SEQUENTIAL FIXED-WIDTH CONFIDENCE INTERVAL ESTIMATION FOR THE PERCENTILES OF A NORMAL DISTRIBUTION

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In this paper we consider sequential fixed-width confidence interval estimation for the percentiles of a normal distribution with unknown mean and variance. We construct two kinds of confidence intervals by using a stopping rule and show their asymptotic consistency. We also try to compare the performance of these confidence intervals by the simulation results.

Key words: Asymptotic consistency, Bias correction, Second-order expansion, Expected sample size, Stopping rule

1. Introduction

Let $X_1, X_2, \dots$ be independent and identically distributed normal random variables with unknown mean $\mu \in (-\infty, \infty)$ and variance $\sigma^2 \in (0, \infty)$. We want to estimate $\theta = a\mu + b\sigma$ with given constants a and b , including the p -th percentiles.

In this paper, we will construct sequential confidence intervals of θ with preassigned fixed-width $2d$ and coverage probability $1 - \alpha$. For a parameter of a negative exponential distribution, Isogai et al. (1995) and Hirose et al. (1997) have considered this problem. The problem of sequential fixed-width confidence interval estimation for the mean has been studied by Chow and Robbins (1965), Woodroffe (1977), Callaert and Janssen (1981), Jurečková (1991) and Mukhopadhyay and Datta (1996), among others. Ghosh and Mukhopadhyay (1989) considered the problem of minimum risk point estimation of θ . We are not aware, however, of the problem of the confidence interval estimation for θ .

The aim of this paper is to construct sequential confidence intervals of θ . In Section 2 we will propose two kinds of sequential confidence intervals, corresponding to the natural sequential estimator of θ and its bias-corrected version, respectively, and show their asymptotic consistency as d approaches zero. We will also derive the second order asymptotic expansion of the expected sample size. Section 3 gives simulation results to compare the performance of these confidence intervals. The proofs of the results in this paper will be presented in Section 4.

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2. Fixed-width confidence intervals

In this section we propose a stopping rule in order to stop sampling and construct sequential confidence intervals of $\theta = a\mu + b\sigma$ with preassigned fixed-width $2d$ and coverage probability $1 - \alpha$ by using the stopping rule. Then we show their asymptotic consistency as d tends to zero. We also derive the second order asymptotic expansion of the expected sample size.

Let $d > 0$ and $\alpha \in (0, 1)$ be given. We choose the constant $z = z_\alpha > 0$ such that $\Phi(z) - \Phi(-z) = 1 - \alpha$ where Φ denotes the standard normal distribution function. For $n \geq 2$ set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i, \quad s_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2, \\ \hat{\theta}_n = a\bar{X}_n + bs_n \quad \text{and} \quad I_n = I_n(d) = [\hat{\theta}_n - d, \hat{\theta}_n + d].$$

Throughout this paper we assume

$$(2.1) \quad m \equiv 2a^2 + b^2 > 0.$$

We will now derive the asymptotic expression of the probability $P\{\theta \in I_n\}$ for large n . For $n \geq 1$ put

$$(2.2) \quad Y_n = \frac{1}{n(n+1)\sigma^2} \left(\sum_{i=1}^n (X_i - X_{n+1}) \right)^2 \quad \text{and} \quad \bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i.$$

Then it is well known that $Y_1, Y_2, \dots$ are independent and identically distributed random variables having a chi-square distribution with one degree of freedom and that

$$(2.3) \quad s_n^2 = \sigma^2 \bar{Y}_{n-1}.$$

For $n \geq 2$ set

$$(2.4) \quad U_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}, \quad V_n = \frac{\sqrt{n-1}(\bar{Y}_{n-1} - 1)}{\sqrt{2}}, \\ W_n = {}^T(U_n, V_n) \quad \text{and} \quad c = {}^T(c_1, c_2) \quad \text{for any real numbers } c_1 \text{ and } c_2,$$

where T denotes transpose. Then it follows from (2.3) and (2.4) that

$$(2.5) \quad \sqrt{n}(\hat{\theta}_n - \theta) = a\sigma U_n + \sqrt{\frac{2n}{n-1}} \cdot \frac{b\sigma^2}{s_n + \sigma} V_n$$

and U_n and V_n are independent because of the independency of $\bar{X}_n$ and s_n^2 . For any real numbers c_1, c_2 and t , by a central limit theorem and the independency of U_n and V_n we have with $i^2 = -1$

