FURTHER CONSTRUCTIONS OF NESTED GROUP DIVISIBLE DESIGNS

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Some methods of construction for nested group divisible designs are given. Cyclic nested group divisible designs are further discussed. Some individual plans are also tabulated with 4 new E-optimal designs.

1. Introduction

Nested group divisible (NGD) designs, which are 3-associate partially balanced incomplete block (PBIB) designs, were introduced by Roy (1953), and subsequently studied by Raghavarao (1960), Roy (1962), Hinkelmann and Kempthorne (1963), Bhagwandas, Sinha and Kageyama (1992), Duan and Kageyama (1993), etc. These designs are useful as 3-factor experiments, having balance as well as orthogonal factorial structure. In this paper, several methods of construction of such designs are provided by breaking up the groups or the blocks of given group divisible (GD) designs. Additionally, a new type of nested GD design, which we call a cyclic nested group divisible (cyclic NGD or CNGD) design, is discussed. Finally some individual plans are also tabulated. This tabulation includes 4 new E-optimal nested GD designs.

An NGD design $(\mathcal{X}, \mathcal{A}, \mathcal{G}, \mathcal{B})$ is an arrangement with v treatments into b blocks such that $v=|\mathcal{X}|$, $b=|\mathcal{B}|$, $p=|\mathcal{A}|$, $m=|\mathcal{G}|$, and

- (i) each treatment $x \in \mathcal{X}$ is replicated r times;
- (ii) each block $B \in \mathcal{B}$ contains $k (< v)$ treatments;
- (iii) there exists an association scheme on $v = pmn$ treatments, which are divided into p sets $S \in \mathcal{A}$ of m groups $G \in \mathcal{G}$ of n treatments each, such that
 - (a) any two treatments from the same set and the same group are first associates;
 - (b) any two treatments from the same set but different groups are second associates;
 - (c) any two treatments from different sets are third associates;
- (iv) any two treatments which are i th associates occur together in λ_i blocks, $i=1, 2, 3$.

Let n_i denote the number of treatments which are i th associates of any treatment. Then $n_1 = n-1$, $n_2 = (m-1)n$, $n_3 = (p-1)mn$.

The present association scheme can be displayed in a $pm \times n$ array as

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$$\begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(p)} \end{bmatrix} \quad \text{with} \quad A^{(i)} = \begin{bmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \cdots & a_{1n}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} & \cdots & a_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{(i)} & a_{m2}^{(i)} & \cdots & a_{mn}^{(i)} \end{bmatrix}, \quad i=1, 2, \dots, p,$$

where $a_{st}^{(i)}$ denotes the t th treatment of the s th group in the i th set. This representation is helpful for the proof of the theorems given in Section 2.

We shall denote the usual group divisible (GD) design by $\text{GD}(v, b, r, k; m, n; \lambda_1, \lambda_2)$ and the present NGD design by $\text{NGD}(v, b, r, k; m, n, p; \lambda_1, \lambda_2, \lambda_3)$. Note that the NGD design with $p=1$ is the same as a GD design.

In an NGD design, if $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$, the design reduces to a GD design by combining the first and the second or the second and the third associate classes. When $\lambda_1 = \lambda_2 = \lambda_3$, the design reduces to a balanced incomplete block (BIB) design by combining all of the associate classes.

The following notations are here used: $\mathbf{1}_s$ is an s -dimensional column vector with all elements unity, I_s is the identity matrix of order s , $J_{s \times t}$ is an $s \times t$ matrix with all elements unity, $J_{s \times s}$ is specially denoted by J_s , $0_{s \times t}$ denotes an $s \times t$ matrix with all elements zero, $A \otimes B$ denotes the Kronecker product of two matrices A and B , A' denotes the transpose of A , and $Z_s = \{0, 1, \dots, s-1\}(\text{mod } s)$.

2. Breaking up groups of GD designs

In this section, we describe several patterned methods of construction by breaking up groups of a GD design.

