ANALYSIS OF VARIANCE OF BALANCED FRACTIONAL $r^m \times s^n$ FACTORIAL DESIGNS OF RESOLUTION V

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In this paper, attention is focused on the analysis of variance and hypothesis testing of balanced fractional $r^m \times s^n$ factorial designs of resolution V. These designs are derived from asymmetrical balanced arrays of strength 4 using the algebraic structure of the multidimensional relationship.

1. Introduction

The analysis of variance (ANOVA) is the standard technique for analyzing data or observations obtained from an experiment (e.g., Scheffé [18], and Seber [19]). The ANOVA of balanced fractional s^m factorial (s^m -BFF) designs of resolution $2l+1$ have been studied by Kuwada [11, 14] for $s=2$ and $1 \leq l \leq m/2$, and $s=3$ and $l=2$, and Hyodo and Kuwada [6] for general s and $l=2$. The ANOVA and the testing hypotheses of various designs have been considered by several researchers (e.g., Kuwada [12, 13]).

The concept of a balanced array was first introduced by Chakravarti [2] as a generalization of an orthogonal array. Under certain conditions, a balanced array of strength $2l$ turns out to be an s^m -BFF design of resolution $2l+1$ (see Srivastava [20], Yamamoto et al. [24], Kuwada [9], and Kuwada and Nishii [15]). The characteristic polynomials or roots of the information matrices of s^m -BFF designs of resolution $2l+1$ were obtained by Srivastava and Chopra [22] for $s=l=2$, Yamamoto et al. [25] for $s=2$ and $1 \leq l \leq m/2$, Hyodo [5] for $s=2$ and $m/2 < l \leq m$, Kuwada [8,10] and Srivastava and Ariyaratna [21] for $s=3$ and $l=2$, and Kuwada and Nishii [16] for general s and $l=2$. Special cases of 3^m -BFF designs have been considered by Hoke [3, 4] and Srivastava and Chopra [23].

Using the algebraic structure of the multidimensional relationship (MDR), Nishii [17] obtained an explicit expression for the characteristic polynomial of the information matrix of an $r^m \times s^n$ -BFF design of resolution V derived from an asymmetrical balanced array (AB-array) of strength 4. This design permits estimation of the general mean, all main effects and all two-factor interactions. This relationship and its algebra were discussed by James [7] in an experimental design. Using the MDR and its algebra, this paper presents the ANOVA and the hypothesis testing of $r^m \times s^n$ -BFF designs of resolution V, which are derived from AB-arrays.

Received June, 1994. Revised April, 1995. Accepted May, 1995.

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2. Preliminaries

Consider a fractional $r^m \times s^n$ factorial experiment with $m+n$ factors $F_1, F_2, \dots, F_m$ and $G_1, G_2, \dots, G_n$, where $m, n \geq 2$. Let $\theta(\mathbf{a}_0; \mathbf{b}_0)' = (\{\theta(\phi; \phi)\})$, $\theta(\mathbf{a}_i; \mathbf{b}_0)' = (\{\theta(u^i; \phi) | 1 \leq u \leq m\})$, $\theta(\mathbf{a}_0; \mathbf{b}_j)' = (\{\theta(\phi; v^j) | 1 \leq v \leq n\})$, $\theta(\mathbf{a}_{ii}; \mathbf{b}_0)' = (\{\theta(u_1^i u_2^i; \phi) | 1 \leq u_1 < u_2 \leq m\})$, $\theta(\mathbf{a}_0; \mathbf{b}_{jj})' = (\{\theta(\phi; v_1^j v_2^j) | 1 \leq v_1 < v_2 \leq n\})$, $\theta(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)' = (\{\theta(u_1^{i_1} u_2^{i_2}; \phi) | 1 \leq u_3 \neq u_4 \leq m\})$, $\theta(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})' = (\{\theta(\phi; v_3^{j_1} v_4^{j_2}) | 1 \leq v_3 \neq v_4 \leq n\})$, and $\theta(\mathbf{a}_i; \mathbf{b}_j)' = (\{\theta(u^i; v^j) | 1 \leq u \leq m, 1 \leq v \leq n\})$, where A' denotes the transpose of a matrix A , and $1 \leq i \leq r-1, 1 \leq j \leq s-1, 1 \leq i_1 < i_2 \leq r-1$ and $1 \leq j_1 < j_2 \leq s-1$. Here $\theta(\phi; \phi)$ is the general mean, $\theta(u^i; \phi)$ and $\theta(\phi; v^j)$ are the i th and j th degree components of the main effects of factors F_u and G_v , respectively, and $\theta(u_1^i u_2^i; \phi)$, $\theta(\phi; v_1^j v_2^j)$, $\theta(u_3^{i_1} u_4^{i_2}; \phi)$, $\theta(\phi; v_3^{j_1} v_4^{j_2})$ and $\theta(u^i; v^j)$ are the i th $\times$ i th, j th $\times$ j th, i_1 th $\times$ i_2 th, j_1 th $\times$ j_2 th and i th $\times$ j th ones of the two-factor interactions of F_{u_1} and F_{u_2} , G_{v_1} and G_{v_2} , F_{u_3} and F_{u_4} , G_{v_3} and G_{v_4} , and F_u and G_v , respectively. Further, let θ be a $\nu(m, n) \times 1$ vector of unknown factorial effects to be estimated, where

$$\begin{aligned} \theta' = & \{\theta(\mathbf{a}_0; \mathbf{b}_0)'\}; \{\theta(\mathbf{a}_i; \mathbf{b}_0)'\}; \{\theta(\mathbf{a}_0; \mathbf{b}_j)'\}; \{\theta(\mathbf{a}_{ii}; \mathbf{b}_0)'\}; \\ & \{\theta(\mathbf{a}_0; \mathbf{b}_{jj})'\}; \{\theta(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)'\}; \{\theta(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})'\}; \{\theta(\mathbf{a}_i; \mathbf{b}_j)'\} \end{aligned}$$

and

$$\nu(m, n) = \sum_{i+j \leq 2} \binom{m}{i} \binom{n}{j} (r-1)^i (s-1)^j.$$

Here $\binom{p}{q} = 0$ if and only if $q < 0$ or $p < q$. Now we consider a fraction $[T^{(1)}; T^{(2)}]$ ($=T$, say) with N assemblies or treatment combinations, where $T^{(1)}$ and $T^{(2)}$ are an $N \times m$ matrix with $0, 1, \dots, r-1$ and an $N \times n$ one with $0, 1, \dots, s-1$, respectively. Then the linear model for a fraction T may be written as

$$(2.1) \quad \mathbf{y}(T) = E_T \theta + \mathbf{e}_T,$$

where $\mathbf{y}(T)$ and E_T are a vector of N observations and the design matrix of size $N \times \nu(m, n)$, respectively, and $\mathbf{e}_T$ is an error vector distributed as $N(\mathbf{0}_N, \sigma^2 I_N)$. Here $\mathbf{0}_p$ and I_p are the $p \times 1$ vector with all zero and the identity matrix of order p , respectively. The normal equation for estimating θ is given by $M_T \hat{\theta} = E_T' \mathbf{y}(T)$, where $M_T (= E_T' E_T)$ is the information matrix of order $\nu(m, n)$. If the M_T is nonsingular, then the BLUE of θ and its variance-covariance matrix are given by $\hat{\theta} = M_T^{-1} E_T' \mathbf{y}(T)$ and $\text{Var}[\hat{\theta}] = \sigma^2 M_T^{-1}$, respectively. Under (2.1), if the M_T^{-1} is invariant under any permutation on m factors in $T^{(1)}$ and on n ones in $T^{(2)}$, then T is called an $r^m \times s^n$ -BFF design of resolution V, where $T = [T^{(1)}; T^{(2)}]$ (see Nishii [17]).