$$E[\exp(it^T c W_n)] \rightarrow E[\exp(it^T c W)] \quad \text{as } n \rightarrow \infty,$$

which implies

$${}^T c W_n \xrightarrow{L} {}^T c W \quad \text{as } n \rightarrow \infty,$$

where " $\xrightarrow{L}$ " stands for convergence in law and $W = {}^T(U, V)$ denotes a two-dimensional normal random vector with mean vector ${}^T(0, 0)$ and covariance

matrix I (the unit matrix). Thus by the Cramér-Wold device (see Billingsley (1968), p. 49) we get

$$(2.6) \quad W_n \xrightarrow{L} W \quad \text{as } n \rightarrow \infty,$$

which yields

$$(2.7) \quad a\sigma U_n + \frac{b\sigma}{\sqrt{2}} V_n \xrightarrow{L} Z \quad \text{as } n \rightarrow \infty,$$

where $Z \equiv a\sigma U + (b\sigma/\sqrt{2}) V$ has a normal distribution with mean 0 and variance $m\sigma^2/2$. Since $s_n^2 \xrightarrow{a.s.} \sigma^2$ as $n \rightarrow \infty$, we have by (2.6)

$$\left| \sqrt{\frac{2n}{n-1}} \cdot \frac{b\sigma^2}{s_n + \sigma} V_n - \frac{b\sigma}{\sqrt{2}} V_n \right| \xrightarrow{p} 0 \quad \text{as } n \rightarrow \infty,$$

where “ $\xrightarrow{a.s.}$ ” and “ $\xrightarrow{p}$ ” denote almost sure convergence and convergence in probability, respectively. Hence it follows from (2.5) and (2.7) that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} Z$ as $n \rightarrow \infty$, which, together with (2.5), yields

$$\sqrt{\frac{2n}{m\sigma^2}} (\hat{\theta}_n - \theta) \xrightarrow{L} N(0, 1) \quad \text{as } n \rightarrow \infty,$$

where $N(0, \beta)$ denotes a normal random variable with mean 0 and variance β . Therefore

$$\begin{aligned} P\{\theta \in I_n\} &= P\left\{ \left| \sqrt{\frac{2n}{m\sigma^2}} (\hat{\theta}_n - \theta) \right| \leq \sqrt{\frac{2n}{m\sigma^2}} d \right\} \\ &\sim \Phi\left(\sqrt{\frac{2n}{m\sigma^2}} d\right) - \Phi\left(-\sqrt{\frac{2n}{m\sigma^2}} d\right) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Now for large n

$$\sqrt{\frac{2n}{m\sigma^2}} d \geq z \quad \text{or equivalently} \quad n \geq \frac{mz^2}{2d^2} \sigma^2$$

implies that $P\{\theta \in I_n\} \geq 1 - \alpha$. Let

$$(2.8) \quad n^* = \frac{mz^2}{2d^2} \sigma^2.$$

Here, for simplicity we assume that n^* is a positive integer. Clearly n^* is the asymptotic optimal fixed sample size. Since σ^2 is unknown, so is n^* . Hence there is no fixed sample size procedure. Thus we propose the following stopping rule:

$$(2.9) \quad N = N_d(L) = \inf \left\{ n \geq k : n \geq \frac{mz^2}{2d^2} s_n^2 L_n \right\},$$

where $k \geq 2$ is the starting sample size, L_x is a positive continuous function on $(0, \infty)$ for which

$$(2.10) \quad L_x = 1 + Lx^{-1} + o(x^{-1}) \quad \text{as } x \rightarrow \infty \quad \text{with a constant } L.$$

REMARK 2.1. Woodroffe (1977) had considered this problem with $a=1$ and $b=0$. In this case we get $m=2$ in (2.1). Then the stopping rule N in (2.9) with $m=2$ and $L=2d$ is the same as in (4.3) of Woodroffe (1977).

REMARK 2.2. By Lemma 1 of Chow and Robbins (1965) we have

$$P\{N < \infty\} = 1 \quad \text{for all } d > 0, \\ N \xrightarrow{a.s.} \infty \quad \text{and} \quad N/n^* \xrightarrow{a.s.} 1 \quad \text{as } d \rightarrow 0.$$

Before providing the results, we note that all the proofs of the results in this section will be provided in the final section. Now, the natural sequential estimator of θ is given by

$$(2.11) \quad \hat{\theta}_N^{(1)} = a\bar{X}_N + b s_N.$$

First, we shall evaluate the bias of the sequential estimator $\hat{\theta}_N^{(1)}$. The following lemma presents the bias of the estimator s_N of σ .