THEOREM 2.1. *The existence of a $\text{GD}(pmn, b', r', k'; p, mn; \lambda'_1, \lambda'_2)$ and a $\text{GD}(mn, b'', r'', k'; m, n; \lambda''_1, \lambda''_2)$ implies the existence of an $\text{NGD}(pmn, b' + pb'', r' + r'', k'; m, n, p; \lambda'_1 + \lambda''_1, \lambda'_1 + \lambda''_2, \lambda'_2)$.*

PROOF. Let N_1 be the $pmn \times b'$ incidence matrix of the first GD design and N_2 be the $mn \times b''$ incidence matrix of the second one. Then it can be shown that

$$N = [N_1 : I_p \otimes N_2]$$

is the incidence matrix of the required NGD design. $\square$

REMARK 2.1. If $\lambda'_2 = \lambda'_1 + \lambda''_2$, we obtain a $\text{GD}(pmn, b' + pb'', r' + r'', k'; pm, n; \lambda'_1 + \lambda''_1, \lambda'_2)$, which generalizes Theorem 3.2 of Duan and Kageyama (1995).

THEOREM 2.2. *The existence of a $\text{GD}(pmn, b', r', k'; p, mn; \lambda'_1, \lambda'_2)$ and a $\text{GD}(mn, b'', r'', k' - (p-1)mn; m, n; \lambda''_1, \lambda''_2)$ implies the existence of an $\text{NGD}(pmn, b' + pb'', r' + r'' + (p-1)b'', k'; m, n, p; \lambda'_1 + \lambda''_1 + (p-1)b'', \lambda'_1 + \lambda''_2 + (p-1)b'', \lambda'_2 + 2r'' + (p-2)b'')$.*

PROOF. Let N_1 be the $pmn \times b'$ incidence matrix of the first GD design and N_2 be the $mn \times b''$ incidence matrix of the second one. Then it follows that

$$N = [N_1 : I_p \otimes N_2 + (J_p - I_p) \otimes J_{mn \times b''}]$$

is the incidence matrix of the required NGD design. $\square$

THEOREM 2.3. *The existence of a $GD(pmn, b', r', k'; p, mn; \lambda_1, \lambda_2)$ and a $GD(mn, b'', r'', k' - mn; m, n; \lambda_1'', \lambda_2'')$ implies the existence of an $NGD(pmn, b' + p(p-1)b'', r' + (p-1)b'' + (p-1)r'', k'; m, n, p; \lambda_1 + (p-1)b'' + (p-1)\lambda_1'', \lambda_1 + (p-1)b'' + (p-1)\lambda_2'', \lambda_2 + 2r'')$ for any positive integer $p \geq 3$.*

PROOF. Let N_1 and N_2 be the $pmn \times b'$ and $mn \times b''$ incidence matrices of the first and the second GD designs, respectively. It can then be shown that

$$\left[N_1 \left| \begin{array}{ccc} \mathbf{1}_{p-1} \otimes N_2 & (1, 0, \dots, 0) \otimes J & \\ I_{p-1} \otimes J & \mathbf{1}_{p-1} \otimes N_2 & \dots \\ & [O : I_{p-2} \otimes J] & \mathbf{1}_{p-1} \otimes N_2 \end{array} \right. \right]$$

is the incidence matrix of the required NGD design. $\square$

A GD design with $v=2k$ is said to be self-complementary. Note that a self-complementary design has the same block structure as its complement. Kageyama and Tanaka (1981) constructed several series of self-complementary GD designs. They can be applied to obtain the following observations. Let $\bar{N} = J - N$.

THEOREM 2.4. *The existence of a $GD(3mn, b', r', k'; 3, mn; \lambda_1, \lambda_2)$ and a self-complementary $GD(mn, b'', r'', k'/3; m, n; \lambda_1'', \lambda_2'')$ implies the existence of an $NGD(3mn, b' + 4b'', r' + 4r'', k'; m, n, 3; \lambda_1 + 4\lambda_1'', \lambda_1 + 4\lambda_2'', \lambda_2 + 2r'')$.*

PROOF. Let N_1 and N_2 be the incidence matrices of the first and the second GD designs, respectively. Then it can be shown that

$$\left[N_1 \left| \begin{array}{cccc} N_2 & \bar{N}_2 & N_2 & N_2 \\ N_2 & N_2 & \bar{N}_2 & N_2 \\ N_2 & N_2 & N_2 & \bar{N}_2 \end{array} \right. \right]$$

is the incidence matrix of the required NGD design. $\square$

THEOREM 2.5. *The existence of a $GD(4mn, b', r', k'; 4, mn; \lambda_1, \lambda_2)$ and a self-complementary $GD(mn, b'', r'', k'/4; m, n; \lambda_1'', \lambda_2'')$ implies the existence of an $NGD(4mn, b' + 4b'', r' + 4r'', k'; m, n, 4; \lambda_1 + 4\lambda_1'', \lambda_1 + 4\lambda_2'', \lambda_2 + 2r'')$.*