Now we define an AB-array (see Nishii [17]):

DEFINITION 2.1. A matrix $[T^{(1)}; T^{(2)}]$ ($=T$, say) of size $N \times (m+n)$ is called an AB-array of strength t and size N having $(m+n)$ constraints, r and s levels, and index set $\{\lambda(\{i_p\}; \{j_q\}) | \sum_{p=0}^{r-1} i_p + \sum_{q=0}^{s-1} j_q = t\}$, if for any nonnegative integers $t_1 (\leq m)$ and $t_2 (\leq n)$ satisfying $t_1 + t_2 = t$, every $N \times t$ submatrix $[T_0^{(1)}; T_0^{(2)}]$ ($=T_0$, say) of T is such that every $(0, 1, \dots, r-1)$ -vector with i_p p 's

($0 \leq p \leq r-1$) in $T_0^{(s)}$ and every $(0, 1, \dots, s-1)$ -vector with j_q q 's ($0 \leq q \leq s-1$) in $T_0^{(s)}$ occur exactly $\lambda(\{i_p\}; \{j_q\})$ times as a row of T_0 , where $T_0^{(k)}$ ($k=1, 2$) are of size $N \times tk$ and are composed of tk columns of $T^{(k)}$. This array is written as $\text{ABA}(N, (m, n), (r, s), t; \{\lambda(\{i_p\}; \{j_q\})\})$ for brevity.

An $\text{ABA}(N, (m, n), (r, s), 4; \{\lambda(\{i_p\}; \{j_q\})\})$ yields an $r^m \times s^n$ -BFF design of resolution V under the information matrix being nonsingular (see Nishii [17]). Thus throughout this paper, we consider a design T which is an $\text{ABA}(N, (m, n), (r, s), 4; \{\lambda(\{i_p\}; \{j_q\})\})$. Using the MDR and its algebra, the information matrix M_T of T is given by

$$(2.2) \quad M_T = \sum_{\beta=0}^5 \sum_{a, b \in Z_\beta} k_\beta(a, b) D_\beta^*(a, b) + \sum_{(u:u), (v:v) \in Z_f} k_{f_{uv}}(u, v) D_{f_{uv}}^*(u, v) \\ + \sum_{(x:x), (y:y) \in Z_g} k_{g_{xy}}(x, y) D_{g_{xy}}^*(x, y),$$

where k 's and D^* 's are some linear functions of indices $\lambda(\{i_p\}; \{j_q\})$ of an AB-array and the matrices of order $\nu(m, n)$ given by some linear combinations of the relationship matrices of the MDR, respectively (see Nishii [17]). Here Z_β ($0 \leq \beta \leq 5$), Z_f and Z_g are the finite sequences such that the 1st, $\{1+i\}$ th, $\{r+j\}$ th, $\{r+s-1+i\}$ th, $\{2r+s-2+j\}$ th, $\left\{2(r+s)-3+(r-1)(i-1)-\binom{i_1+1}{2}+i_2\right\}$ th, $\left\{2(r+s)-3+\binom{r-1}{2}+(s-1)(j_1-1)-\binom{j_1+1}{2}+j_2\right\}$ th and $\left\{2(r+s)-3+\binom{r-1}{2}+\binom{s-1}{2}+(s-1)(i-1)+j\right\}$ th elements of Z_0 are $(a_0; b_0)$, $(a_i; b_0)$, $(a_0; b_j)$, $(a_{ii}; b_0)$, $(a_0; b_{jj})$, $(a_{i_1 i_2}; b_0)$, $(a_0; b_{j_1 j_2})$ and $(a_i; b_j)$, respectively, (i) if $m \geq 4$, the i th and the $\left\{(r-1)i_1-\binom{i_1+1}{2}+i_2\right\}$ th ones of Z_1 are $(a_{ii}; b_0)$ and $(a_{i_1 i_2}; b_0)$, respectively, and (ii) if $m=2, 3$, Z_1 vanishes, (i) if $m \geq 3$, the $\left\{(r-1)(i_1-1)-\binom{i_1+1}{2}+i_2\right\}$ th one of Z_2 is $(a_{i_1 i_2}; b_0)$, and (ii) if $m=2$, Z_2 vanishes, (i) if $n \geq 4$, the j th and $\left\{(s-1)j_1-\binom{j_1+1}{2}+j_2\right\}$ th ones of Z_3 are $(a_0; b_{jj})$ and $(a_0; b_{j_1 j_2})$, respectively, and (ii) if $n=2, 3$, Z_3 vanishes, (i) if $n \geq 3$, the $\left\{(s-1)(j_1-1)-\binom{j_1+1}{2}+j_2\right\}$ th one of Z_4 is $(a_0; b_{j_1 j_2})$ and (ii) if $n=2$, Z_4 vanishes, the $\{(s-1)(i-1)+j\}$ th one of Z_5 is $(a_i; b_j)$, the i th, $\{r-1+i\}$ th, $\left\{2(r-1)+(r-1)(i_1-1)-\binom{i_1+1}{2}+i_2\right\}$ th, $\left\{2(r-1)+\binom{r-1}{2}+(r-1)(i_1-1)-\binom{i_1+1}{2}+i_2\right\}$ th and $\{r(r-1)+(s-1)(i-1)+j\}$ th ones of Z_f are $((a_i; b_0): 1)$, $((a_{ii}; b_0): 2)$, $((a_{i_1 i_2}; b_0): 3)$, $((a_{i_1 i_2}; b_0): 4)$ and $((a_i; b_j): 1)$, respectively, and the j th, $\{s-1+j\}$ th, $\left\{2(s-1)+(s-1)(j_1-1)-\binom{j_1+1}{2}+j_2\right\}$ th, $\left\{2(s-1)+\binom{s-1}{2}+(s-1)(j_1-1)-\binom{j_1+1}{2}+j_2\right\}$ th and $\{s(s-1)+(s-1)(i-1)+j\}$ th ones of Z_g are $((a_0; b_j): 1)$, $((a_0; b_{jj}): 2)$, $((a_0; b_{j_1 j_2}): 3)$, $((a_0; b_{j_1 j_2}): 4)$ and $((a_i; b_j): 1)$, respectively. The cardinalities of Z_0, Z_1 (if

$m \geq 4$), Z_2 (if $m \geq 3$), Z_3 (if $n \geq 4$), Z_4 (if $n \geq 3$), Z_5 , Z_f and Z_g are given by $\binom{r+s}{2}$ ($=\tau_0$, say), $\binom{r}{2}$ ($=\tau_1$, say), $\binom{r-1}{2}$ ($=\tau_2$, say), $\binom{s}{2}$ ($=\tau_3$, say), $\binom{s-1}{2}$ ($=\tau_4$, say), $(r-1)(s-1)$ ($=\tau_5$, say), $(r-1)(r+s-1)$ ($=\tau_f$, say) and $(s-1)(r+s-1)$ ($=\tau_g$, say), respectively.

The information matrix M_T of T is isomorphic to the symmetric matrices K_0, K_1 (if $m \geq 4$), K_2 (if $m \geq 3$), K_3 (if $n \geq 4$), K_4 (if $n \geq 3$), K_5 , K_f and K_g of order $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_f$ and τ_g , respectively, where $K_\beta = ||k_\beta(\mathbf{a}, \mathbf{b})||$ ($0 \leq \beta \leq 5$), $K_f = ||k_{f_{uv}}(\mathbf{u}, \mathbf{v})||$ and $K_g = ||k_{g_{xy}}(\mathbf{x}, \mathbf{y})||$ (see Nishii [17]). Thus we can obtain the following (see Nishii [17]):

PROPOSITION 2.1. *Let T be an $\text{ABA}(N, (m, n), (r, s), 4; \{\lambda(\{i_p\}; \{j_q\})\})$. Then a necessary and sufficient condition for T to be an $r^m \times s^n$ -BFF design of resolution V is that K_0, K_1 (if $m \geq 4$), K_2 (if $m \geq 3$), K_3 (if $n \geq 4$), K_4 (if $n \geq 3$), K_5 , K_f and K_g are positive definite.*

PROPOSITION 2.2. *Let T be an array given by Proposition 2.1. Then the characteristic polynomial $\Psi_T(x)$ of the information matrix M_T of T is given by*

$$\Psi_T(x) = \det[M_T - xI_{\nu(m,n)}] \\ = \prod_{\beta=0}^5 \{\det[K_\beta - xI_{\tau_\beta}]\}^{\phi_\beta} \{\det[K_f - xI_{\tau_f}]\}^{\phi_f} \{\det[K_g - xI_{\tau_g}]\}^{\phi_g},$$

where $\phi_0 = 1$, $\phi_1 = m(m-3)/2$ (if $m \geq 4$), $\phi_2 = \binom{m-1}{2}$ (if $m \geq 3$), $\phi_3 = n(n-3)/2$ (if $n \geq 4$), $\phi_4 = \binom{n-1}{2}$ (if $n \geq 3$), $\phi_5 = (m-1)(n-1)$, $\phi_f = m-1$ and $\phi_g = n-1$.

