NOTE ON DESIGNS IN SEROLOGY

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Some new series of neighbor designs, which are useful in serology, are presented in this paper. These are based on a principle of Rees [5].

1. Introduction

Rees [5] introduced the concept and name of a “neighbor” design which finds its application in serology like virus research. He stated that “A technique used in virus research requires the arrangement in circles of samples from a number of virus preparations in such a way that over the whole set a sample from each virus preparation appears next to a sample from every other virus preparation”. For example, in a circular plate (1, 2, 3, 5), a sample 1 occurs next to both samples 5 and 2; a sample 2 occurs next to both samples 1 and 3; a sample 3 occurs next to both samples 2 and 5; and a sample 5 occurs next to both samples 3 and 1. That is, on a circular plate, every sample has as neighbors two other (adjacent) samples.

Thus, a neighbor design with parameters $v$, $b$, $r$, $k$ and $\lambda'$ is an arrangement of $v$ different treatments (antigens) into $b$ blocks (circular plates around an antiseraum) such that (i) each block has $k$ treatments, not necessarily all distinct, (ii) each treatment occurs $r$ times in the design, not necessarily on $r$ different blocks, and (iii) every treatment is a neighbor of each other treatment exactly $\lambda'$ times. It is known that $r=\lambda'(v-1)/2$ and $b=\lambda'v(v-1)/2k$ hold. As an example, we present a design with parameters $v=8$, $b=r=7$, $k=8$ and $\lambda'=2$:

\[(0, 1, 2, 7, 3, 6, 4, 5), \ (0, 2, 3, 1, 4, 7, 5, 6), \ (0, 3, 4, 2, 5, 1, 6, 7), \ (0, 4, 5, 3, 6, 2, 7, 1), \ (0, 5, 6, 4, 7, 3, 1, 2), \ (0, 6, 7, 5, 1, 4, 2, 3), \ (0, 7, 1, 6, 2, 5, 3, 4)\]

Diagram of this design is:

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A problem of construction of neighbor designs has been variously discussed by Das and Saha [1], Hwang [2], Hwang and Lin [3], Lawless [4], and Rees [5]. In this note, we shall give some new series of neighbor designs. One of them is also a balanced incomplete block (BIB) design.

2. New series of neighbor designs

Rees [5] has suggested a principle for the construction of basic blocks which generate a neighbor design. This is as follows:

Let 0, 1, 2, ..., v-1 be the v elements (treatments) of G(v), an Abelian group of order v under addition (mod v). Let there be t basic blocks each containing k treatments (not necessarily all distinct), (a_{i1}, a_{i2}, ..., a_{ik}), i=1, 2, ..., t. These t basic blocks, when developed mod v, yield a neighbor design, if the following conditions are satisfied,

(i) the sum of "forward" differences arising from each basic block reduces to zero (mod v);

(ii) among the totality of 2kt "forward" and "backward" differences, reduced mod v, every non-zero element, 1, 2, ..., v-1, of G(v) occurs 2' times,

where a set of forward and backward differences arising out of a block (a_{i1}, a_{i2}, ..., a_{ik}) is respectively given by (a_{i2}-a_{i1}, a_{i3}-a_{i2}, ..., a_{ik}-a_{ik-1}, a_{i1}-a_{i2}) and (a_{i1}-a_{i2}, a_{i2}-a_{i3}, ..., a_{ik-1}-a_{ik}, a_{ik}-a_{i1}).

This principle is used here. For construction of neighbor designs, it is important to arrange the elements in the basic blocks such that conditions (i) and (ii) are satisfied. Some methods of construction in this direction are suggested by Das and Saha [1]. As an application of their method III, we can obtain the following.

THEOREM 1. The existence of a BIB design with parameters v, b, r, k and \lambda based on the v elements, 0, 1, 2, ..., v-1, of a finite Abelian group implies the existence of a neighbor design with parameters v'=v, b'=(b-r)v, r'=2(b-r)k, k'=2k and A'=4(r-\lambda).