LEMMA 2.1. *If $k > 7$ in (2.9) then*

$$E(s_N) - \sigma = -\frac{5\sigma}{4}(n^*)^{-1} + o(d^2) \quad \text{as } d \rightarrow 0.$$

From this lemma we get the bias of $\hat{\theta}_N^{(1)}$.

PROPOSITION 2.1. *If $k > 7$ then*

$$E(\hat{\theta}_N^{(1)}) - \theta = -\frac{5b\sigma}{4}(n^*)^{-1} + o(d^2) \quad \text{as } d \rightarrow 0.$$

Taking Proposition 2.1 into account, we consider the following bias-corrected sequential estimator $\hat{\theta}_N^{(2)}$ of θ :

$$(2.12) \quad \hat{\theta}_N^{(2)} = \hat{\theta}_N^{(1)} + \frac{5bd^2}{2mz^2} s_N^{-1}.$$

We may, of course, be able to consider other bias-corrected sequential estimators. Then the bias of $\hat{\theta}_N^{(2)}$ is given by

PROPOSITION 2.2. *If $k > 7$ then*

$$E(\hat{\theta}_N^{(2)}) - \theta = o(d^2) \quad \text{as } d \rightarrow 0.$$

We shall now construct the confidence intervals for θ . For $j=1, 2$ let

$$(2.13) \quad I_N^{(j)} = I_N^{(j)}(d, L) = [\hat{\theta}_N^{(j)} - d, \hat{\theta}_N^{(j)} + d].$$

Once the sampling stops, $I_N^{(j)}$ ($j=1, 2$) is used as the confidence interval of θ with width $2d$ and coverage probability $1-\alpha$. The asymptotic consistency of $I_N^{(j)}$ ($j=1, 2$) in the sense of Chow and Robbins (1965) can then be shown by the following theorem.

THEOREM 2.1. *If $k > 1$ in (2.9) for $j=1$ and $k > 7$ for $j=2$ then*

$$\lim_{d \rightarrow 0} P\{\theta \in I_N^{(j)}\} = 1 - \alpha \quad \text{for } j=1, 2.$$

Finally we provide the second-order asymptotic expansion of the expected sample size.

THEOREM 2.2. *If $k > 3$ then*

$$E(N) = n^* + \nu + L - 2 + o(1) \quad \text{as } d \rightarrow 0,$$

where ν is the constant given in Lemma 4.1 (ii) of Section 4.

REMARK 2.3. This result with $L=2\Delta$ is the same as in (4.3) of Woodroffe (1977).

3. Simulation results

In this section we present the simulation results for two sequential confidence intervals given in Section 2. Throughout the simulations we consider $\mu=0$, $\sigma^2=1$, $L_x=1+1.18x^{-1}+o(x^{-1})$ in (2.10), $k=10$ in (2.9), $n^*=50, 100, 500, 1000$ in (2.8), and $\alpha=0.05$, that is, the sequential 95% confidence intervals of $\theta=a\mu+b\sigma$. We note that $L=1.18$ in Theorem 2.2 is chosen in order to adjust the expected sample size $E(N)$ to the asymptotic optimal fixed sample size n^* . For $d > 0$ put

$$\theta_j = E(\hat{\theta}_N^{(j)}) \quad \text{and} \quad p_j = P\{\theta \in I_N^{(j)}\} \quad \text{for } j=1, 2.$$

All θ_j and p_j ($j=1, 2$) in the tables below were estimated by repeating the experiment 20,000 times. Table 1, 2 and 3 show the results for $(a, b) = (1, 0.67452)$, $(1, 2)$ and $(1, 3)$, respectively.