3. Algebraic structure of $r^m \times s^n$ -BFF designs

In a practical experiment, the most interesting factorial effects may be the main effects, next the two-factor interactions, and so on. Furthermore, the lower degree components of the main effects (or interactions) may be of more interest than the higher ones. Thus we consider the finite sequences Z_β^* ($0 \leq \beta \leq 5$), Z_f^* and Z_g^* , which are given by some permutation of Z_β, Z_f and Z_g such that the 1st, $\{1+i\}$ th, $\{r+j\}$ th, $\{r+s-1+i\}$ th and $\{2r+s-2+j\}$ th elements of Z_0^* , the i th one of Z_1^* , the j th one of Z_3^* , the i th and $\{r-1+i\}$ th ones of Z_f^* , and the j th and $\{s-1+j\}$ th ones of Z_g^* are the same order as in Z_0, Z_1, Z_3, Z_f and Z_g , respectively, and the $\{2(r+s)-3+f(r-1; i_1, i_2)\}$ th, $\left\{2(r+s)-3+\binom{r-1}{2}+f(s-1; j_1, j_2)\right\}$ th and $\left\{2(r+s)-3+\binom{r-1}{2}+\binom{s-1}{2}+g(r-1, s-1; i, j)\right\}$ th ones of Z_0^* are $(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)$, $(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})$ and $(\mathbf{a}_i; \mathbf{b}_j)$, respectively, the $\{r-1+f(r-1; i_1, i_2)\}$ th one of Z_f^* (if $m \geq 4$) is $(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)$, the $\{f(r-1; i_1, i_2)\}$ th one of Z_f^* (if $m \geq 3$) is $(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)$, the $\{s-1+f(s-1; j_1, j_2)\}$ th one of Z_g^* (if $n \geq 4$) is $(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})$, the $\{f(s-1; j_1, j_2)\}$ th one of Z_g^* (if $n \geq 3$) is $(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})$, the $\{g(r-1, s-1; i, j)\}$ th one of Z_g^* is $(\mathbf{a}_i; \mathbf{b}_j)$, the $\{2(r-1)+2f(r-1; i_1, i_2)-1\}$ th, $\{2(r-1)+2f(r-1; i_1, i_2)\}$ th and $\{r(r-1)+g(r-1, s-1; i, j)\}$ th ones of Z_f^* are $((\mathbf{a}_{i_1 i_2}; \mathbf{b}_0): 3)$, $((\mathbf{a}_{i_1 i_2}; \mathbf{b}_0): 4)$ and $((\mathbf{a}_i; \mathbf{b}_j): 1)$, respectively, and the $\{2(s-1)+2f(s-1;$

$j_1, j_2) - 1\}$ th, $\{2(s-1) + 2f(s-1; j_1, j_2)\}$ th and $\{s(s-1) + g(r-1, s-1; i, j)\}$ th ones of Z_g^* are $((\mathbf{a}_0; \mathbf{b}_{j_1 j_2}): 3)$, $((\mathbf{a}_0; \mathbf{b}_{j_1 j_2}): 4)$ and $((\mathbf{a}_i; \mathbf{b}_j): 1)$, respectively, where

$$f(p; x, y) = [(x+y-1)/2][(x+y-2)/2] + x - \binom{x+y-p}{2},$$

$$g(p, q; x, y) = \binom{x+y-1}{2} + x - \binom{x+y-1-p}{2} - \binom{x+y-q}{2}$$

and $[x]$ denotes the largest integer not exceeding x .

We now consider an $r^m \times s^n$ -BFF design of resolution V derived from an AB-array. Then some properties of $A_\beta^*(\mathbf{a}, \mathbf{b})$ ($0 \leq \beta \leq 5$) of size $\nu(\mathbf{a}) \times \nu(\mathbf{b})$, $A_{f_{uv}}^*(\mathbf{u}, \mathbf{v})$ of size $\nu(\mathbf{u}) \times \nu(\mathbf{v})$ and $A_{g_{xy}}^*(\mathbf{x}, \mathbf{y})$ of size $\nu(\mathbf{x}) \times \nu(\mathbf{y})$, which are given by some linear combinations of the local MDR matrices, are given as follows (see Nishii [17]):

$$\begin{aligned} (3.1a) \quad & A_0^*(\mathbf{a}, \mathbf{b}) = [1/\{\nu(\mathbf{a}) \times \nu(\mathbf{b})\}^{1/2}] G_{\nu(\mathbf{a}) \times \nu(\mathbf{b})} && \text{for } \mathbf{a}, \mathbf{b} \in Z_0^*, \\ (3.1b) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_{f_{11}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_0)\}, \\ (3.1c) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_{g_{11}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_j)\}, \\ (3.1d) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_1^*(\mathbf{a}, \mathbf{a}) + A_{f_{22}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_{ii}; \mathbf{b}_0)\}, \\ (3.1e) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_3^*(\mathbf{a}, \mathbf{a}) + A_{g_{22}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{jj})\}, \\ (3.1f) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_1^*(\mathbf{a}, \mathbf{a}) + A_2^*(\mathbf{a}, \mathbf{a}) + A_{f_{33}}^*(\mathbf{a}, \mathbf{a}) + A_{g_{44}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)\}, \\ (3.1g) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_3^*(\mathbf{a}, \mathbf{a}) + A_4^*(\mathbf{a}, \mathbf{a}) + A_{g_{33}}^*(\mathbf{a}, \mathbf{a}) + A_{g_{44}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})\}, \\ (3.1h) \quad & A_0^*(\mathbf{a}, \mathbf{a}) + A_5^*(\mathbf{a}, \mathbf{a}) + A_{f_{11}}^*(\mathbf{a}, \mathbf{a}) + A_{g_{11}}^*(\mathbf{a}, \mathbf{a}) = I_{\nu(\mathbf{a})} && \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_j)\}, \\ (3.2a) \quad & A_\beta^*(\mathbf{a}, \mathbf{c}) A_\beta^*(\mathbf{c}, \mathbf{b}) = \delta_{\beta\gamma} A_\beta^*(\mathbf{a}, \mathbf{b}), \\ (3.2b) \quad & A_{f_{uw}}^*(\mathbf{u}, \mathbf{w}) A_{f_{wv}}^*(\mathbf{w}, \mathbf{v}) = \delta_{wv} A_{f_{uv}}^*(\mathbf{u}, \mathbf{v}), \\ (3.2c) \quad & A_{g_{xz}}^*(\mathbf{x}, \mathbf{z}) A_{g_{zy}}^*(\mathbf{z}, \mathbf{y}) = \delta_{zy} A_{g_{xy}}^*(\mathbf{x}, \mathbf{y}), \\ (3.2d) \quad & A_\beta^*(\mathbf{a}, \mathbf{c}) A_{f_{cv}}^*(\mathbf{c}, \mathbf{v}) = A_{f_{ac}}^*(\mathbf{a}, \mathbf{c}) A_\beta^*(\mathbf{c}, \mathbf{v}) = A_\beta^*(\mathbf{a}, \mathbf{d}) A_{g_{dv}}^*(\mathbf{d}, \mathbf{v}) \\ & = A_{g_{ad}}^*(\mathbf{a}, \mathbf{d}) A_\beta^*(\mathbf{d}, \mathbf{v}) = A_{f_{aw}}^*(\mathbf{a}, \mathbf{w}) A_{g_{wv}}^*(\mathbf{w}, \mathbf{v}) \\ & = A_{g_{aw}}^*(\mathbf{a}, \mathbf{w}) A_{f_{wv}}^*(\mathbf{w}, \mathbf{v}) = O_{\nu(\mathbf{a}) \times \nu(\mathbf{v})}, \\ (3.3a) \quad & \text{rank}[A_\beta^*(\mathbf{a}, \mathbf{b})] = \phi_\beta && \text{for } \beta = 0, 1, \dots, 5, \\ (3.3b) \quad & \text{rank}[A_{f_{uv}}^*(\mathbf{u}, \mathbf{v})] = \phi_f \end{aligned}$$

and

$$(3.3c) \quad \text{rank}[A_{g_{xy}}^*(\mathbf{x}, \mathbf{y})] = \phi_g,$$

where $G_{p \times q}$ and $O_{p \times q}$ are the $p \times q$ matrices with all unity and with all zero, respectively, and δ_{pq} is the Kronecker delta. Here

$$\begin{aligned} \nu(\mathbf{a}) &= 1 && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_0)\}, \\ &= m && \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_0)\}, \\ &= n && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_j)\}, \end{aligned}$$

$$\begin{aligned}
&= \binom{m}{2} && \text{if } \mathbf{a} \in \{(\mathbf{a}_{ii}; \mathbf{b}_0)\}, \\
&= \binom{n}{2} && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{jj})\}, \\
&= 2 \binom{m}{2} && \text{if } \mathbf{a} \in \{(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)\}, \\
&= 2 \binom{n}{2} && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})\}, \\
&= mn && \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_j)\}.
\end{aligned}$$

