REMARK ON STABILIZING CONSTANTS OF
THE EXTREME STATISTIC

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Stabilizing constants of a distribution function which belongs to the domain
of attraction of $\exp(-e^{-x})$ are given by inverse function of the distribution
function. But with some distributions their exact values can not be expressed
by sample size $n$ in simple forms. For instance in the case of normal distribu-
tion, inadequate sequences based on an approximation of the inverse function
are used sometimes as stabilizing constants. In this paper, we shall show some
properties of stabilizing constants in particular simple necessary and sufficient
condition, and then apply the results obtained to some examples.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with a com-
mon distribution function $F(x)$ and then define the extreme statistic $Z_n=\max\{X_1,
X_2, \ldots, X_n\}$. If there exist sequences of constants $a_n>0$ and $b_n$ such that $(Z_n-b_n)/a_n$
has the limit distribution $M(x)=\exp(-e^{-x})$, or $F^n(a_nx+b_n) \Rightarrow M(x)$, then $F$ is said to
belong to the domain of attraction of $M(x)$ (notation $F \in D(M)$) and $a_n$ and $b_n$ are
called stabilizing constants. In this paper we consider the properties of $a_n$ and $b_n$.

In Section 2, first we show some properties of $a_n$ and $b_n$. Then we give a
specification of the auxiliary function in Gnedenko's characterization (Theorem 6)
in a special case. In Section 3, we apply our procedure to some examples. In
Appendix we quote two theorems due to Gnedenko and a lemma due to Khintchine
which are used in Sections 2 and 3.

2. Main results

In the sequel we use the notation

$$x_0=x_0(F)=\sup\{x|F(x)<1\},$$

which will be called the end-point of the distribution function $F$.

First we prove four theorems useful in choosing stabilizing constants.

Theorem 1. a) Suppose $F \in D(M)$ or there exist $a_n>0$ and $b_n$ such that

$$F^n(a_nx+b_n) \Rightarrow M(x).$$

Then

$$\lim_{n \to \infty} n \cdot (1-F(b_n))=1.$$

b) Conversely, if $F \in D(M)$, then (2.1) holds with $b_n$ which satisfies (2.2) and $a_n=f(b_n)$, where $f$ is defined in Theorem 6.

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Proof. a) By Theorem 7, we have (2.2).

b) By Theorem 6, we obtain

\[
\lim_{n \to \infty} n \cdot (1 - F(a_n x + b_n)) = \lim_{n \to \infty} \frac{1 - F(a_n x + b_n)}{1 - F(b_n)} = e^{-x}
\]
for all real \( x \).

Hence from Theorem 7, we have (2.1).

**Theorem 2.** Let \( F \) and \( G \) be distribution functions. Suppose there exist \( a_n > 0 \), \( b_n \) and \( \alpha_n > 0 \) such that (2.1) holds, and it holds

\[
(2.3) \quad G^\alpha(a_n x + b_n) \xrightarrow{w} \lambda (A x + B),
\]

where \( A > 0 \). Then we have

\[
(2.4) \quad \lim_{n \to \infty} \frac{a_n}{\alpha_n} = A.
\]

**Proof.** For \( s > 0 \), define

\[
\beta(s) = \inf \{ x \mid 1 - F(x) \leq 1/s \}
\]

and

\[
\beta(s) = \inf \{ x \mid 1 - G(x) \leq 1/s \}.
\]

From Lemma 1 and Theorem 2.1.2* of [4], we can suppose without loss of generality \( a_n = b(en) - b(n) \), \( b_n = b(n) = \beta(e^n n) \) and \( \alpha_n = \beta(e^\alpha e^n n) - \beta(e^n n) \). It holds

\[
\beta(e^n [en]) \leq \beta(e^n n) \leq \beta(e^n ([en] + 1)) \quad \text{for all } n,
\]

where \([ \ ]\) is the Gauss symbol. So we have

\[
(2.7) \quad \frac{b([en]) - b(n)}{b(en) - b(n)} \leq \frac{\beta(e^n n) - \beta(e^n [en])}{b(en) - b(n)} \leq \frac{b([en] + 1) - b(n)}{b(en) - b(n)}.
\]

As \( b \) is non-decreasing, we have for fixed \( c \) \( (1 < c < e) \) and sufficiently large \( n \)

\[
1 \leq \frac{b([en]) - b(n)}{b(en) - b(n)} \leq \frac{b([en] + 1) - b(n)}{b(en) - b(n)}.
\]

From Theorem 2.4.1 of [4], we have

\[
\lim_{n \to \infty} \frac{b(cn) - b(n)}{b(en) - b(n)} = \log c.
\]

Hence we have

\[
(2.8) \quad \lim_{n \to \infty} \frac{b([en]) - b(n)}{b(en) - b(n)} = 1.
\]

Similarly

\[
(2.9) \quad \lim_{n \to \infty} \frac{b([en] + 1) - b(n)}{b(en) - b(n)} = 1.
\]

By (2.7), (2.8) and (2.9), it holds
On the other hand from Theorem 2.4.1 of [4], it folds

\begin{equation}
\lim_{n \to \infty} \frac{\beta(e^{e^n}) - \beta(e^n)}{b(e^n) - b(n)} = A.
\end{equation}

By (2.10) and (2.11), we have (2.4).