PROOF. The required design can be given by

$$\left[N_1 \left| \begin{array}{cccc} N_2 & N_2 & N_2 & N_2 \\ N_2 & \bar{N}_2 & N_2 & \bar{N}_2 \\ N_2 & N_2 & \bar{N}_2 & \bar{N}_2 \\ N_2 & \bar{N}_2 & \bar{N}_2 & N_2 \end{array} \right. \right],$$

where N_1 and N_2 are the incidence matrices of the first and the second GD designs, respectively. $\square$

THEOREM 2.6. *The existence of a $GD(3mn, b', r', k'; 3, mn; \lambda_1, \lambda_2)$ and a self-complementary $GD(mn, b'', r'', k'/2; m, n; \lambda_1'', \lambda_2'')$ implies the existence of an $NGD(3mn, b' + 6b'', r' + 4r'', k'; m, n, 3; \lambda_1 + 4\lambda_1'', \lambda_1 + 4\lambda_2'', \lambda_2 + r'')$.*

PROOF. Let N_1 and N_2 be the incidence matrices of the first and the second GD designs, respectively. The pattern as

$$\left[\begin{array}{c|cccccc} & N_2 & N_2 & 0 & N_2 & \bar{N}_2 & 0 \\ N_1 & N_2 & 0 & N_2 & \bar{N}_2 & 0 & N_2 \\ & 0 & N_2 & N_2 & 0 & N_2 & \bar{N}_2 \end{array} \right]$$

can yield the required NGD design. $\square$

THEOREM 2.7. *The existence of a $GD(2mn, b', r', k'; 2, mn; \lambda'_1, \lambda'_2)$ and a self-complementary $GD(mn, b'', r'', k'/2; m, n; \lambda''_1, \lambda''_2)$ implies the existence of an $NGD(2mn, b' + 2b'' + 2s, r' + 2r'' + s, k'; m, n, 2; \lambda'_1 + 2\lambda''_1 + s, \lambda'_1 + 2\lambda''_2 + s, \lambda'_2 + r'')$ for any positive integer s .*

PROOF. The result follows from the pattern as

$$\left[\begin{array}{c|cccc} N_1 & N_2 & N_2 & J_{mn \times s} & 0_{mn \times s} \\ & N_2 & \bar{N}_2 & 0_{mn \times s} & J_{mn \times s} \end{array} \right],$$

where N_1 and N_2 are the incidence matrices of the first and the second GD designs, respectively. $\square$

Note that all of the constructions described above can be generalized by taking t copies of the ingredient designs when breaking up the groups of the first GD design. Also note that the ideas in Bhagwandas, Sinha and Kageyama (1992) are used to break up the groups in some theorems.

3. Breaking up blocks of GD designs

The following construction is combinatorial in some sense.

THEOREM 3.1. *The existence of a $GD(pm, b', r', k'; p, m; \lambda'_1, \lambda'_2)$ and a $GD(k'n, b'', r'', k''; k', n; \lambda''_1, \lambda''_2)$ implies the existence of an $NGD(pmn, b'b'', r'r'', k''; m, n, p; r'\lambda'_1, \lambda'_1\lambda''_2, \lambda'_2\lambda''_2)$.*

PROOF. Let $(i, j), 1 \leq i \leq p, 1 \leq j \leq m$, be the treatment of the first GD design, where (i, j) and (i', j') are first associates if $i = i'$ and $j \neq j'$, and second associates if $i \neq i'$. Now we define n new treatments $(i, j, l), 1 \leq l \leq n$, for each old treatment (i, j) , and an NGD association scheme is stated as follows: (i, j, l) and (i', j', l') are first associates if $i = i', j = j'$ and $l \neq l'$, second associates if $i = i'$ and $j \neq j'$, and third associates if $i \neq i'$. For each block $B' = \{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$, we can construct a $GD(k'n, b'', r'', k''; k', n; \lambda''_1, \lambda''_2)$ on $B' \times \{1, 2, \dots, n\}$ with block set $\mathcal{B}_{B'}$. Then combining all of the $\mathcal{B}_{B'}$ gives the block set of the required PBIB design based on the NGD association scheme. $\square$

This is a rather powerful, though simple method for the construction of NGD designs with small parameters, as shown in Section 5.