From the simulation results we feel that the bias of the estimator $\hat{\theta}_N^{(1)}$ of θ is corrected. However, it appears that the coverage probabilities of the sequential confidence intervals $I_N^{(2)}$ constructed by using $\hat{\theta}_N^{(2)}$ is not always higher than that of $I_N^{(1)}$ by $\hat{\theta}_N^{(1)}$. It is therefore interesting to compare the performance of these confidence intervals by evaluating their coverage probabilities more precisely. Furthermore, we are interested in investigating the mean square error (m.s.e) and the variance for the estimators $\hat{\theta}_N^{(1)}$ and $\hat{\theta}_N^{(2)}$ of θ . The

Table 1. $\theta = \mu + 0.67452\sigma$ (the 75% point)

	n^*			
	50	100	500	1000
θ	0.67452	0.67452	0.67452	0.67452
d	0.307100	0.217152	0.097114	0.068670
$E(N)$	52.0393	101.4368	499.1302	1002.7206
θ_1	0.652689	0.673278	0.668962	0.676934
θ_2	0.670139	0.681779	0.670655	0.677778
p_1	0.84865	0.96585	0.94030	0.94200
p_2	0.84865	0.96585	0.95750	0.93675

Table 2. $\theta = \mu + 2\sigma$

	n^*			
	50	100	500	1000
θ	2.0	2.0	2.0	2.0
d	0.480100	0.339482	0.151821	0.107354
$E(N)$	46.8588	101.0119	500.8920	1003.7228
θ_1	1.878739	1.981119	1.991994	2.003431
θ_2	1.932594	2.006457	1.997006	2.005931
p_1	0.84100	0.94325	0.96615	0.95090
p_2	0.84100	0.94725	0.96615	0.95850

Table 3. $\theta = \mu + 3\sigma$

	n^*			
	50	100	500	1000
θ	3.0	3.0	3.0	3.0
d	0.650058	0.459661	0.205567	0.145357
$E(N)$	51.0504	99.0922	498.2734	1001.9756
θ_1	2.975254	2.940844	2.986245	3.002163
θ_2	3.053375	2.979244	2.993785	3.005915
p_1	0.91880	0.91085	0.95215	0.96315
p_2	0.91880	0.93080	0.96085	0.96315

following results give the m.s.e and the variance of the estimators for θ . The proof of the results in this section will be briefly given in Section 4.

THEOREM 3.1. *If $k > 16$ then as $d \rightarrow 0$*

$$(3.1) \quad E\{\hat{\theta}_N^{(1)} - \theta\}^2 = \frac{m\sigma^2}{2n^*} + \left\{(-\nu - L + 4)\frac{m}{2} + \frac{87}{16}b^2\right\} \frac{\sigma^2}{(n^*)^2} + o(d^4)$$

and

$$(3.2) \quad E\{\hat{\theta}_N^{(2)} - \theta\}^2 = \frac{m\sigma^2}{2n^*} + \left\{(-\nu - L + 4)\frac{m}{2} + \frac{21}{8}b^2\right\} \frac{\sigma^2}{(n^*)^2} + o(d^4).$$

From Theorem 3.1 and Propositions 2.1 and 2.2 we readily get the variance of the estimators $\hat{\theta}_N^{(1)}$ and $\hat{\theta}_N^{(2)}$ of θ .

PROPOSITION 3.1. *If $k > 16$ then as $d \rightarrow 0$*

$$\text{Var}(\hat{\theta}_N^{(1)}) = \frac{m\sigma^2}{2n^*} + \left\{(-\nu - L + 4)\frac{m}{2} + \frac{31}{8}b^2\right\} \frac{\sigma^2}{(n^*)^2} + o(d^4)$$

and

$$\text{Var}(\hat{\theta}_N^{(2)}) = \frac{m\sigma^2}{2n^*} + \left\{(-\nu - L + 4)\frac{m}{2} + \frac{21}{8}b^2\right\} \frac{\sigma^2}{(n^*)^2} + o(d^4).$$

REMARK 3.1. Since the results of Theorem 3.1 and Proposition 3.1, indicate that both the m.s.e and the variance of the bias-corrected estimator $\hat{\theta}_N^{(2)}$ are

a little bit better than those of the natural estimator $\hat{\theta}_N^{(1)}$, it seems that the bias-correction is efficient in the case of sequential point estimation for θ . However, it appears that the variances are much larger with respect to the bias of the estimators $\hat{\theta}_N^{(1)}$ and $\hat{\theta}_N^{(2)}$. Therefore we feel that this fact justifies the simulation results.