It follows from (3.1a) through (3.1h) that

$$\begin{aligned}
\boldsymbol{\theta}(\mathbf{a}) &= A_0^*(\mathbf{a}, \mathbf{a})\boldsymbol{\theta}(\mathbf{a}) && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_0)\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_{j_{11}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) && \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_0)\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_{i_{11}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_j)\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_1^*(\mathbf{a}, \mathbf{a}) + A_{j_{22}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) && \text{if } \mathbf{a} \in \{(\mathbf{a}_{ii}; \mathbf{b}_0)\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_2^*(\mathbf{a}, \mathbf{a}) + A_{i_{22}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) && \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{jj})\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_1^*(\mathbf{a}, \mathbf{a}) + A_2^*(\mathbf{a}, \mathbf{a}) + A_{j_{33}}^*(\mathbf{a}, \mathbf{a}) + A_{i_{44}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) \\
&&& \text{if } \mathbf{a} \in \{(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_3^*(\mathbf{a}, \mathbf{a}) + A_4^*(\mathbf{a}, \mathbf{a}) + A_{j_{33}}^*(\mathbf{a}, \mathbf{a}) + A_{i_{44}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) \\
&&& \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})\}, \\
&= \{A_0^*(\mathbf{a}, \mathbf{a}) + A_5^*(\mathbf{a}, \mathbf{a}) + A_{j_{11}}^*(\mathbf{a}, \mathbf{a}) + A_{i_{11}}^*(\mathbf{a}, \mathbf{a})\}\boldsymbol{\theta}(\mathbf{a}) \\
&&& \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_j)\}.
\end{aligned}$$

Thus the linear model (2.1) is rewritten as

$$\begin{aligned}
\mathbf{y}(T) &= \sum_{\beta=0}^5 \sum_{\mathbf{a} \in Z_\beta^*} E(\mathbf{a}) A_\beta^*(\mathbf{a}, \mathbf{a})\boldsymbol{\theta}(\mathbf{a}) + \sum_{(\mathbf{u}; \mathbf{u}) \in Z_f^*} E(\mathbf{u}) A_{f_{uu}}^*(\mathbf{u}, \mathbf{u})\boldsymbol{\theta}(\mathbf{u}) \\
&\quad + \sum_{(\mathbf{x}; \mathbf{x}) \in Z_g^*} E(\mathbf{x}) A_{g_{xx}}^*(\mathbf{x}, \mathbf{x})\boldsymbol{\theta}(\mathbf{x}) + \mathbf{e}r,
\end{aligned}$$

where $E(\mathbf{a})$ are $N \times \nu(\mathbf{a})$ submatrices of E_T corresponding to $\boldsymbol{\theta}(\mathbf{a})$ for $\mathbf{a} \in Z_0^*$. The (3.1a), (3.2) and (3.3) show that (I) every element of the vector $A_\beta^*(\mathbf{a}, \mathbf{a})\boldsymbol{\theta}(\mathbf{a})$ represents the average of $\boldsymbol{\theta}(\mathbf{a})$, (II) the elements of $A_\beta^*(\mathbf{a}, \mathbf{a})\boldsymbol{\theta}(\mathbf{a})$ ($1 \leq \beta \leq 5$), $A_{f_{uu}}^*(\mathbf{u}, \mathbf{u})\boldsymbol{\theta}(\mathbf{u})$ and $A_{g_{xx}}^*(\mathbf{x}, \mathbf{x})\boldsymbol{\theta}(\mathbf{x})$ represent the contrasts of the elements of $\boldsymbol{\theta}(\mathbf{a})$, $\boldsymbol{\theta}(\mathbf{u})$ and $\boldsymbol{\theta}(\mathbf{x})$, respectively, and any two contrasts are orthogonal, and (III) there exist ϕ_β ($0 \leq \beta \leq 5$), ϕ_f and ϕ_g independent parametric functions of $\boldsymbol{\theta}(\mathbf{a})$ in $A_\beta^*(\mathbf{a}, \mathbf{a})\boldsymbol{\theta}(\mathbf{a})$, of $\boldsymbol{\theta}(\mathbf{u})$ in $A_{f_{uu}}^*(\mathbf{u}, \mathbf{u})\boldsymbol{\theta}(\mathbf{u})$ and of $\boldsymbol{\theta}(\mathbf{x})$ in $A_{g_{xx}}^*(\mathbf{x}, \mathbf{x})\boldsymbol{\theta}(\mathbf{x})$, respectively.

Let

$$\begin{aligned}
F_\beta(\mathbf{a}, \mathbf{b}) &= E(\mathbf{a}) A_\beta^*(\mathbf{a}, \mathbf{b}) E(\mathbf{b})' && \text{for } \mathbf{a}, \mathbf{b} \in Z_\beta^* \ (0 \leq \beta \leq 5), \\
F_{f_{uv}}(\mathbf{u}, \mathbf{v}) &= E(\mathbf{u}) A_{f_{uv}}^*(\mathbf{u}, \mathbf{v}) E(\mathbf{v})' && \text{for } (\mathbf{u}; \mathbf{u}), (\mathbf{v}; \mathbf{v}) \in Z_f^*
\end{aligned}$$

and

$$F_{g_{xy}}(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}) A_{g_{xy}}^*(\mathbf{x}, \mathbf{y}) E(\mathbf{y})' \quad \text{for } (\mathbf{x}; \mathbf{x}), (\mathbf{y}; \mathbf{y}) \in Z_g^*.$$

Then the following can be proved easily (e.g., Hyodo and Kuwada [6]):

LEMMA 3.1.

$$\begin{aligned} F_{\beta}(\mathbf{a}, \mathbf{c})F_{\tau}(\mathbf{d}, \mathbf{b}) &= \delta_{\beta\tau} k_{\beta}(\mathbf{c}, \mathbf{d})F_{\beta}(\mathbf{a}, \mathbf{b}), \\ F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{w})F_{\mathcal{J}_{pv}}(\mathbf{p}, \mathbf{v}) &= k_{\mathcal{J}_{vp}}(\mathbf{w}, \mathbf{p})F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{v}), \\ F_{\sigma_{xz}}(\mathbf{x}, \mathbf{z})F_{\sigma_{qy}}(\mathbf{q}, \mathbf{y}) &= k_{\sigma_{zq}}(\mathbf{z}, \mathbf{q})F_{\sigma_{xy}}(\mathbf{x}, \mathbf{y}) \end{aligned}$$

and

$$\begin{aligned} F_{\beta}(\mathbf{a}, \mathbf{b})F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{v}) &= F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{v})F_{\beta}(\mathbf{a}, \mathbf{b}) = F_{\beta}(\mathbf{a}, \mathbf{b})F_{\sigma_{xy}}(\mathbf{x}, \mathbf{y}) \\ &= F_{\sigma_{xy}}(\mathbf{x}, \mathbf{y})F_{\beta}(\mathbf{a}, \mathbf{b}) = F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{v})F_{\sigma_{xy}}(\mathbf{x}, \mathbf{y}) = F_{\sigma_{xy}}(\mathbf{x}, \mathbf{y})F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{v}) \\ &= \mathbf{0}_{N \times N}. \end{aligned}$$