4. Cyclic nested group divisible designs

Now we deal with a special type of nested GD designs which we call a cyclic nested group divisible design. A construction method is given. This is

based on the technique of Jimbo and Kuriki (1983) and Jimbo and Vanstone (1984).

Let $(\mathcal{X}, \mathcal{A}, \mathcal{G}, \mathcal{B})$ be an $\text{NGD}(v, b, r, k; m, n, p; \lambda_1, \lambda_2, \lambda_3)$, and σ be a permutation on $\mathcal{X}$. For any block $B = \{b_1, \dots, b_k\} \in \mathcal{B}$, any group $G = \{x_1, \dots, x_n\} \in \mathcal{G}$ and any set $S = \{G_1, \dots, G_m\} \in \mathcal{A}$, define $B^\sigma = \{b_1^\sigma, \dots, b_k^\sigma\}$, $G^\sigma = \{x_1^\sigma, \dots, x_n^\sigma\}$ and $S^\sigma = \{G_1^\sigma, \dots, G_m^\sigma\}$. If $\mathcal{B}^\sigma = \{B^\sigma : B \in \mathcal{B}\} = \mathcal{B}$, $\mathcal{G}^\sigma = \{G^\sigma : G \in \mathcal{G}\} = \mathcal{G}$ and $\mathcal{A}^\sigma = \{S^\sigma : S \in \mathcal{A}\} = \mathcal{A}$, then σ is called an automorphism of $(\mathcal{X}, \mathcal{A}, \mathcal{G}, \mathcal{B})$. If there is an automorphism σ of order $v = pmn = |\mathcal{X}|$, then the $\text{NGD}(v, b, r, k; m, n, p; \lambda_1, \lambda_2, \lambda_3)$ is said to be cyclic, and denoted by $\text{CNGD}(v, b, r, k; m, n, p; \lambda_1, \lambda_2, \lambda_3)$.

In a CNGD design, if $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$, the design can be reduced to a cyclic GD design $\text{CGD}(v, b, r, k; p, mn; \lambda_1, \lambda_3)$ or $\text{CGD}(v, b, r, k; pm, n; \lambda_1, \lambda_2)$ by combining the first and the second, or the second and the third associate classes, as mentioned in Section 1. Clearly, in a $\text{CGD}(mn, b, r, k; m, n; \lambda_1, \lambda_2)$ each group must be the subgroup $mZ_n = \{0, m, 2m, \dots, (n-1)m\}$ of Z_{mn} or its coset (see Mukerjee, Jimbo and Kageyama (1987)). Note that when $\lambda_1 = \lambda_2 = \lambda$ in the CGD, it is called a cyclic balanced incomplete block design, denoted by $\text{CB}(k, \lambda; v)$.

For a block B of a $\text{CGD}(v, b, r, k; m, n; \lambda_1, \lambda_2)$, the orbit of B is defined by $\{B + y \pmod{v} : y \in Z_v\}$. The length of an orbit is its cardinality. An orbit of length v is said to be full; otherwise it is called a short orbit. A base block of an orbit R is any arbitrarily chosen block $B \in R$.

In order to describe our construction, we also need the concept of a difference matrix. Let $D = (d_{ij})$, $i = 1, 2, \dots, k; j = 1, 2, \dots, g\lambda$, be a matrix with entries from Z_g such that the list $(d_{ii} - d_{ij})_{i=1, \dots, g\lambda}$ contains each element of Z_g precisely λ times for $i \neq j$ and $i, j \in \{1, 2, \dots, k\}$. Then D is called a difference matrix and denoted by $\text{DM}(k, \lambda; g)$.