The converse of Theorem 2 becomes the following theorem.

\textbf{THEOREM 3.} \ Let \( F \) and \( G \) be distribution functions. Suppose there exist \( a_n > 0, b_n, a_n > 0 \) and \( \beta_n \) such that (2.1) holds, and it holds

\begin{equation}
G^\prime(a_n x + \beta_n) \to A(x).
\end{equation}

a) If \( x_0(F) = x_0(G) = \infty \) and for \( 0 \leq c \leq \infty \),

\begin{equation}
\lim_{n \to \infty} \frac{a_n}{\alpha_n} = c,
\end{equation}

then

\begin{equation}
\lim_{n \to \infty} \frac{b_n}{\beta_n} = c.
\end{equation}

b) If \( x_0(F) = x_0(G) = x_\beta < \infty \) and for \( 0 \leq c \leq \infty \), (2.13) holds, then

\begin{equation}
\lim_{n \to \infty} \frac{x_\beta - b_n}{x_\beta - \beta_n} = c.
\end{equation}

\textbf{PROOF.} \ For \( s > 0 \), define

\begin{equation}
\alpha(s) = b(es) - b(s)
\end{equation}

and

\begin{equation}
\beta(s) = \beta(es) - \beta(s)
\end{equation}

where \( b(s) \) and \( \beta(s) \) are defined as in (2.5) and (2.6), respectively. From Lemma 1, we can suppose without loss of generality \( a_n = a(n), b_n = b(n), a_n = a(n) \) and \( \beta_n = \beta(n) \).

a) From Theorem 1.4.1 and Corollary 1.4.1 of [4], we have

\begin{equation}
b(s) = \frac{\alpha(t)}{t} \bigg|_{1}^{s} dt
\end{equation}

and

\begin{equation}\beta(s) = \frac{\alpha(t)}{t} \bigg|_{1}^{s} dt
\end{equation}

Suppose (2.13) holds then by Lemma 1 of [5], it holds

\begin{equation}
\lim_{n \to \infty} \frac{\alpha(n)}{\alpha(s)} = c.
\end{equation}

Hence we have
b) From Theorem 1.4.1 and Corollary 1.4.1 of [4], we have

\[ x_n - b(s) \sim \int_1^x \frac{a(t)}{t} \, dt \]

and

\[ x_n - \beta(s) \sim \int_1^x \frac{a(t)}{t} \, dt \]

so that a similar argument as in a) proves b).

**Corollary 1.** Suppose \( F \) is a distribution function with infinite end-point. Then \( 1-F(\log x) \) is \((-a)\)-varying (see the definition in Haan [4], p. 4) if and only if there exist \( b_n \) such that

\[ \frac{1}{a} \log n, \text{ where } a > 0. \]

**Proof.** It is clear from Theorem 3 and Theorem 8 of [3].

**Theorem 4.** Let \( F \) and \( G \) be distribution functions. Suppose \( F \in D(A) \) or there exist \( a_n > 0 \) and \( b_n \) such that (2.1) holds. Then a necessary and sufficient condition that there exist \( a_n > 0 \) such that

\[ G^n(\alpha_n x + b_n) \xrightarrow{w} A(x) \]

is that \( x_n(F) = x_n(G) (=x_n, \text{ say}), \alpha_n / a_n \to A, \text{ and} \)

\[ \lim_{x \to 0} \frac{1-G(x)}{1-F(x)} = e^{-B}. \]

**Proof.** Sufficiency. From Theorem 7, we have

\[ \lim_{n \to \infty} n \cdot (1-F(\alpha_n x + b_n)) = e^{-x}. \]

So it holds that \( a_n x + b_n \to x_0 \) as \( n \to \infty \). Hence we have

\[ \lim_{n \to \infty} \frac{1-G(\alpha_n x + b_n)}{1-F(\alpha_n x + b_n)} = e^{-B}. \]

Therefore it holds that

\[ \lim_{n \to \infty} n \cdot (1-G(\alpha_n x + b_n)) = e^{-(x+B)}. \]

By \( \lim_{n \to \infty} \alpha_n / a_n = A > 0 \), we have (2.16).

