A CONDITION FOR THE EXISTENCE OF CERTAIN RESOLVABLE 2-DESIGNS*

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A resolvable 2-(v=2k, k, λ) design (i.e., a resolvable BIB design with parameters v=2k, b, r, k, λ) is characterized as follows: A necessary condition for the existence of a resolvable 2-(v=2k, k, λ) design is that r - λ = 1 is even. This necessary condition is also sufficient for the practical range of values of k. That design is also a 3-design (i.e., a doubly BIB design).

Statements

We consider a 2-(v=2k, k, λ) design with parameters b and r, in which vr = bk and r(k-1) = λ(v-1) clearly hold. In this case, we get r = 2λ + λ/(k-1) which yields that the parameters of a 2-(v=2k, k, λ) design are given as follows:

\[ v = 2k, \quad b = 2l(2k-1), \quad r = l(2k-1), \quad k, \quad λ = l(k-1) \]

for a positive integer l.

When this design is resolvable, we have

**Theorem 1.** A necessary condition for the existence of a resolvable 2-(v=2k, k, λ) design is that r - λ = 1 is an even number.

**Proof.** Let N be the incidence matrix of a resolvable 2-design with parameters v = 2k, b = 2l(2k-1), r = l(2k-1), k and λ = l(k-1). Then, from the resolvability of N, after renumbering blocks of the design we can put N as

\[
N = \begin{bmatrix}
E_{a \times r} & O_{a \times r} \\
N_{1} & N_{1}^t
\end{bmatrix},
\]

where \(E_{a \times t}\) (\(O_{a \times t}\)) is an s \times t matrix whose all elements are unity (zero) and \(N_{1}^t\) is the complement of \(N_{1}\) (i.e., \(N_{1}^t = E_{n \times t} - N_{1}\)). In this case, we now show that \(N_{1}\) is the incidence matrix of a 2-design with parameters \(v^* = 2k-1, b^* = l(2k-1), k^* = k-1, r^* = l(k-1)\) and \(λ^* = l(k-2)/2\). It is clear that \(v^* = v-1 = 2k-1, b^* = r = l(2k-1), k^* = k-1\) and \(r^* = λ = l(k-1)\). Let \(λ_{aβ}\) be the coincidence number of \(a\)-th treatment and \(β\)-th treatment in \(N_{1}\). Then, from a coincidence property of \(N\), we obtain \(λ_{aβ} + (b^* - 2r^* + λ_{aβ}) = λ^*\) and so \(λ_{aβ} = l(k-2)/2\) which is obviously constant for all \(α, (≠)β\). Hence, \(λ^* = l(k-2)/2 = l(k-2)/2 = l(\alpha - λ)/2 - l\) which implies that \(r - λ = 1\) is an even number.

**Remark.** (i) If \(l = 1\), then the design is necessarily affine resolvable. (ii) \(N_{1}\) is a 2-(v=2k+1, k, λ*) design. From the above proof and Theorem 1 of Kimberley

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[2], \( N \) is also a \( 3-(v=2k, k, \lambda^*) \) design.

**Corollary 1.** A resolvable \( 2-(v=2k, k, \lambda) \) design is a 3-design.

**Corollary 2.** The existence of a resolvable \( 2-(v=2k, k, \lambda) \) design implies the existence of a \( 2-(2k-1, k-1, \lambda') \) design with \( \lambda' = \lambda(k-2)/2(k-1) \).

It may be conjectured that the necessary condition of Theorem 1 is also sufficient. For example, it follows from Theorems 5.1 and 6.1 of Sprott [3] that when \( v-1 \) is a prime or a prime power, a resolvable \( 2-(v=2k, k, \lambda) \) design can be constructed for the following cases: (i) any \( l \) and \( k=2u_1 \) (even); (ii) \( l=2u_2 \) (even) and any \( k \). Furthermore, there exists an affine resolvable 2-design with parameters \( v = 2^{d+1}, k=2^d, b=2(2^{d+1}-1), r=2^{d+1}-1 \) and \( \lambda=2^d-1 \) for a positive integer \( d \), which also includes individual examples of a composite number \( v-1 \) (cf. Kageyama [1]). These two facts show that the necessary condition is sufficient for \( k \leq 10 \).

**Theorem 2.** A necessary and sufficient condition for the existence of a resolvable \( 2-(v=2k, k, \lambda) \) design with \( k \leq 10 \) is that \( r-\lambda \) is an even number.

Note that for many other values of \( k \geq 11 \) the necessary condition of Theorem 1 is sufficient.

In view of Theorem 1 the following resolvable designs with parameters as given below are non-existent.

1. \( v=6, \ k=3, \ b=10, \ r=5, \ \lambda=2 \).
2. \( v=6, \ k=3, \ b=30, \ r=15, \ \lambda=6 \).
3. \( v=10, \ k=5, \ b=18, \ r=9, \ \lambda=4 \).
4. \( v=10, \ k=5, \ b=54, \ r=27, \ \lambda=12 \).
5. \( v=14, \ k=7, \ b=26, \ r=13, \ \lambda=6 \).
6. \( v=14, \ k=7, \ b=78, \ r=39, \ \lambda=18 \).
7. \( v=18, \ k=9, \ b=34, \ r=17, \ \lambda=8 \).
8. \( v=18, \ k=9, \ b=102, \ r=51, \ \lambda=24 \).

Designs of (1), (3), (5), and (7) are examples when \( l=1 \). Hence, the non-existence of these designs can also be shown by use of the concept of affine resolvability.

**References**