Let $K_{\beta}^*(t)^{-1} = ||\eta_{\beta}^t(\mathbf{a}, \mathbf{b})||$ ($0 \leq \beta \leq 5$), $K_{\mathcal{J}}^*(p)^{-1} = ||\eta_{\mathcal{J}_{uv}}^p(\mathbf{u}, \mathbf{v})||$ and $K_{\sigma}^*(q)^{-1} = ||\eta_{\sigma_{xy}}^q(\mathbf{x}, \mathbf{y})||$, where $K_{\beta}^*(t)$, $K_{\mathcal{J}}^*(p)$ and $K_{\sigma}^*(q)$ are the first $(t+1) \times (t+1)$, $(p+1) \times (p+1)$ and $(q+1) \times (q+1)$ submatrices of K_{β}^* , $K_{\mathcal{J}}^*$ and K_{σ}^* which are given by permuting the rows and columns of K_{β} , $K_{\mathcal{J}}$ and K_{σ} according as Z_{β}^* , $Z_{\mathcal{J}}^*$ and Z_{σ}^* , respectively, and $0 \leq t \leq \tau_{\beta}-1$, $0 \leq p \leq \tau_{\mathcal{J}}-1$ and $0 \leq q \leq \tau_{\sigma}-1$. Further, let

$$\begin{aligned} P_{\beta}^t &= \sum_{k=0}^{t-1} \sum_{\mathbf{a}, \mathbf{b} \in Z_{\beta}^{t-k}} (-1)^k \eta_{\beta}^{t-k}(\mathbf{a}, \mathbf{b}) F_{\beta}(\mathbf{a}, \mathbf{b}), \\ P_{\mathcal{J}}^p &= \sum_{k=0}^{p-1} \sum_{(\mathbf{u}:\mathbf{u}), (\mathbf{v}:\mathbf{v}) \in Z_{\mathcal{J}}^{p-k}} (-1)^k \eta_{\mathcal{J}_{uv}}^{p-k}(\mathbf{u}, \mathbf{v}) F_{\mathcal{J}_{uv}}(\mathbf{u}, \mathbf{v}) \end{aligned}$$

and

$$P_{\sigma}^q = \sum_{k=0}^{q-1} \sum_{(\mathbf{x}:\mathbf{x}), (\mathbf{y}:\mathbf{y}) \in Z_{\sigma}^{q-k}} (-1)^k \eta_{\sigma_{xy}}^{q-k}(\mathbf{x}, \mathbf{y}) F_{\sigma_{xy}}(\mathbf{x}, \mathbf{y}),$$

where $\eta_{\beta}^{-1}(\mathbf{a}, \mathbf{b}) = \eta_{\mathcal{J}_{uv}}^{-1}(\mathbf{u}, \mathbf{v}) = \eta_{\sigma_{xy}}^{-1}(\mathbf{x}, \mathbf{y}) = 0$, and Z_{β}^{*t} , $Z_{\mathcal{J}}^{*p}$ and Z_{σ}^{*q} are the subsequences of Z_{β}^* , $Z_{\mathcal{J}}^*$ and Z_{σ}^* which are composed of the first $(t+1)$ elements of Z_{β}^* , of the first $(p+1)$ ones of $Z_{\mathcal{J}}^*$ and of the first $(q+1)$ ones of Z_{σ}^* , respectively. Note that $Z_{\beta}^{*\tau_{\beta}-1} = Z_{\beta}^*$ ($0 \leq \beta \leq 5$), $Z_{\mathcal{J}}^{*\tau_{\mathcal{J}}-1} = Z_{\mathcal{J}}^*$ and $Z_{\sigma}^{*\tau_{\sigma}-1} = Z_{\sigma}^*$. Then the following can be proved along the lines similar to Lemma 3.4 of Hyodo and Kuwada [6].

LEMMA 3.2. (i) The P_{β}^t ($0 \leq t \leq \tau_{\beta}-1$, $0 \leq \beta \leq 5$), $P_{\mathcal{J}}^p$ ($0 \leq p \leq \tau_{\mathcal{J}}-1$), P_{σ}^q ($0 \leq q \leq \tau_{\sigma}-1$) and P_e are symmetric and mutually orthogonal idempotent matrices, where $P_e = I_N - \sum_{\beta=0}^5 \sum_{t=0}^{\tau_{\beta}-1} P_{\beta}^t - \sum_{p=0}^{\tau_{\mathcal{J}}-1} P_{\mathcal{J}}^p - \sum_{q=0}^{\tau_{\sigma}-1} P_{\sigma}^q$.

(ii) $\text{rank}[P_{\beta}^t] = \phi_{\beta}$, $\text{rank}[P_{\mathcal{J}}^p] = \phi_{\mathcal{J}}$, $\text{rank}[P_{\sigma}^q] = \phi_{\sigma}$ and $\text{rank}[P_e] = N - \nu(m, n)$.

Let

$$\begin{aligned} R_{\beta}^t &= R_{L(t-1; \beta) \perp} L(t; \beta) \quad (0 \leq \beta \leq 5), \quad R_{\mathcal{J}}^p = R_{L(p-1; \mathcal{J}) \perp} L(p; \mathcal{J}), \\ R_{\sigma}^q &= R_{L(q-1; \sigma) \perp} L(q; \sigma) \quad \text{and} \quad R_e = R_{E_T^{\perp}}, \end{aligned}$$

where $L(t; \beta)$ are the matrices obtained by the first $(t+1)$ submatrices $E(\mathbf{a})A_{\beta}^{\dagger}(\mathbf{a}, \mathbf{a})$ of the matrix $[E(\mathbf{a})A_{\beta}^{\dagger}(\mathbf{a}, \mathbf{a}) | \mathbf{a} \in Z_{\beta}^*]$ ($0 \leq \beta \leq 5$), $L(p; \mathcal{J})$ are the first $(p+1)$ ones of $[E(\mathbf{u})A_{\mathcal{J}_{uv}}^{\dagger}(\mathbf{u}, \mathbf{u}) | (\mathbf{u}:\mathbf{u}) \in Z_{\mathcal{J}}^*]$ and $L(q; \sigma)$ are the first $(q+1)$ ones of $[E(\mathbf{x})A_{\sigma_{xy}}^{\dagger}(\mathbf{x}, \mathbf{x}) | (\mathbf{x}:\mathbf{x}) \in Z_{\sigma}^*]$. Note that $L(-1; \beta) = L(-1; \mathcal{J}) = L(-1; \sigma) = \mathbf{0}_N$. Here $R_{A^{\perp}}(B)$ is the orthocomplement subspace of $R(A)$, which is the linear subspace spanned by the column vectors of a matrix A , relative to $R(B)$ for the case $R(A) \subset R(B)$, and $R_{C^{\perp}}$ is the orthocomplement subspace of $R(C)$ relative to the N -dimensional vector space R^N . Then from Lemma 3.2, we have the following:

THEOREM 3.1.

$$R^N = \bigoplus_{\beta=0}^5 \bigoplus_{t=0}^{t_{\beta}-1} R_{\beta}^t \bigoplus_{p=0}^{p_{\beta}-1} R_{\beta}^p \bigoplus_{q=0}^{q_{\beta}-1} R_{\beta}^q \bigoplus R_e,$$

where $\bigoplus$ denotes the direct sum.

Note that P_{β}^t ($0 \leq t \leq \tau_{\beta}-1$; $0 \leq \beta \leq 5$), P_{β}^p ($0 \leq p \leq \tau_{\beta}-1$) P_{β}^q ($0 \leq q \leq \tau_{\beta}-1$) and P_e are the projections of R^N onto R_{β}^t , R_{β}^p , R_{β}^q and R_e , respectively.

4. ANOVA and hypothesis testing

Let $S_{\beta}^t = \mathbf{y}(T)' P_{\beta}^t \mathbf{y}(T)$ ($0 \leq \beta \leq 5$), $S_{\beta}^p = \mathbf{y}(T)' P_{\beta}^p \mathbf{y}(T)$, $S_{\beta}^q = \mathbf{y}(T)' P_{\beta}^q \mathbf{y}(T)$ and $S_e = \mathbf{y}(T)' P_e \mathbf{y}(T)$, which correspond to the sum of squares (SS) due to $A_{\beta}^t(\mathbf{a}, \mathbf{a})\theta(\mathbf{a})$, $A_{\beta}^p(\mathbf{u}, \mathbf{u})\theta(\mathbf{u})$, $A_{\beta}^q(\mathbf{x}, \mathbf{x})\theta(\mathbf{x})$ and an error term, respectively. Here $\mathbf{a}$, $(\mathbf{u}: \mathbf{u})$ and $(\mathbf{x}: \mathbf{x})$ are the first $\{t+1\}$ th element of Z_{β}^t , the first $\{p+1\}$ th one of Z_{β}^p and the first $\{q+1\}$ th one of Z_{β}^q , respectively. Furthermore we assume that $N > \nu(m, n)$. Then Theorem 3.1 yields the following:

THEOREM 4.1.

$$\mathbf{y}(T)' \mathbf{y}(T) = \sum_{\beta=0}^5 \sum_{t=0}^{t_{\beta}-1} S_{\beta}^t + \sum_{p=0}^{p_{\beta}-1} S_{\beta}^p + \sum_{q=0}^{q_{\beta}-1} S_{\beta}^q + S_e.$$