PROOF. Let 0, 1, 2, ..., v-1, the v elements of a finite Abelian group G(v) of order v, be the v treatments of a BIB design with parameters v, b, r, k and \lambda. Consider b-r blocks each with k treatments in which the treatment 0 does not occur, as follows;

B_i=(a_{i1}, a_{i2}, ..., a_{ik}), \quad i=1, 2, ..., b-r,

which can have a property (A) that (B_i-B_i), i=1, 2, ..., b-r, mod v, contains all the non-zero elements of G(v) each occurring 2(r-\lambda) times. Take these b-r blocks and insert zeroes prior to every treatment in each of them, as

(*) \quad B'_i=(0, a_{i1}, 0, a_{i2}, ..., 0, a_{ik}), \quad i=1, 2, ..., b-r,

which yield, when developed mod v, the required neighbor design, because it follows from a property (A) that among 4k(b-r) forward and backward differences arising from (*), each non-zero element of G(v) occurs exactly 4(r-\lambda) times.

Furthermore, respective sets of blocks containing the treatment 0 and of blocks not containing the treatment 0 yield, from the same (constructive) approach as
Theorem 1 for a complement of a BIB design with parameters $v$, $b$, $r$, $k$ and $\lambda$, the following.

**Theorem 2.** If there exists a BIB design with parameters $v$, $b$, $r$, $k$ and $\lambda$ based on the $v$ elements of a finite Abelian group, then there are neighbor designs with the following parameters:

(i) $v'' = v$, $b'' = (b-r)v$, $r'' = 2(v-k-1)(b-r)$, $k'' = 2(v-k-1)$, $\lambda'' = 4(b-2r+\lambda)$;

(ii) $v''' = v$, $b''' = vr$, $r''' = 2r(v-k)$, $k''' = 2(v-k)$, $\lambda''' = 4(r-\lambda)$.

Theorems 1 and 2 are new, but the idea of these theorems are similar to that of Theorem 5.4 of Das and Saha [1].

Applications of Theorems 1 and 2 to known series of the corresponding BIB designs yield a number of new series of neighbor designs. For example, Sprott [6] showed that if $v = 2m(2^\lambda - 1) + 1 = p$, where $p$ is prime, then the BIB design with parameters $v = 2m(2^\lambda - 1) + 1$, $b = mv$, $r = 2m\lambda$, $k = 2\lambda$, $\lambda$ can be constructed via the initial blocks $(0, x^i, x^{i+2m}, \ldots, x^{i+(2^{\lambda-1} - 1)m})$, $i = 0, 1, \ldots, m-1$, where $x$ is a primitive element of a Galois field of order $v$. Hence, from Theorems 1 and 2, this series of BIB designs yields three series of neighbor designs with the following parameters:

when $2m(2^\lambda - 1) + 1$ is a prime or a prime power,

1. $v' = 2m(2^\lambda - 1) + 1$, $b' = m(v' - 2\lambda)v'$, $r' = 4m(v' - 2\lambda)\lambda$, $k' = 4\lambda$, $\lambda' = 4(2m-1)\lambda$;

2. $v'' = 2m(2^\lambda - 1) + 1$, $b'' = m(v'' - 2\lambda)v''$, $r'' = 4m(2m-1)\lambda - m)v'' - 2\lambda$, $k'' = 4(2m-1)\lambda - m)$, $\lambda'' = 4(mv'' - 4m\lambda + \lambda)$;

3. $v''' = 2m(2^\lambda - 1) + 1$, $b''' = 2m\lambda v'''$, $r''' = 4m\lambda(2m-1)(2\lambda - 1)$, $k''' = 2(2m-1)\lambda$.

Note that $2m(2^\lambda - 1) + 1$ is a prime or a prime power for a number of positive integers $m$ and $\lambda$.

As another individual example, we have the following.

**Example.** Consider a BIB design with parameters $v = b = 13$, $r = k = 4$ and $\lambda = 1$ based on the 13 treatments, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, whose a difference set is given by $(0, 1, 3, 9) \mod 13$. In this case, Theorems 1 and 2 yield that, when developed mod 13,

(i) 9 basic blocks, $(0, 1, 0, 2, 0, 4, 0, 10)$, $(0, 2, 0, 3, 0, 5, 0, 11)$, $(0, 3, 0, 4, 0, 6, 0, 12)$, $(0, 1, 0, 5, 0, 6, 0, 8)$, $(0, 2, 0, 6, 0, 7, 0, 9)$, $(0, 3, 0, 7, 0, 8, 0, 10)$, $(0, 4, 0, 8, 0, 9, 0, 11)$, $(0, 5, 0, 9, 0, 10, 0, 12)$, $(0, 1, 0, 7, 0, 11, 0, 12)$, generate a neighbor design with parameters $v' = 13$, $b' = 117$, $r' = 72$, $k' = 8$ and $\lambda' = 12$;

