j=0}^{i-1} c_{ij} \phi_j(\theta),
\]

where $\left\{ c_{ij} \right\}_{0 \leq j < i \leq m+1}$ is a set of nonnegative real numbers. Notice that a value $M(\theta)$ does not depend on $c_{m+1,0}$, since $\Phi_{m+1,0}(\theta) = 0$ on $\Theta$. Then $c_{m+1,0}$ may be an arbitrary number but we define $c_{m+1,0} = 0$ so as to coincide with (2.2). For a nonempty subset $S$ of $\mathbb{R}^m$, we define a subset $\partial S$ of $\mathbb{R}^m$ by

\[
\partial S = \overline{S} \cap S^c
\]

where $\overline{S}$ and $S^c$ are the closure and the complement of $S$ respectively.

We consider the following extremum problem:

(1) Choose $\theta \in \Theta$ so that $\theta$ maximizes $M(\theta)$.

We shall find out whether the problem (1) has an optimal solution or not and give a sufficient condition for the existence of an optimal solution of the problem (1). Notice that even if $M(\theta)$ is concave, the problem (1) has no optimal solution in general.

G. Kulldorff has dealt with the problem similar to the problem (1) in Kulldorff [4] and [5]. But he has solved the problem when the data are grouped, one of the variables $\mu$ and $\sigma$ is known and the tolerance intervals are mutually disjoint.

4. Existence of an optimal solution

Let $F(\theta)$ denote the $m$-dimensional row vector with components $\Phi_1(\theta), \cdots, \Phi_m(\theta)$. We see that the image $F(\theta)$ of $\Theta$ under $F$ is a bounded subset of $\mathbb{R}^m$, since $0 < \Phi(\theta) < 1$ on $\Theta$. In order to determine the boundary value of $M(\theta)$, we need the following proposition which was proved in Nakamura and Kariya [6]:

**Proposition 1.** Let $L_i = \{z = (z_1, \cdots, z_m) \in \mathbb{R}^m; z_j = 0 (j < i), 0 \leq z_i \leq 1 \text{ and } z_j = 1 (i \leq j)\}$ for $i = 1, \cdots, m$ and let $L_0 = \{z = (z_1, \cdots, z_m) \in \mathbb{R}^m; z_1 = \cdots = z_m \text{ and } 0 \leq z_i \leq 1\}$. Then $\partial F(\theta)$
Remark. If \( m \geq 3 \), then \( F(\Theta) \) is not convex set and it has no interior point.

For each \( k \) \((0 \leq k \leq m)\), denote by \( L_k \) the set excluding its terminal points from \( L \). Then we have

**Lemma 1.** For any sequence \( \{\theta_n\} \) in \( \Theta \) with \( F(\theta_n) \rightarrow z \in L_k \) \((1 \leq k \leq m)\) as \( n \rightarrow \infty \), \( \{M(\theta_n)\} \) is an unbounded sequence if and only if the following condition \((C_k)\) is fulfilled:

\[
(C_k) \quad c_{ij} \neq 0 \quad \text{for some pair} \quad (i, j) \quad \text{such that} \quad 0 \leq j < i < k \quad \text{or} \quad k < j < i \leq m + 1.
\]

**Proof.** Let \( \{\theta_n\} \) be a sequence in \( \Theta \) such that \( F(\theta_n) \rightarrow z = (z_1, \cdots, z_m) \in L_k \) as \( n \rightarrow \infty \). Then we have \( \lim_{n \to \infty} \Phi_{ij}(\theta_n) = -\infty \) if \( 0 \leq j < i < k \) or \( k < j < i \leq m \), \( \lim_{n \to \infty} \Phi_{ik}(\theta_n) = \ln z_k \) if \( j < k \) and \( \lim_{n \to \infty} \Phi_{ik}(\theta_n) = \ln(1-z_k) \) if \( k < i \), which prove our statement.

By the same reasoning, we have

**Lemma 2.** For any sequence \( \{\theta_n\} \) in \( \Theta \) with \( F(\theta_n) \rightarrow z \in L_0 \) as \( n \rightarrow \infty \), \( \{M(\theta_n)\} \) is an unbounded sequence if and only if the following condition \((C_0)\) is fulfilled:

\[
(C_0) \quad c_{ij} \neq 0 \quad \text{for some pair} \quad (i, j) \quad \text{such that} \quad 1 \leq j < i \leq m.
\]