THEOREM 4.2. An unbiased estimator of σ^2 is given by

$$\hat{\sigma}^2 = S_e / \{N - \nu(m, n)\}.$$

Let $Z_{\beta}^{*(t)}$ ($0 \leq \beta \leq 5$), $Z_{\beta}^{*(p)}$ and $Z_{\beta}^{*(q)}$ be the subsequences of Z_{β}^t , Z_{β}^p and Z_{β}^q which are composed of the $\{t+1\}$ th to τ_{β} th elements of Z_{β}^t , of the $\{p+1\}$ th to τ_{β} th ones of Z_{β}^p , and of the $\{q+1\}$ th to τ_{β} th ones of Z_{β}^q , respectively. The noncentrality parameters $\lambda_{\beta}^t/\sigma^2$ ($0 \leq \beta \leq 5$), $\lambda_{\beta}^p/\sigma^2$ and $\lambda_{\beta}^q/\sigma^2$, of the quadratic forms S_{β}^t/σ^2 , S_{β}^p/σ^2 and S_{β}^q/σ^2 are defined by $E[\mathbf{y}(T)'] P_{\beta}^t E[\mathbf{y}(T)]/\sigma^2$, $E[\mathbf{y}(T)'] P_{\beta}^p \times E[\mathbf{y}(T)]/\sigma^2$ and $E[\mathbf{y}(T)'] P_{\beta}^q E[\mathbf{y}(T)]/\sigma^2$, respectively, where $E[\mathbf{y}]$ denotes the expected value of a random vector $\mathbf{y}$. From the definitions of η 's, we get $\sum_{\mathbf{a} \in Z_{\beta}^{*(t)}} \eta_{\beta}^t(\mathbf{b}, \mathbf{c}) k_{\beta}(\mathbf{a}, \mathbf{b}) = \delta_{\mathbf{ac}}$ if $\mathbf{a} \in Z_{\beta}^{*(t)}$, $\sum_{(\mathbf{v}: \mathbf{v}) \in Z_{\beta}^{*(p)}} \eta_{\beta}^p(\mathbf{v}, \mathbf{w}) k_{\beta}(\mathbf{u}, \mathbf{v}) = \delta_{\mathbf{uw}}$ if $(\mathbf{u}: \mathbf{u}) \in Z_{\beta}^{*(p)}$ and $\sum_{(\mathbf{v}: \mathbf{v}) \in Z_{\beta}^{*(q)}} \eta_{\beta}^q(\mathbf{y}, \mathbf{z}) k_{\beta}(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{xz}}$ if $(\mathbf{x}: \mathbf{x}) \in Z_{\beta}^{*(q)}$, where $\delta_{\mathbf{ab}} = 1$ or 0 according as $\mathbf{a} = \mathbf{b}$ or not. Thus we can obtain the following:

THEOREM 4.3. The noncentrality parameters of the quadratic forms S_{β}^t/σ^2 ($0 \leq t \leq \tau_{\beta}-1$, $0 \leq \beta \leq 5$), S_{β}^p/σ^2 ($0 \leq p \leq \tau_{\beta}-1$) and S_{β}^q/σ^2 ($0 \leq q \leq \tau_{\beta}-1$) are given by

$$\lambda_{\beta}^t/\sigma^2 = \sum_{\mathbf{a}, \mathbf{b} \in Z_{\beta}^{*(t)}} \{c_{\beta}^t(\mathbf{a}, \mathbf{b})/\sigma^2\} \theta(\mathbf{a})' A_{\beta}^t(\mathbf{a}, \mathbf{b}) \theta(\mathbf{b}) \quad \text{for } 0 \leq \beta \leq 5,$$

$$\lambda_{\beta}^p/\sigma^2 = \sum_{(\mathbf{u}: \mathbf{u}), (\mathbf{v}: \mathbf{v}) \in Z_{\beta}^{*(p)}} \{c_{\beta}^p(\mathbf{u}: \mathbf{u}, \mathbf{v}: \mathbf{v})/\sigma^2\} \theta(\mathbf{u})' A_{\beta}^p(\mathbf{u}, \mathbf{v}) \theta(\mathbf{v})$$

and

$$\lambda_{\beta}^q/\sigma^2 = \sum_{(\mathbf{x}: \mathbf{x}), (\mathbf{y}: \mathbf{y}) \in Z_{\beta}^{*(q)}} \{c_{\beta}^q(\mathbf{x}: \mathbf{x}, \mathbf{y}: \mathbf{y})/\sigma^2\} \theta(\mathbf{x})' A_{\beta}^q(\mathbf{x}, \mathbf{y}) \theta(\mathbf{y}),$$

respectively, where

$$c_{\beta}^t(\mathbf{a}, \mathbf{b}) = \sum_{k=0}^1 \sum_{i, j \in Z_{\beta}^{*(t)-k}} (-1)^k \eta_{\beta}^{t-k}(\mathbf{i}, \mathbf{j}) k_{\beta}(\mathbf{a}, \mathbf{i}) k_{\beta}(\mathbf{j}, \mathbf{b})$$

$$= 0$$

$$c_{\beta}^p(\mathbf{u}: \mathbf{u}, \mathbf{v}: \mathbf{v}) = \sum_{k=0}^1 \sum_{(i: i), (j: j) \in Z_{\beta}^{*(p)-k}} (-1)^k \eta_{\beta}^{p-k}(\mathbf{i}, \mathbf{j}) k_{\beta}(\mathbf{u}, \mathbf{i}) k_{\beta}(\mathbf{j}, \mathbf{v})$$

$$= 0$$

and

$$c_{ij}^q(\mathbf{x}: \mathbf{x}, \mathbf{y}: \mathbf{y}) = \sum_{k=0}^1 \sum_{(i:i), (j:j) \in Z_{ij}^{q-k}} (-1)^k \eta_{ij}^{q-k}(\mathbf{i}, \mathbf{j}) k_{\sigma_{xi}}(\mathbf{x}, \mathbf{i}) k_{\sigma_{jy}}(\mathbf{j}, \mathbf{y})$$

$$= 0 \quad \text{if } (\mathbf{x}: \mathbf{x}), (\mathbf{y}: \mathbf{y}) \in Z_{ij}^{*(q)},$$

$$\quad \text{otherwise.}$$

The ANOVA of an $r^m \times s^n$ -BFF design of resolution V derived from an AB-array is summarized in Table below:

Table. ANOVA of $r^m \times s^n$ -BFF designs of resolution V

Source	SS	d.f.	Noncentrality parameters
$A_{\beta}^{\#}(\mathbf{a}, \mathbf{a})\theta(\mathbf{a})$	S_{β}^i	ϕ_{β}	$\lambda_{\beta}^i/\sigma^2$
$A_{f_{uu}}^{\#}(\mathbf{u}, \mathbf{u})\theta(\mathbf{u})$	S_f^p	ϕ_f	λ_f^p/σ^2
$A_{g_{xx}}^{\#}(\mathbf{x}, \mathbf{x})\theta(\mathbf{x})$	S_g^q	ϕ_g	λ_g^q/σ^2
Error	S_e	$N - \nu(m, n)$	
Total	$\mathbf{y}(T)' \mathbf{y}(T)$	N	

$\mathbf{a}$: the $\{t+1\}$ th element of Z_{β}^* ($0 \leq t \leq \tau_{\beta}-1$, $0 \leq \beta \leq 5$)

$(\mathbf{u}: \mathbf{u})$: the $\{p+1\}$ th element of Z_f^* ($0 \leq p \leq \tau_f-1$)

$(\mathbf{x}: \mathbf{x})$: the $\{q+1\}$ th element of Z_g^* ($0 \leq q \leq \tau_g-1$)