THEOREM 4.1. *If there exist a $\text{CGD}(mn, b, r, k; m, n; \lambda_1, \lambda_2)$ ($\lambda_1 \leq \lambda_2$) without short orbit, and a $\text{DM}(k, \lambda; g)$, then there exists a $\text{CNGD}(mng, b_1, r_1, k; n, g, m; 0, \lambda_1\lambda, \lambda_2\lambda)$. Furthermore, if there exists a $\text{CGD}(ng, b', r', k; n, g; \lambda'_1, \lambda'_2)$, then there exists a $\text{CNGD}(mng, b_2, r_2, k; n, g, m; \lambda'_1, \lambda'_2 + \lambda_1\lambda, \lambda_2\lambda)$.*

PROOF. For $v = mn$, let the vg treatments be arranged in a rectangular of v rows and g columns as follows:

$$\begin{array}{cccc}
 0 & v & \cdots & (g-1)v \\
 1 & v+1 & \cdots & (g-1)v+1 \\
 \vdots & \vdots & & \vdots \\
 m-1 & v+m-1 & \cdots & (g-1)v+m-1 \\
 m & v+m & \cdots & (g-1)v+m \\
 \vdots & \vdots & & \vdots \\
 2m-1 & v+2m-1 & \cdots & (g-1)v+2m-1 \\
 2m & v+2m & \cdots & (g-1)v+2m \\
 \vdots & \vdots & & \vdots \\
 mn-1 & v+mn-1 & \cdots & (g-1)v+mn-1
 \end{array}$$

Two treatments x and y are first associates if $x - y \in vZ_g$; second associ-

ates if $x - y \in mZ_{ng} - vZ_g$; and third associates if $x - y \in Z_{vg} - mZ_{ng}$. This is a nested association scheme with m sets of n groups of g treatments each.

Now let $D = (d_{ij})$ be a $DM(k, \lambda; g)$. For each base block $B = \{b_1, \dots, b_k\}$ of the $CGD(mn, b, r, k; m, n; \lambda_1, \lambda_2)$, construct the following λg base blocks:

$$B_j = \{b_i + d_{ij}v : i = 1, 2, \dots, k\}$$

for $j = 1, 2, \dots, \lambda g$. Then every treatment of $vZ_g - \{0\} = \{v, \dots, (g-1)v\}$ does not occur with the treatment 0, every treatment of $mZ_{ng} - vZ_g$ occurs together with the treatment 0 exactly $\lambda_1\lambda$ times, and every treatment of $Z_{vg} - mZ_{ng}$ occurs together with the treatment 0 exactly $\lambda_2\lambda$ times.

Next, for each base block $C = \{c_1, \dots, c_k\}$ of the $CGD(ng, b', r', k; n, g; \lambda'_1, \lambda'_2)$, let $mC = \{mc_1, \dots, mc_k\}$ be a base block of the desired design. Then any treatment of $mZ_{ng} - vZ_g$ occurs together with the treatment 0 λ'_2 times, and every treatment of $vZ_g - \{0\}$ occurs together with the treatment 0 λ'_1 times. This completes the proof. $\square$

Note that the representation of the association scheme defined above is different from that in Section 1.

This theorem generalizes Theorem 1 of Jimbo and Kuriki (1983), Theorem 1 of Jimbo and Vanstone (1984) and Theorem 3.5 of Jimbo (1993). For example, by letting $\lambda'_1 = \lambda\lambda_2$, $\lambda'_2 = \lambda(\lambda_2 - \lambda_1)$, we obtain a $CB(k, \lambda\lambda_2; vg)$, which was obtained firstly by Jimbo (1993).

For more information on cyclic GD designs and difference matrices, see the references mentioned in this section.

5. NGD designs with $r, k \leq 10$

A large number of GD designs with $r, k \leq 10$ are available in Clatworthy (1973) and Sinha (1991). Such plans can be used to produce NGD designs through the theorems presented here. For example, a $GD(4, 4, 2, 2; 2, 2; 0, 1)$, SR1, and a $GD(4, 8, 4, 2; 2, 2; 2, 1)$, R1, listed in Clatworthy (1973) can produce an $NGD(8, 32, 8, 2; 2, 2, 2; 4, 0, 1)$ by use of Theorem 3.1. We list some NGD designs with $r, k \leq 10$ as in Table 1. Notations S, SR, R in the Source of the table denote GD designs from Clatworthy (1973). Note that a similar table was also listed in Duan and Kageyama (1993). However, they have no overlap between them.