Necessity. By Theorem 2 we have \( \lim_{n \to \infty} \alpha_n / a_n = A \). By (2.1), (2.16) and Theorem 7, we have (2.18) and (2.19), hence we get \( b_n \to x_n(F) \) and \( b_n \to x_n(G) \). So it holds \( x_n(F) = x_n(G) = x_0 \). From Lemma 1, we can suppose without loss of generality \( b_n \) is
defined as in (A.2), and hence $b_n \uparrow x_0$ as $n \to \infty$. For any $x$ sufficiently near $x_0$ ($x < x_0$), there exist an $n$ such that $b_n \leq x \leq b_{n+1}$. Then we have

$$
(2.20) \quad \frac{1-G(b_{n+1})}{1-F(b_n)} \leq \frac{1-G(x)}{1-F(b_n)} \leq \frac{1-G(b_n)}{1-F(b_{n+1})}.
$$

By (2.18) and (2.19), the left hand side of (2.20) is

$$
(2.21) \quad \frac{1-G(b_{n+1})}{1-F(b_n)} = \frac{n}{n+1} \cdot \frac{(n+1)(1-G(b_{n+1}))}{n \cdot (1-F(b_n))} \to e^{-b} \quad \text{as } n \to \infty.
$$

Similarly we have

$$
(2.22) \quad \frac{1-G(b_n)}{1-F(b_{n+1})} \to e^{-b} \quad \text{as } n \to \infty.
$$

From (2.20), (2.21) and (2.22), we obtain (2.17).

**Corollary 2** (Resnick [6], Lemma 2.5). Suppose (2.1) holds. Then a necessary and sufficient condition that

$$
G^\circ(a_n x + b_n) \xrightarrow{w} \Lambda(Ax + B) \quad \text{and} \quad A > 0
$$

is $A = 1$, $x_0(F) = x_0(G)$, and (2.17).

**Remark 1.** Theorem 4 is an extension of Lemma 2.5 of [6] with a simpler proof.

**Remark 2.** Given a distribution function $G \in D(A)$, from Theorem 1 and Corollary 2 we can present the following procedure to find stabilizing constants of $G$. We first choose an appropriate simple distribution function $F$ such that (2.17) holds for some $B$, seek $b_n$ which satisfy (2.2) and then define $a_n = f(b_n)$. The explicit form of the function $f$ is given by Theorem 2.5.1 of [4].

The following theorem determines the transformation of the stabilizing constants induced by a transformation of the underlying random variable.

**Theorem 5.** Let $a_n$ and $b_n$ be stabilizing constants of a distribution function $F \in D(A)$ with infinite end-point. Define

$$
F^*(x) = \begin{cases} 
0 & \text{if } x < 0 \\
F(g(x)) & \text{if } x \geq 0,
\end{cases}
$$

where $g$ is a function which has a $\rho$-varying derivative ($-1 < \rho < \infty$). Then $F^* \in D(A)$, and

$$
(2.23) \quad F^*(a^*_n x + b^*_n) \xrightarrow{w} \Lambda(x)
$$

where $a^*_n = a_n g^{-1}(b_n)/((\rho + 1)b_n)$ and $b^*_n = g^{-1}(b_n)$.

**Proof.** From Theorem 1.5.6 b) of [4], it holds $F^* \in D(A)$, and $f^*(t) = f(g(t))/((\rho + 1)g(t)$, where $f$ and $f_*$ are auxiliary functions of $F$ and $F^*$, respectively. The inverse function $g^{-1}$ of $g$ well determined, because $g$ is continuous and strictly increasing. By $\inf \{x | 1 - F^*(x) \leq 1/n\} = g^{-1}(b_n)$ and without loss of generality from Lemma 1 we may take $f^*(b^*_n) = a_n g^{-1}(b_n)/((\rho + 1)b_n)$, hence we have (2.23).
Corollary 3. Suppose \( a > 0 \) and \( \theta > 0 \). For a distribution function \( F \) with infinite endpoint we have

\[
\frac{1-F(t+x^{1-a}/a\theta)}{1-F(t)} \to e^{-x} \quad \text{as} \quad t \to \infty,
\]

if and only if \( 1-F((\log x)^{1/\theta}) \) is \((-\theta)\)-varying.

Proof. It is clear from Theorem 8 of [3] and Theorem 5.

3. Examples

Now we shall determine stabilizing constants for some examples, using the results of Section 2.