4. Proofs

In this section all the proofs of the results in Sections 2 and 3 are given. By (2.2), (2.3), (2.8) and (2.9) we get

$$N = \inf \left\{ n \geq k : \sum_{i=1}^{n-1} Y_i \leq (n^*)^{-1} (n-1)^2 l_{n-1} \right\},$$

where $Y_n (n=1, 2, \dots)$ are given in (2.2), $l_x = 1 + lx^{-1} + o(x^{-1})$ as $x \rightarrow \infty$ with the constant $l=1-L$ and L is the constant given in (2.10). Let

$$(4.1) \quad t = t_d(l) = \inf \left\{ n \geq k-1 : \sum_{i=1}^n Y_i \leq (n^*)^{-1} n^2 l_n \right\}.$$

Then

$$(4.2) \quad N = t+1 \quad \text{and} \quad s_N^2 = \sigma^2 \bar{Y}_t.$$

We introduce notation used in this section. Let

$$(4.3) \quad t^* = (n^*)^{-1/2} (t - n^*), \quad S_n = \sum_{i=1}^n Y_i, \quad R = S_t - t, \quad T = (n^*)^{-1/2} R, \\ \eta = t/n^*, \quad r = (n^*)^{-1} t^2 l_t - S_t = \eta t l_t - S_t$$

and let χ_n^2 denote a random variable having a chi-square distribution with n degrees of freedom. Throughout the remainder of this paper, we consider the case where $d \rightarrow 0$ and let $d_0 > 0$ be sufficiently small. Before proving the results in Section 2, we give four lemmas. From (4.1), Theorems 2.1-2.4 and Lemmas 2.1 and 2.3 of Woodroffe (1977) we have

LEMMA 4.1. *The following facts hold.*

$$(i) \quad t \xrightarrow{a.s.} \infty, \quad \eta \xrightarrow{a.s.} 1 \quad \text{and} \quad \bar{Y}_t \xrightarrow{a.s.} 1.$$

$$(ii) \quad t^* \xrightarrow{L} N(0, 2), \quad r \xrightarrow{L} H$$

and t^* and r are asymptotically independent where H is the random variable given in Theorem 2.1 of Woodroffe (1977),

$$\nu = E(H) = \frac{3}{2} - \sum_{n=1}^{\infty} n^{-1} E\{\max(0, S_n - 2n)\} \\ \doteq 0.820 \text{ by Table 2.1 of Woodroffe (1977).}$$

$$(iii) \quad \text{For any fixed } s > 0, \quad E(\eta^s) = O(1) \quad \text{and} \quad E(r^s) = O(1).$$

$$(iv) \quad P\left(\eta \leq \frac{1}{2}\right) = O(d^{k-1}).$$

$$(v) \quad \text{If } k-1 > s \text{ for any fixed } s > 0 \text{ then } \{|t^*|^s, 0 < d \leq d_0\} \text{ is uniformly integrable.}$$

(vi) If $k > 3$ then $E(t) = n^* + \nu - l - 2 + o(1)$.

From Theorem 2 of Chow et al. (1979), together with Lemma 4.1 (iii), we get

LEMMA 4.2. For any $s > 0$, $\{|T|^s, 0 < d \leq d_0\}$ is uniformly integrable.

LEMMA 4.3. If $k-1 \geq 2s$ for any fixed $s > 0$ then $E(\eta^{-s}) = O(1)$.

PROOF. Since $\eta \geq (n^*)^{-1}$ and $k-1-2s \geq 0$, by Lemma 4.1 (iv) and (2.8) we have

$$\begin{aligned} E(\eta^{-s}) &= E\left\{\eta^{-s} I\left(\eta \leq \frac{1}{2}\right)\right\} + E\left\{\eta^{-s} I\left(\eta > \frac{1}{2}\right)\right\} \\ &\leq (n^*)^s P\left(\eta \leq \frac{1}{2}\right) + 2^s = O(d^{k-1-2s}) + 2^s = O(1), \end{aligned}$$

where $I(A)$ denotes the indicator function of A . Thus the proof is complete.