Sinha and Shah (1988) obtained a class of E-optimal 3-concurrence most balanced designs, which may be considered as a generalization of GD designs, to provide a unified approach for identifying certain available classes of GD designs as E-optimal. Sinha and Kageyama (1992) described some methods of constructing E-optimal nested group divisible designs and provided two tables of such designs with $r \leq 20$ and $k \leq 10$. The designs with an asterisk in the present table show such E-optimal designs, four of which are new. Note that the E-optimal design of No. 1 is given in Table 2.1 by Sinha and Kageyama (1992).

Table 1. NGD designs with $r, k \leq 10$

No	v	r	k	b	m	n	p	λ_1	λ_2	λ_3	Source	Remark
1*	8	8	2	32	2	2	2	0	2	1	SR9, SR2	Th 2.1
2	8	8	2	32	2	2	2	4	0	1	SR1, R1	Th 3.1
3	8	9	2	36	2	2	2	1	2	1	SR9, R3	Th 2.1
4	8	10	2	40	2	2	2	0	1	2	SR10, SR1	Th 2.1
5	8	10	2	40	2	2	2	0	3	1	SR9, SR3	Th 2.1
6	8	10	2	40	2	2	2	2	0	2	SR1, R3	Th 3.1
7	8	10	2	40	2	2	2	6	0	1	SR1, R2	Th 3.1
8	12	6	3	24	2	2	3	0	2	1	S1, R54	Th 3.1
9	12	6	3	24	2	2	3	4	0	1	SR18, R42	Th 3.1
10	12	9	3	36	2	2	3	6	2	1	R42, R42	Th 3.1
11	16	9	3	48	2	2	4	0	3	1	S6, R54	Th 3.1
12	16	9	3	48	2	2	4	6	0	1	R54, R42	Th 3.1
13	18	9	3	54	2	3	3	0	2	1	R42, SR23	Th 3.1
14	18	9	3	54	3	2	3	6	0	1	SR23, R42	Th 3.1
15	12	10	3	40	3	2	2	2	3	1	R74, SR18	Th 2.1
16	18	10	3	60	2	3	3	4	0	1	SR18, R59	Th 3.1
17	18	10	3	60	3	2	3	0	2	1	S30, SR19	Th 2.1
18	12	4	4	12	2	2	3	4	0	1	SR18, S1	Th 3.1
19	12	6	4	18	2	2	3	6	2	1	S1, S6	Th 3.1
20	16	6	4	24	2	2	4	6	0	1	R45, S1	Th 2.1
21*	18	6	4	27	2	3	3	0	2	1	S1, SR41	Th 3.1
22	18	6	4	27	3	2	3	6	0	1	SR23, S1	Th 3.1
23	12	8	4	24	2	2	3	0	4	2	S1, SR36	Th 3.1
24	12	8	4	24	2	2	3	8	0	2	SR18, S2	Th 3.1
25	12	8	4	24	2	3	2	6	0	2	SR1, R94	Th 3.1
26	18	8	4	36	2	3	3	6	0	1	SR18, R104	Th 3.1
27	18	8	4	36	3	2	3	4	2	1	S21, R109	Th 3.1
28*	24	8	4	48	2	4	3	0	2	1	S1, SR44	Th 3.1
29	24	8	4	48	4	2	3	8	0	1	SR26, S1	Th 3.1
30	24	9	4	54	2	3	4	0	3	1	S6, SR41	Th 3.1
31	24	9	4	54	3	2	4	9	0	1	SR41, S6	Th 3.1
32*	8	10	4	20	2	2	2	2	4	3	SR38, SR1	Th 2.7
33	12	10	4	30	2	2	3	6	4	2	S1, R97	Th 3.1
34	18	10	4	45	3	2	3	10	2	1	S21, S11	Th 3.1
35	24	10	4	60	2	2	6	10	0	1	R70, S1	Th 3.1
36*	30	10	4	75	2	5	3	0	2	1	S1, SR46	Th 3.1
37	30	10	4	75	5	2	3	10	0	1	SR28, S1	Th 3.1
38	12	10	5	24	2	2	3	4	6	3	S1, R134	Th 3.1
39	16	10	5	32	2	4	2	8	0	2	SR1, R133	Th 3.1
40	18	10	5	36	2	3	3	2	4	2	S1, R145	Th 3.1
41	18	10	5	36	2	3	3	8	0	2	SR18, R137	Th 3.1
42	18	10	5	36	3	2	3	0	4	2	S21, R144	Th 3.1
43	24	10	5	48	2	4	3	8	0	1	SR18, R143	Th 3.1
44	36	10	5	72	3	4	3	0	2	1	S21, R153	Th 3.1
45	18	4	6	12	2	3	3	4	0	1	SR18, S21	Th 3.1
46	12	6	6	12	2	2	3	6	4	2	S1, S18	Th 3.1
47	18	6	6	18	2	3	3	6	2	1	S1, S27	Th 3.1
48	24	6	6	24	2	3	4	6	0	1	R54, S21	Th 3.1
49	27	6	6	27	3	3	3	6	0	1	SR23, S21	Th 3.1
50	18	8	6	24	2	3	3	8	0	2	SR18, S22	Th 3.1