Example 1. Consider the following distribution functions

\[
F_n(x) = 1-e^{-cx} \quad x \geq 0
\]

and

\[
G_n(x) = 1-ax^c e^{-cx} \quad x \geq x_1,
\]

where \( a, a, c (> 0) \) and \( b \) are constants and \( x_1 \) is the constant such that \( G_n(x) \), \( (G_n(x))' \geq 0 \) for all \( x \geq x_1 \). Here we determine stabilizing constants of \( F_n \) and \( G_n \), using Theorems 1, 5 and Corollary 3. First we consider \( F_n \), then it is easily seen that

\[
F_n(a_n x + b_n) \Rightarrow \Lambda(x),
\]

where \( a_n = 1/c \) and \( b_n = (\log n)/c \). By Theorem 5, we have

\[
F_n(a^*_n x + b^*_n) \Rightarrow \Lambda(x),
\]

where \( a^*_n = ((\log n)/c)^{1-a/c} \) and \( b^*_n = ((\log n)/c)^{1/c} \). By Corollary 2.4.3 of [4] we have \( G_n \in D(A) \). Define

\[
\beta_n = ((\log n)/c)^{1/a} + \frac{1}{ac} \frac{(b/a) (\log (\log n))/c + \log a}{((\log n)/c)^{1-1/a}}
\]

then \( n (1-G_n(\beta_n)) \to 1 \) as \( n \to \infty \). By Corollary 3, we take \( \alpha_n = \beta_n^{1-a/c} \). It holds \( a^*_n \sim \alpha_n \), by Lemma 1 we have

\[
G^*_n(a^*_n x + \beta_n) \Rightarrow \Lambda(x).
\]

The following example is found frequently, for which, however, inadequate sequences are sometimes claimed to be stabilizing constants, as in Example 9.3.2 of David [1].

Example 2. Consider the standard normal distribution function \( \Phi(x) = \int_{-\infty}^{x} \phi(t) dt \), where \( \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2) \). It is well known that \( \Phi \in D(A) \) and asymptotically for large \( x \)

\[
1-\Phi(x) \sim \phi(x)/x.
\]

So we consider the distribution function defined by
where \( x_i \) is the solution of the equation \( \phi(x) = x \). Then from Example 1, \( b_n = \sqrt{2} \log n - (\log \sqrt{2} \log n + \log \sqrt{2\pi} / \sqrt{2} \log n) \) and \( a_n = 1 / \sqrt{2} \log n \) are stabilizing constants. It is noted that \( b'_n = \sqrt{2} \log n \) which is suggested in [1] satisfies \( b'_n \sim b_n \) but not (2). In fact

\[
\lim_{n \to \infty} n \cdot (1 - F(b'_n)) = 0
\]

so that it is not a stabilizing constant.

Appendix
Here we quote two theorems due to Gnedenko [3] concerning the domain of attraction of \( A \), and a lemma due to Khintchine ([2], p. 246) concerning the stabilizing constants.

**Theorem 6.** \( F \in D(A) \) if and only if there exists a function \( f: \mathbb{R} \to \mathbb{R}^+ \) such that

\[
\lim_{t \to x_0^+} \frac{1 - F(t + x \cdot f(t))}{1 - F(t)} = e^{-x}
\]

for all real \( x \), where \( \mathbb{R}^+ = (0, \infty) \) and \( x_0 \) is the end-point of \( F \).

**Theorem 7 (Lemma 4 and Theorem 6 of [3]).** \( F \in D(A) \) or there exist \( a_n > 0 \) and \( b_n \) such that (2.1) holds if and only if

\[
\lim_{n \to \infty} n \cdot (1 - F(a_n x + b_n)) = e^{-x}
\]

for all real \( x \).

Moreover (A.1) holds with

\[
\begin{cases}
  b_n = \inf \{ x | F(x) \geq 1 - 1/n \} \\
  a_n = \inf \{ x | F(x) \geq 1 - 1/n \} - b_n
\end{cases}
\]

for \( n = 1, 2, 3, \ldots \).

**Lemma 1.** Let \( U \) and \( V \) be two distribution functions neither of which is concentrated at one point. If for a sequence \( \{F_n\} \) of distribution functions and constants \( a_n > 0 \), \( b_n \), and \( \beta_n \)

\[
F_n(a_n x + b_n) \xrightarrow{m} U(x), \quad F_n(\alpha_n x + \beta_n) \xrightarrow{m} V(x)
\]

then

\[
\lim_{n \to \infty} \alpha_n / a_n = A > 0, \quad \lim_{n \to \infty} \beta_n / b_n = B
\]

and

\[
V(x) = U(Ax + B)
\]

is true. Conversely, if (A.4) holds then each of the two relations (A.3) implies the other and (A.5).
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