LEMMA 4.4. Let ξ be a random variable lying between 0 and 1. If $k-1 > 2s$ for any fixed $s > 1$ then

$$\sup_{d>0} E\{|\bar{Y}_t|^{-s}\} < \infty \quad \text{and} \quad \sup_{d>0} E\{|1 + \xi(\bar{Y}_t - 1)|^{-s}\} < \infty.$$

PROOF. Let $\mathfrak{F}_n$ be the σ -algebra generated by $\{\bar{Y}_n, \bar{Y}_{n+1}, \dots\}$. Then it is well known that $\{\bar{Y}_n, \mathfrak{F}_n, n \geq k-1\}$ is a reversed martingale. Since $n\bar{Y}_n$ has the same distribution as χ_n^2 , it is easy to see that for any $p > -n/2$

$$(4.4) \quad E\{(\bar{Y}_n)^p\} = (2/n)^p (\Gamma(p + n/2) / \Gamma(n/2)).$$

Hence by Doob's maximal inequality (e.g., Chow and Teicher (1988), p. 247) and (4.4) we get

$$(4.5) \quad E\{|\bar{Y}_t|^{-s}\} \leq (s/(s-1))^s E\{(\bar{Y}_{k-1})^{-s}\} < \infty$$

for $k-1 > 2s$, which gives the first assertion of the lemma since the right-hand side of (4.5) does not depend on d . It follows from (4.5) and the convexity of the function x^{-s} on $(0, \infty)$ that

$$\sup_{d>0} E\{|1 + \xi(\bar{Y}_t - 1)|^{-s}\} \leq 1 + \sup_{d>0} E\{|\bar{Y}_t|^{-s}\} < \infty,$$

which shows the second assertion. Therefore the proof of the lemma is complete.

We are now in the position to prove the results in Section 2.

PROOF of LEMMA 2.1. It is clear that

$$(4.6) \quad s_N - \sigma = \frac{1}{2\sigma} \{(s_N^2 - \sigma^2) - (s_N - \sigma)^2\}.$$

Let

$$(4.7) \quad Q = \eta t(l_t - 1) - r.$$

Then it follows from (4.3) and (4.7) that

$$(4.8) \quad R = (\eta - 1)t + Q.$$

Since $E(S_t) = E(t)$ by Wald's lemma, we have, by (4.2) and (4.3),

$$(4.9) \quad \begin{aligned} \frac{1}{2\sigma} E(s_N^2 - \sigma^2) &= \frac{\sigma}{2} E\{(t^{-1} - (n^*)^{-1})R\} \\ &= -\frac{\sigma}{2} (n^*)^{-1} E\{\eta^{-1}(\eta - 1)R\}. \end{aligned}$$

From (4.3) and (4.8) we get

$$(4.10) \quad \eta^{-1}(\eta - 1)R = (t^*)^2 + \eta^{-1}(\eta - 1)Q.$$

It follows from (2.10) and Lemma 4.1 that

$$(4.11) \quad Q \xrightarrow{L} l - H$$

and $(t^*)^2 \xrightarrow{L} 2\chi_1^2$, which, together with (4.10), yield

$$(4.12) \quad \eta^{-1}(\eta - 1)R \xrightarrow{L} 2\chi_1^2.$$

By (4.3), Hölder's inequality and Lemmas 4.1-4.3 we have

$$\begin{aligned} E\{|\eta^{-1}(\eta - 1)R|^{3/2}\} &= E\{|\eta^{-1}t^*T|^{3/2}\} \\ &\leq \{E(\eta^{-3})\}^{1/2} \{E(|t^*|^6)\}^{1/4} \{E(|T|^6)\}^{1/4} = O(1), \end{aligned}$$

which implies the uniform integrability of $\{\eta^{-1}(\eta - 1)R, 0 < d \leq d_0\}$. Hence from (4.9) and (4.12) we get

$$(4.13) \quad \frac{1}{2\sigma} E(s_N^2 - \sigma^2) = -\sigma(n^*)^{-1} + o(d^2).$$

By Taylor's expansion and (4.2) we have

$$s_N - \sigma = \frac{1}{2} \{\sigma^2 + \xi(s_N^2 - \sigma^2)\}^{-1/2} (s_N^2 - \sigma^2) = \frac{\sigma}{2} B_t^{-1/2} (\bar{Y}_t - 1),$$

where ξ is a random variable lying between 0 and 1 and $B_t \equiv 1 + \xi(\bar{Y}_t - 1)$. Thus we get $(s_N - \sigma)^2 = (\sigma^2/4) B_t^{-1} (\bar{Y}_t - 1)^2$, which, together with (4.3), implies

$$(4.14) \quad -\frac{1}{2\sigma} E(s_N - \sigma)^2 = -\frac{\sigma}{8n^*} E(\eta^{-1}t^{-1}R^2B_t^{-1}).$$