Consider the hypotheses $\cap_{\mathbf{a} \in Z_{\beta}^{*(t)}} H_{\beta}^{\mathbf{a}} (=H_{\beta}^{(t)}$, say), $\cap_{\mathbf{a} \in Z_{\beta}^{*(t)}} H_{\beta}^{\mathbf{a}} (=H_{\beta}^{(t)}$, say) (if $m \geq 4$), $\cap_{\mathbf{a} \in Z_{\beta}^{*(t)}} H_{\beta}^{\mathbf{a}} (=H_{\beta}^{(t)}$, say) (if $m \geq 3$), $\cap_{\mathbf{a} \in Z_{\beta}^{*(t)}} H_{\beta}^{\mathbf{a}} (=H_{\beta}^{(t)}$, say) (if $n \geq 4$), $\cap_{\mathbf{a} \in Z_{\beta}^{*(t)}} H_{\beta}^{\mathbf{a}} (=H_{\beta}^{(t)}$, say) (if $n \geq 3$), $\cap_{\mathbf{a} \in Z_{\beta}^{*(t)}} H_{\beta}^{\mathbf{a}} (=H_{\beta}^{(t)}$, say), $\cap_{(\mathbf{u}: \mathbf{u}) \in Z_f^{*(p)}} H_f^{\mathbf{u}} (=H_f^{(p)}$, say) and $\cap_{(\mathbf{x}: \mathbf{x}) \in Z_g^{*(q)}} H_g^{\mathbf{x}} (=H_g^{(q)}$, say), where $H_{\beta}^{\mathbf{a}}$ ($0 \leq \beta \leq 5$), $H_f^{\mathbf{u}}$ and $H_g^{\mathbf{x}}$ are the hypotheses such that $A_{\beta}^{\#}(\mathbf{a}, \mathbf{a})\theta(\mathbf{a}) = \mathbf{0}_{\nu(\mathbf{a})}$, $A_{f_{uu}}^{\#}(\mathbf{u}, \mathbf{u})\theta(\mathbf{u}) = \mathbf{0}_{\nu(\mathbf{u})}$ and $A_{g_{xx}}^{\#}(\mathbf{x}, \mathbf{x})\theta(\mathbf{x}) = \mathbf{0}_{\nu(\mathbf{x})}$, respectively. First we are interested in testing the hypotheses such that $H_{\beta}^{(\tau_{\beta}-1)}$ against $H_{\beta}^{\mathbf{a}}$ ($0 \leq \beta \leq 5$), $H_f^{(\tau_f-1)}$ against $H_f^{\mathbf{u}}$ and $H_g^{(\tau_g-1)}$ against $H_g^{\mathbf{x}}$, where $H_{\beta}^{\mathbf{a}}$, $H_f^{\mathbf{u}}$ and $H_g^{\mathbf{x}}$ are the hypotheses such that $A_{\beta}^{\#}(\mathbf{a}, \mathbf{a})\theta(\mathbf{a}) \neq \mathbf{0}_{\nu(\mathbf{a})}$ for $\mathbf{a}$ being the last element of Z_{β}^* , $A_{f_{uu}}^{\#}(\mathbf{u}, \mathbf{u})\theta(\mathbf{u}) \neq \mathbf{0}_{\nu(\mathbf{u})}$ for $(\mathbf{u}: \mathbf{u})$ being the last one of Z_f^* and $A_{g_{xx}}^{\#}(\mathbf{x}, \mathbf{x})\theta(\mathbf{x}) \neq \mathbf{0}_{\nu(\mathbf{x})}$ for $(\mathbf{x}: \mathbf{x})$ being the last one of Z_g^* , respectively. Next, if $H_{\beta}^{(\tau_{\beta}-1)}$ ($H_f^{(\tau_f-1)}$ or $H_g^{(\tau_g-1)}$) is accepted, then consider the hypotheses that $H_{\beta}^{(\tau_{\beta}-2)}$ against $H_{\beta}^{(\tau_{\beta}-1)}$ ($H_f^{(\tau_f-2)}$ against $H_f^{(\tau_f-1)}$ or $H_g^{(\tau_g-2)}$ against $H_g^{(\tau_g-1)}$), and so on. This method is the so-called nested test procedure (e.g., Anderson [1] and Seber [19]). This means that the hypothesis $H_{\beta}^{(t)}$ against $H_{\beta}^{\mathbf{a}}$ ($H_f^{(p)}$ against $H_f^{\mathbf{u}}$ or $H_g^{(q)}$ against $H_g^{\mathbf{x}}$) is accepted only if all the tests $H_{\beta}^{(\tau_{\beta}-1)}$ against $H_{\beta}^{\mathbf{a}}$, $H_{\beta}^{(\tau_{\beta}-2)}$ against $H_{\beta}^{(\tau_{\beta}-1)}$, $\dots$, $H_{\beta}^{(t)}$ against $H_{\beta}^{(t+1)}$ ($H_f^{(\tau_f-1)}$ against $H_f^{\mathbf{u}}$, $H_f^{(\tau_f-2)}$ against $H_f^{(\tau_f-1)}$, $\dots$, $H_f^{(p)}$ against $H_f^{(p+1)}$ or $H_g^{(\tau_g-1)}$ against $H_g^{\mathbf{x}}$, $H_g^{(\tau_g-2)}$ against $H_g^{(\tau_g-1)}$, $\dots$, $H_g^{(q)}$ against $H_g^{(q+1)}$) are not significant. Note that Theorem 4.3 implies that $H_{\beta}^{(t)}$ ($H_f^{(p)}$ or $H_g^{(q)}$) is accepted if and only if $\lambda_{\beta}^i = 0$ ($\lambda_f^p = 0$ or $\lambda_g^q = 0$). The test statistics for the nested method are given by

$$\frac{S_{\beta}^i/\phi_{\beta}}{\{S_e + S_{\beta}^{(t)}\}/\{N - \nu(m, n) + (\tau_{\beta} - 1 - t)\phi_{\beta}\}} (=F_{\beta}^i, \text{ say}) \quad \text{for } \beta = 0, 1, \dots, 5,$$

$$\frac{S_f^p/\phi_f}{\{S_e + S_f^{(p)}\}/\{N - \nu(m, n) + (\tau_f - 1 - p)\phi_f\}} (= F_f^p, \text{ say})$$

and

$$\frac{S_g^q/\phi_g}{\{S_e + S_g^{(q)}\}/\{N - \nu(m, n) + (\tau_g - 1 - q)\phi_g\}} (= F_g^q, \text{ say}),$$

where $S_\beta^{(i)} = \sum_{i=t+1}^{t+p-1} S_\beta^i$, $S_f^{(p)} = \sum_{j=p+1}^{t-1} S_f^j$ and $S_g^{(q)} = \sum_{k=q+1}^{t-1} S_g^k$. The nested procedure is continued until a significant test is obtained for each β ($0 \leq \beta \leq 5$), f and g . Note that F_β^i , F_f^p and F_g^q have all central or noncentral F distributions with ϕ_β and $\{N - \nu(m, n) + (\tau_\beta - 1 - t)\phi_\beta\}$ d.f. and λ_β^i/σ^2 , with ϕ_f and $\{N - \nu(m, n) + (\tau_f - 1 - p)\phi_f\}$ d.f. and λ_f^p/σ^2 and with ϕ_g and $\{N - \nu(m, n) + (\tau_g - 1 - q)\phi_g\}$ d.f. and λ_g^q/σ^2 depending on which of the $H_\beta^{(i)}$, $H_f^{(p)}$ and $H_g^{(q)}$ are true, respectively.

Finally we consider the hypotheses $\cap_{\mathbf{a} \in Z_0^{(h)}} H^{\mathbf{a}} (= H^{(h)}, \text{ say})$, where $H^{\mathbf{a}}$ are the hypotheses that every elementary contrast of $\boldsymbol{\theta}(\mathbf{a})$ is zero, i.e.,

$$\begin{aligned} H^{\mathbf{a}} &= H_3^{\mathbf{a}} \cap H_f^{\mathbf{a}:1} \cap H_g^{\mathbf{a}:1} & \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_j)\}, \\ &= H_3^{\mathbf{a}} \cap H_4^{\mathbf{a}} \cap H_g^{\mathbf{a}:3} \cap H_g^{\mathbf{a}:4} & \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{j_1 j_2})\}, \\ &= H_1^{\mathbf{a}} \cap H_2^{\mathbf{a}} \cap H_f^{\mathbf{a}:3} \cap H_f^{\mathbf{a}:4} & \text{if } \mathbf{a} \in \{(\mathbf{a}_{i_1 i_2}; \mathbf{b}_0)\}, \\ &= H_3^{\mathbf{a}} \cap H_g^{\mathbf{a}:2} & \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_{jj})\}, \\ &= H_1^{\mathbf{a}} \cap H_f^{\mathbf{a}:2} & \text{if } \mathbf{a} \in \{(\mathbf{a}_{ii}; \mathbf{b}_0)\}, \\ &= H_g^{\mathbf{a}:1} & \text{if } \mathbf{a} \in \{(\mathbf{a}_0; \mathbf{b}_j)\}, \\ &= H_f^{\mathbf{a}:1} & \text{if } \mathbf{a} \in \{(\mathbf{a}_i; \mathbf{b}_0)\}. \end{aligned}$$