Table 1. (continued)

No	v	r	k	b	m	n	p	λ_1	λ_2	λ_3	Source	Remark
51	18	8	6	24	3	2	3	0	4	2	S21, SR66	Th 3.1
52	36	8	6	48	4	3	3	8	0	1	SR26, S21	Th 3.1
53	16	9	6	24	2	2	4	9	6	2	S6, S18	Th 3.1
54	24	9	6	36	2	3	4	9	3	1	S6, S27	Th 3.1
55	24	9	6	36	3	2	4	9	0	2	SR41, S18	Th 3.1
56	36	9	6	54	3	3	4	9	0	1	SR41, S27	Th 3.1
57	18	10	6	30	3	2	3	10	4	2	S21, S28	Th 3.1
58	27	10	6	45	3	3	3	10	2	1	S21, S38	Th 3.1
59	36	10	6	60	2	3	6	10	0	1	R70, S21	Th 3.1
60	45	10	6	75	3	5	3	0	2	1	S21, SR75	Th 3.1
61	45	10	6	75	5	3	3	10	0	1	SR28, S21	Th 3.1
62	24	4	8	12	2	4	3	4	0	1	SR18, S53	Th 3.1
63	24	6	8	18	2	4	3	6	2	1	S1, S61	Th 3.1
64	32	6	8	24	2	4	4	6	0	1	R54, S53	Th 3.1
65	36	6	8	27	3	4	3	6	0	1	SR23, S53	Th 3.1
66	24	8	8	24	2	4	3	8	0	2	SR18, S54	Th 3.1
67	48	8	8	48	4	4	3	8	0	1	SR26, S53	Th 3.1
68	32	9	8	36	2	4	4	9	3	1	S6, S61	Th 3.1
69	48	9	8	54	3	4	4	9	0	1	SR41, S61	Th 3.1
70	36	10	8	45	3	4	3	10	2	1	S21, S69	Th 3.1
71	48	10	8	60	2	4	6	10	0	1	R70, S53	Th 3.1
72	60	10	8	75	5	4	3	10	0	1	SR28, S53	Th 3.1
73	18	6	9	12	2	3	3	6	4	2	S1, S82	Th 3.1
74	24	9	9	24	2	3	4	9	6	2	S6, S18	Th 3.1
75	36	9	9	36	3	3	4	9	0	2	SR41, S82	Th 3.1
76	27	10	9	30	3	3	3	10	4	2	S21, S86	Th 3.1
77	30	4	10	12	2	5	3	4	0	1	SR18, S100	Th 3.1
78	30	6	10	18	2	5	3	6	2	1	S1, S106	Th 3.1
79	40	6	10	24	2	5	4	6	0	1	R54, S100	Th 3.1
80	45	6	10	27	3	5	3	6	0	1	SR23, S100	Th 3.1
81	30	8	10	24	2	5	3	8	0	2	SR18, S101	Th 3.1
82	60	8	10	48	4	5	3	8	0	1	SR26, S100	Th 3.1
83	40	9	10	36	2	5	4	9	3	1	S6, S106	Th 3.1
84	60	9	10	54	3	5	4	9	0	1	SR41, S106	Th 3.1
85	45	10	10	45	3	5	3	10	2	1	S21, S114	Th 3.1
86	60	10	10	60	2	5	6	10	0	1	R70, S100	Th 3.1
87	75	10	10	75	5	5	3	10	0	1	SR28, S100	Th 3.1

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