(ii) 9 basic blocks, $(0, 3, 0, 5, 0, 6, 0, 7, 0, 8, 0, 9, 0, 11, 0, 12)$, $(0, 1, 0, 2, 0, 4, 0, 6, 0, 7, 0, 8, 0, 9, 0, 10, 0, 11)$, $(0, 2, 0, 3, 0, 4, 0, 7, 0, 9, 0, 10, 0, 11, 0, 12)$, $(0, 1, 0, 3, 0, 4, 0, 5, 0, 8, 0, 10, 0, 11, 0, 12)$, $(0, 1, 0, 2, 0, 4, 0, 5, 0, 6, 0, 9, 0, 11, 0, 12)$, $(0, 1, 0, 2, 0, 3, 0, 5, 0, 6, 0, 7, 0, 10, 0, 12)$, $(0, 1, 0, 2, 0, 3, 0, 4, 0, 6, 0, 7, 0, 8, 0, 11)$, $(0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 8, 0, 9, 0, 10)$, generate a neighbor design with parameters $v'' = 13$, $b'' = 117$, $r'' = 144$, $k'' = 16$ and $\lambda'' = 24$. 

(iii) 4 basic blocks, \((0, 2, 0, 4, 0, 5, 0, 6, 0, 7, 0, 8, 0, 10, 0, 11, 0, 12), (0, 1, 0, 2, 0, 3, 0, 6, 0, 8, 0, 9, 0, 10, 0, 11, 0, 12), (0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 7, 0, 8, 0, 9, 0, 10, 0, 11),\) generate a neighbor design with parameters \(v'''=13, b'''=52, r'''=72, k'''=18\) and \(\lambda'''=12\).

It is interesting to note that for blocks developed mod \(a\) of a neighbor design, if each element, \(x\), in the blocks is changed to \(a-1-x\) (mod \(a\)), then the changed blocks may generate the same or another neighbor design.

Lawless \([4]\) has noted that if a BIB design with parameters \(v, b, r, k\) and \(\lambda\) has the "neighbor design" property, then its parameters are written as

\[v, \ b=\lambda'v(v-1)/2k, \ r=\lambda'(v-1)/2, \ k, \ \lambda=\lambda'(k-1)/2\]

and has further pointed out that BIB designs of series \(A\) and \(B\) of Sprott \([6]\) are also neighbor designs.

As another new series of neighbor designs which are also BIB designs, from Theorem 2.1 of Sprott \([7]\), when \(k=\lambda=3\), we can obtain the following.

**Theorem 3.** If \(2m+1=p^t\), where \(p\) is a prime, then the BIB and neighbor design with parameters \(v=2m+1, b=mv, r=3m, k=\lambda=3\) and \(\lambda'=3\) can be constructed from the basic blocks

\[(0, x^i, x^{i+m}), \quad i=0, 1, \ldots, m-1,\]

where \(x\) is a primitive element of a Galois field of order \(v\).

**Proof.** It is sufficient to show that this design is also a neighbor design. The "backward" and "forward" differences between successive treatments in the basic blocks, \((0, x^i, x^{i+m})\) are \(\pm x^i, \pm x^i(x^m-1), \pm x^i\). Since \(x^{2m}=1\), we get \(x^m=-1\). Then the backward and forward differences can be written as \(x^i, x^{i+m}, x^{i}(x^m-1), x^{i+m}(x^m-1), x^i, x^{i+m}\). As \(i\) ranges over \(0, 1, \ldots, m-1\), these differences obviously give each non-zero element of a Galois field of order \(v\) exactly three times. Thus, blocks \((0, x^i, x^{i+m}), \ i=0, 1, \ldots, m-1\), when developed, show that each treatment occurs next to each other treatment exactly three times.

Note that \(2m+1\) is a prime or a prime power for many positive integers \(m\) \((m=1, 2, 3, 4, 5, 6, 8, 9, \ldots)\).

**References**