Note that the $H^{\mathbf{a}}$ are equivalent to $[I_{\nu(\mathbf{a})} - \{1/\nu(\mathbf{a})\}G_{\nu(\mathbf{a}) \times \nu(\mathbf{a})}]\boldsymbol{\theta}(\mathbf{a}) = \mathbf{0}_{\nu(\mathbf{a})}$. Consider the hypothesis $H^{(\tau_0-1)}$ against H^A , where H^A is the hypothesis that at least one of the elementary contrasts of $\boldsymbol{\theta}(\mathbf{a}_{\tau-1}; \mathbf{b}_{s-1})$ is not zero. Next if $H^{(\tau_0-1)}$ is accepted, then consider $H^{(\tau_0-2)}$ against $H^{(\tau_0-1)}$, and so on. This is also the nested test procedure. For this method, the test statistics are given by

$$\frac{S^h/\phi^h}{\{S_e + S^{(h)}\}/\{N - \nu(m, n) + \xi(h)\}} (= F^h, \text{ say}) \quad \text{for } h=1, 2, \dots, \tau_0-1,$$

where

$$\begin{aligned} S^h &= S_3^{h-2(r+s)+3-\binom{r-1}{2}-\binom{s-1}{2}} + S_f^{h-2s+1+\binom{r-1}{2}-\binom{s-1}{2}} + S_g^{h-2r+1-\binom{r-1}{2}+\binom{s-1}{2}} \\ &\quad \text{if } 2(r+s)-3+\binom{r-1}{2}+\binom{s-1}{2} \leq h \leq \tau_0-1, \\ &= S_3^{h-2r-s+2-\binom{r-1}{2}} + S_4^{h-2(r+s)+3-\binom{r-1}{2}} + S_g^{2h-2(2r+s)+4-2\binom{r-1}{2}} + S_g^{2h-2(2r+s)+5-2\binom{r-1}{2}} \\ &\quad \text{if } 2(r+s)-3+\binom{r-1}{2} \leq h \leq 2(r+s)-4+\binom{r-1}{2}+\binom{s-1}{2}, \\ &= S_1^{h-r-2s+2} + S_2^{h-2(r+s)+3} + S_f^{2h-2(r+2s)+4} + S_f^{2h-2(r+2s)+5} \\ &\quad \text{if } 2(r+s)-3 \leq h \leq 2(r+s)-4+\binom{r-1}{2}, \\ &= S_3^{h-2r-s+2} + S_g^{h-2r+1} \quad \text{if } 2r+s-2 \leq h \leq 2(r+s)-4, \end{aligned}$$

$$\begin{aligned}
&= S_1^{h-r-s+1} + S_f^{h-s} && \text{if } r+s-1 \leq h \leq 2r+s-3, \\
&= S_o^{h-r} && \text{if } r \leq h \leq r+s-2, \\
&= S_f^{h-1} && \text{if } 1 \leq h \leq r-1, \\
S(h) &= S_o^{(h-2(r+s)+3-\binom{r-1}{2}-\binom{s-1}{2})} + S_f^{(h-2s+1+\binom{r-1}{2}-\binom{s-1}{2})} + S_o^{(h-2r+1-\binom{r-1}{2}+\binom{s-1}{2})} \\
&&& \text{if } 2(r+s)-3+\binom{r-1}{2}+\binom{s-1}{2} \leq h \leq \tau_0-1, \\
&= S_3^{(h-2r-s+2-\binom{r-1}{2})} + S_4^{(h-2(r+s)+3-\binom{r-1}{2})} + S_o^{(2h-2(2r+s)+5-2\binom{r-1}{2})} \\
&&& \text{if } 2(r+s)-3+\binom{r-1}{2} \leq h \leq 2(r+s)-4+\binom{r-1}{2}+\binom{s-1}{2}, \\
&= S_1^{(h-r-2s+2)} + S_2^{(h-2(r+s)+3)} + S_f^{(2h-2(r+2s)+5)} \\
&&& \text{if } 2(r+s)-3 \leq h \leq 2(r+s)-4+\binom{r-1}{2}, \\
&= S_3^{(h-2r-s+2)} + S_o^{(h-2r+1)} \\
&&& \text{if } 2r+s-2 \leq h \leq 2(r+s)-4, \\
&= S_1^{(h-r-s+1)} + S_f^{(h-s)} && \text{if } r+s-1 \leq h \leq 2r+s-3, \\
&= S_o^{(h-r)} && \text{if } r \leq h \leq r+s-2, \\
&= S_f^{(h-1)} && \text{if } 1 \leq h \leq r-1, \\
\phi^h &= \phi_s + \phi_f + \phi_o && \text{if } 2(r+s)-3+\binom{r-1}{2}+\binom{s-1}{2} \leq h \leq \tau_0-1, \\
&= \phi_3 + \phi_4 + 2\phi_o && \text{if } 2(r+s)-3+\binom{r-1}{2} \leq h \leq 2(r+s)-4+\binom{r-1}{2}+\binom{s-1}{2}, \\
&= \phi_1 + \phi_2 + 2\phi_f && \text{if } 2(r+s)-3 \leq h \leq 2(r+s)-4+\binom{r-1}{2}, \\
&= \phi_3 + \phi_o && \text{if } 2r+s-2 \leq h \leq 2(r+s)-4, \\
&= \phi_1 + \phi_f && \text{if } r+s-1 \leq h \leq 2r+s-3, \\
&= \phi_o && \text{if } r \leq h \leq r+s-2, \\
&= \phi_f && \text{if } 1 \leq h \leq r-1
\end{aligned}$$

and

$$\begin{aligned}
\xi(h) &= \left\{ \tau_s - h + 2(r+s) - 4 + \binom{r-1}{2} + \binom{s-1}{2} \right\} \phi_s + \left\{ \tau_f - h + 2s - 2 - \binom{r-1}{2} \right. \\
&\quad \left. + \binom{s-1}{2} \right\} \phi_f + \left\{ \tau_o - h + 2r - 2 + \binom{r-1}{2} - \binom{s-1}{2} \right\} \phi_o \\
&&& \text{if } 2(r+s)-3+\binom{r-1}{2}+\binom{s-1}{2} \leq h \leq \tau_0-1, \\
&= \left\{ \tau_3 - h + 2r + s - 3 + \binom{r-1}{2} \right\} \phi_3 + \left\{ \tau_4 - h + 2(r+s) - 4 + \binom{r-1}{2} \right\} \phi_4 \\
&\quad + \left\{ \tau_o - 2h + 2(2r+s) - 6 + 2\binom{r-1}{2} \right\} \phi_o
\end{aligned}$$

$$\begin{aligned}
& \text{if } 2(r+s)-3+\binom{r-1}{2} \leq h \leq 2(r+s)-4+\binom{r-1}{2}+\binom{s-1}{2}, \\
& = (\tau_1-h+r+2s-3)\phi_1 + \{\tau_2-h+2(r+s)-4\}\phi_2 + \{\tau_f-2h+2(r+2s)-6\}\phi_f \\
& \quad \text{if } 2(r+s)-3 \leq h \leq 2(r+s)-4+\binom{r-1}{2}, \\
& = (\tau_3-h+2r+s-3)\phi_3 + (\tau_g-h+2r-2)\phi_g \\
& \quad \text{if } 2r+s-2 \leq h \leq 2(r+s)-4, \\
& = (\tau_1-h+r+s-2)\phi_1 + (\tau_f-h+s-1)\phi_f \\
& \quad \text{if } r+s-1 \leq h \leq 2r+s-3, \\
& = (\tau_g-h+r-1)\phi_g \\
& \quad \text{if } r \leq h \leq r+s-2, \\
& = (\tau_f-h)\phi_f \quad \text{if } 1 \leq h \leq r-1.
\end{aligned}$$

All of the above equations also have F distributions, and the nested procedure is continued until a significant test is obtained. The F^h have central or non-central F distributions with ϕ^h and $\{N-\nu(m, n)+\xi(h)\}$ d.f., and noncentrality parameters λ^h/σ^2 depending on which of the $H^{(h)}$ are true, where λ^h are given by changing of S_*^a 's into λ_*^a 's in S^h .

Acknowledgments

The authors would like to thank the editor and the referees for their useful comments and suggestions.

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