For each \( \lambda \in (0, 1) \) we define

\[
\tilde{M}_k(\lambda) \equiv \left( \sum_{j=0}^{k-1} c_{kj} \right) \ln \lambda + \left( \sum_{i=k+1}^{m+1} c_{ik} \right) \ln(1-\lambda) \quad (k = 1, \cdots, m)
\]

and

\[
\tilde{M}_0(\lambda) \equiv \left( \sum_{i=1}^{m} c_{i0} \right) \ln \lambda + \left( \sum_{j=1}^{m+1} c_{0j} \right) \ln(1-\lambda).
\]

Now we compute the boundary value of \( M(\theta) \) in a sense of the following:

**Lemma 3.** Assume that \( \{c_{ij}\} \) does not satisfy a condition \((C_k)\), and let \( \{\theta_n\} \) be a sequence in \( \Theta \) with \( F(\theta_n) \rightarrow z = (z_1, \cdots, z_m) \in L_k \) as \( n \rightarrow \infty \). Then

(i) \( M(\theta_n) \rightarrow \tilde{M}_k(z_k) \) as \( n \rightarrow \infty \) when \( k \geq 1 \).

(ii) \( M(\theta_n) \rightarrow \tilde{M}_m(z_1) \) as \( n \rightarrow \infty \) when \( k = 0 \).

**Proof.** It follows from our assumption that \( \{M(\theta_n)\} \) is a convergent sequence. Assume that \( k \geq 1 \); then we have

\[
\lim_{n \to \infty} M(\theta_n) = \sum_{j=0}^{k-1} c_{kj} \lim_{n \to \infty} \Phi_{kj}(\theta_n) + \sum_{i=k+1}^{m+1} c_{ik} \lim_{n \to \infty} \Phi_{ik}(\theta_n)
\]

\[
= \left( \sum_{j=0}^{k-1} c_{kj} \right) \ln z_k + \left( \sum_{i=k+1}^{m+1} c_{ik} \right) \ln(1-z_k) = \tilde{M}_k(z_k).
\]

By the similar argument, (ii) follows.

Let us define a function \( f(\lambda) \) defined on \((0, 1)\) by

\[
f(\lambda) \equiv a \ln \lambda + b \ln(1-\lambda),
\]

where \( a \) and \( b \) are positive real numbers.

**Proposition 2.** There exists an unique value \( \bar{\lambda} \) of \( \lambda \) maximizing \( f(\lambda) \) over \((0, 1)\).
Moreover $\lambda = a/(a+b)$.

Let $\lambda \in (0, 1)$ and let $\Phi^{-1}$ be the inverse function of the standard normal distribution function. For each real number $x$, denote by $D_x(x)$ the half-line in $\Theta$: $\{\theta = (\mu, \sigma) \in \Theta; x - \mu = \sigma \Phi^{-1}(\lambda)\}$. For $\theta = (\mu, \sigma) \in D_x(x)$, denote by $W_{x, \sigma}(\theta)$ the directional derivative of $M(\theta)\sigma^2$ at $\theta$ along the half-line $D_x(x)$, i.e.,

$$W_{x, \sigma}(\theta) = \lim_{\sigma' \to \sigma, \theta' = (\mu', \sigma') \in D_x(x)} (M(\theta') - M(\theta))/\sigma'. $$

We can easily compute

$$\sigma^2 W_{x, \sigma}(\theta) = \sum_{i=1}^n \sum_{j=1}^m c_{ij}[(x-x_i)\phi_i(\theta) - (x-x_j)\phi_j(\theta)][\Phi_i(\theta) - \Phi_j(\theta)]^{-1} + \sum_{i=1}^n c_{ii}(x-x_i)\phi_i(\theta)[\Phi_i(\theta)]^{-1} - \sum_{j=1}^m c_{mj+1,j} (x-x_j)\phi_j(\theta)[1 - \Phi_j(\theta)]^{-1}.$$  

The following proposition shows that problem (1) has no optimal solution in general:

**Proposition 3.** Let $\{c_{ij}\}$ does not satisfy at least one of the conditions (Ck) ($k=1, 2, \cdots, m$), and put $I = \{k; \{c_{ij}\}$ does not satisfy the condition (Ck) and $k \geq 1\}$. Then

$$M(\theta) < \min_{\lambda \in I} \sup \{\tilde{M}_k(\lambda); \lambda \in (0, 1)\}$$

for all $\theta \in \Theta$.

**Proof.** We set $\tilde{M}_k = \sup \{\tilde{M}_k(\lambda); \lambda \in (0, 1)\}$, $a_k = \sum_{j=0}^{k-1} c_{kj}$ and $b_k = \sum_{i=k+1}^{m+1} c_{ik}$. At first we show the inequality (3) in the case $I$ contains only one element $k$. If either $a_k$ or $b_k$ is zero, then $\tilde{M}_k = 0$, so that the inequality (3) is obvious. Hence we may assume that both $a_k$ and $b_k$ are not zero. By Proposition 2 we see that there exists $\tilde{\lambda} \in (0, 1)$ such that $\tilde{M}_k(\tilde{\lambda}) = \tilde{M}_k$. Let $\tilde{\theta}' \in \Theta$ and $\lambda = \Phi_k(\tilde{\theta}')$. It then follows from the condition (Ck) and (2) that

$$\sigma^2 W_{x, \sigma}(\theta) = \sum_{1 \leq i, j \leq m} c_{ij}[(x-x_i)\phi_i(\theta) - (x-x_j)\phi_j(\theta)][\Phi_i(\theta) - \Phi_j(\theta)]^{-1} + \sum_{i=k}^m c_{ii}(x-x_i)\phi_i(\theta)[\Phi_i(\theta)]^{-1} - \sum_{j=1}^k c_{mj+1,j} (x-x_j)\phi_j(\theta)[1 - \Phi_j(\theta)]^{-1},$$

which implies that $W_{x, \sigma}(\theta) < 0$ since $a_k \neq 0$ and $b_k \neq 0$, and hence $W_{x, \sigma}(\theta) < 0$ for all $\theta \in D_x(x)$. Since

$$\lim_{x_1, x_2 \in \Theta, \theta = (\mu, \sigma) \in D_x(x_1)} F(\theta) = (0, \cdots, 0, \lambda, 1, \cdots, 1) \in \tilde{D}_1,$$

it follows from Lemma 3 that $\lim_{x_1, x_2 \in \Theta, \theta = (\mu, \sigma) \in D_x(x_1)} M(\theta) = \tilde{M}_k(\lambda)$. With these results we conclude

$$M(\theta) < \tilde{M}_k(\lambda)$$

for all $\theta \in D_x(x)$. Therefore $M(\theta') < \tilde{M}_k$, which derives the inequality (3). Next we shall establish the inequality (3) in the case $I$ consists of more than two elements. Let $k$ and $k'$ ($k < k'$) be elements of $I$. Then it immediately follows that $a_k = 0$ and $b_k = 0$, so that $\tilde{M}_k = \tilde{M}_k = 0$. Hence the inequality (3) is clear. This completes the proof.

For $\lambda \in (0, 1)$ and $x \in R$, we set
and

\begin{equation}
G(\lambda, x) = g(\lambda) [A(x) - (A(x) + B(x))\lambda] \nonumber.
\end{equation}

We have

**Lemma 4.** Assume that \(\{c_{ij}\}\) does not satisfy the condition \((C_0)\). Then \(\sigma^*M_{i,s}(\theta) \rightarrow G(\lambda, x)\) as \(\sigma \rightarrow \infty\).

**Proof.** By the condition \((C_0)\) and the formula (2) we have

\[\sigma^*M_{i,s}(\theta) = \sum_{i=0}^{m} c_{ij}(x_i - x_{i,j})\phi_i(\theta)\Phi_i(\theta) - \sum_{j=1}^{m} c_{m+1,j}(x_i - x_{j})\phi_j(\theta)[1 - \Phi_j(\theta)]^{-1}.\]

Noting that \(t(x_i, \theta) \to \Phi^{-1}(\lambda)\) as \(\sigma \to \infty\) and \(\Phi_i(\theta) \to \lambda\) as \(\sigma \to \infty\), we have

\[\sigma^*M_{i,s}(\theta) \rightarrow G(\lambda, x)\] as \(\sigma \rightarrow \infty\)

by the above equality.

We are now in position to give some conditions under which problem (1) has an optimal solution.

**Theorem 1.** Assume that \(\{c_{ij}\}\) fulfills the condition \((C_0)\). Then problem (1) has an optimal solution if and only if \(\{c_{ij}\}\) satisfies all of the conditions \((C_1), \ldots, (C_m)\).

**Proof.** Suppose that \(\{c_{ij}\}\) does not satisfy the condition \((C_k)\) for some \(k \geq 1\). It follows from Lemma 3 and Proposition 3 that

\[\sup \{M(\theta); \theta \in \Theta\} = \sup \{\tilde{M}(\lambda); \lambda \in (0, 1)\}\]

and

\[M(\theta) < \sup \{\tilde{M}(\lambda); \lambda \in (0, 1)\}\],

which mean that problem (1) has no optimal solution. Hence the “only if” part is proved. Next we show the “if” part. Assume that \(\{c_{ij}\}\) fulfills all of the conditions \((C_1), \ldots, (C_m)\). Choose an arbitrary \(\theta_0 \in \Theta\) and fix it. We show that \(S = \{\theta \in \Theta; M(\theta) \leq M(\theta_0)\}\) is a compact subset of \(\Theta\). Let \(\theta_n\) be a sequence in \(S\) and suppose that \(\{\theta_n\}\) has no cluster points in \(S\). Then \(\{\theta_n\}\) has no cluster points in \(\Theta\). We can find a subsequence \(\{\theta_{n'}\}\) of \(\{\theta_n\}\) such that \(\lim_{n' \to \infty} F(\theta_{n'}) \in \partial F(\theta)\) (see [6; Theorem 1]). Since \(\{c_{ij}\}\) fulfills all of the conditions \((C_0), (C_1), \ldots, (C_m)\), it follows from Lemma 1 and Lemma 2 that \(M(\theta_n) \to -\infty\) as \(n' \to \infty\) if \(\lim_{n' \to \infty} F(\theta_{n'}) \in \bigcup_{k=0}^{m} L_k\). Assume that \(\lim_{n' \to \infty} F(\theta_{n'})\) is a terminal point of \(L_k\) \((0 \leq k \leq m)\). Then it can be easily seen that \(M(\theta_{n'}) \to -\infty\) as \(n' \to \infty\). Therefore we obtain \(M(\theta_k) \leq -\infty\). This is a contradiction. Thus \(S\) is compact. Since \(M(\theta)\) is continuous in \(\theta\), we can conclude that problem (1) has an optimal solution.

**Theorem 2.** Assume that \(\{c_{ij}\}\) does not satisfy the condition \((C_3)\), and let \(\{c_{ij}\}\) satisfy all of the conditions \((C_1), (C_2), \ldots, (C_m)\). If \(a = \sum_{i=1}^{m} x_i c_{1,i} \neq 0\), \(b = \sum_{j=1}^{m} c_{m+1,j} \neq 0\) and the inequality

\begin{equation}
\left(\sum_{i=1}^{m} x_i (c_{1,i} + c_{m+1,i})\right) \left(\sum_{i=1}^{m} x_i c_{1,i}\right) < 0
\end{equation}

\(a\)
holds, then problem (1) has an optimal solution.

**Proof.** Let \( x \in R \) and set \( \lambda^* = \frac{a}{a+b} \). Then we have from (5)

\[
G(\lambda^*, x) = xg(x)\left(x - \frac{(a+b)\lambda^*}{a}\right) + g(x)\left(\sum_{i=1}^{m} x_i (c_{i0} + c_{i+1,1})\right) \lambda^* - \sum_{i=1}^{m} x_i c_{i0}
\]

\[
= g(x)(a+b)^{-1}\left[ a \sum_{i=1}^{m} x_i (c_{i0} + c_{i+1,1}) -(a+b) \sum_{i=1}^{m} x_i c_{i0}\right] < 0.
\]

It follows by Lemma 4

\[
M^{*,2}(\theta) < 0
\]

for sufficiently large \( \sigma \) with \( \theta = (\mu, \sigma) \in D_{\nu}(x) \). This implies that there exists \( \theta_0 \in D_{\nu}(x) \) such as \( M_0(\lambda^*) < M(\theta_0) \). Let \( \{ \theta_n \} \) be a sequence in \( S \) and assume that this sequence has no cluster point in \( S \). By the same reasoning as in the proof of Theorem 1, there exists a subsequence \( \{ \theta_n \} \) of \( \{ \theta_n \} \) such that \( z = \lim_{n \to \infty} F(\theta_n) \in \partial F(\theta) \). If \( z \in \bigcup_{k=1}^{n} D_{\nu}(x) \), it can be easily seen that \( M(\theta_n) \to -\infty \) as \( n \to \infty \), which contradicts the fact: \( \{ \theta_n \} \subset S \). If \( z \in \bigcap_{k=1}^{n} D_{\nu}(x) \), then we have from Lemma 3 that \( M(\theta_n) \to M_0(z_1) \) as \( n \to \infty \), where \( z = (z_1, \cdots, z_m) \). This and \( M_0(\lambda^*) < M(\theta_0) \) derive the strict inequality: \( M(\theta_0) < M(\theta_0) \), which is a contradiction. Therefore we conclude that \( S \) is a compact set. The existence of optimal solution of problem (1) follows from the continuity of \( M(\theta) \). This completes the proof.

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**References**