In view of Anscombe's theorem (e.g., Corollary 1.4 of Woodroffe (1982)) we have

$$(4.15) \quad R/\sqrt{2t} \xrightarrow{L} N(0, 1),$$

which, together with Lemma 4.1, yields

$$(4.16) \quad \eta^{-1}t^{-1}R^2B_t^{-1} \xrightarrow{L} 2\chi_1^2.$$

Choose $\alpha > 1$ and $q > 1$ such that $k - 1 > 6\alpha q$ and $p^{-1} + q^{-1} = 1$. By (4.3), Hölder's inequality and Lemmas 4.2-4.4 we get

$$\begin{aligned} E\{|\eta^{-1}t^{-1}R^2B_t^{-1}|^q\} &= E\{|\eta^{-2}T^2B_t^{-1}|^q\} \\ &\leq \{E(|T|^{2ap})\}^{1/p} \{E(\eta^{-3aq})\}^{2/3q} \{E(|B_t|^{-3aq})\}^{1/3q} = O(1), \end{aligned}$$

which implies the uniform integrability of $\{\eta^{-1}t^{-1}R^2B_t^{-1}, 0 < d \leq d_0\}$. Hence (4.16) gives $E(\eta^{-1}t^{-1}R^2B_t^{-1}) = 2 + o(1)$, which, together with (4.14), yields

$$(4.17) \quad -\frac{1}{2\sigma}E(s_N - \sigma)^2 = -\frac{\sigma}{4}(n^*)^{-1} + o(d^2).$$

Thus from (4.6), (4.13) and (4.17) we obtain the lemma. This completes the proof of Lemma 2.1.

PROOF OF PROPOSITION 2.1. Since $\bar{X}_n$ and $(s_2, \dots, s_n)$ are independent, $\bar{X}_n$ and $I(N=n)$ are independent, given $N=n$. Thus

$$E(\bar{X}_N) = \sum_{n=k}^{\infty} E(\bar{X}_n | N=n) P(N=n) = \mu.$$

From this result, (2.11), and Lemma 2.1, we obtain the proposition. This completes the proof of Proposition 2.1.

We need the following lemma in order to prove Proposition 2.2.

LEMMA 4.5. *If $k > 6$ then $E(s_N^{-1}) = \sigma^{-1} + o(d)$.*

Since the proof of Lemma 4.5 is similar to that of Lemma 2.1, we omit it.

PROOF OF PROPOSITION 2.2. Proposition 2.2 is an immediate consequence of (2.8), (2.12), Proposition 2.1 and Lemma 4.5.

PROOF OF THEOREM 2.1. For $n \geq 2$ let

$$\begin{aligned} A_n &= (n^*/n)^{1/2}, \quad B = a(2/m)^{1/2}, \\ C &= b/\sqrt{m}, \quad C_n = \{2b\sigma/((s_n + \sigma)\sqrt{m})\}(n/(n-1))^{1/2}, \\ D^{(1)} &= 0 \quad \text{and} \quad D^{(2)} = D_n^{(2)} = \{5bd/(2mzs_n)\}(n/n^*)^{1/2}. \end{aligned}$$

From (2.5), (2.8) and (2.12) we have that for $j=1, 2$

$$zd^{-1}(\hat{\theta}_N^{(j)} - \theta) = A_N(BU_N + C_N V_N + D^{(j)}),$$

which, together with (2.13), yields that for $j=1, 2$

$$(4.18) \quad P\{\theta \in I_N^{(j)}\} = P\{-z \leq A_N(BU_N + C_N V_N + D^{(j)}) \leq z\}.$$

It follows from (2.6) that

$$(4.19) \quad V_n \xrightarrow{L} V \quad \text{and} \quad BU_n + CV_n \xrightarrow{L} BU + CV \quad \text{as } n \rightarrow \infty.$$

From Example 1.8 and Lemma 1.4 of Woodroffe (1982), $\{V_n, n \geq 2\}$ and $\{BU_n + CV_n, n \geq 2\}$ are uniformly continuous in probability. Hence by (4.19) and Anscombe's theorem (e.g., Theorem 1.4 of Woodroffe (1982)) we have

$$(4.20) \quad V_N \xrightarrow{L} V \quad \text{and} \quad BU_N + CV_N \xrightarrow{L} BU + CV \quad \text{as } d \rightarrow 0.$$

Since $N \xrightarrow{a.s.} \infty$, $N/n^* \xrightarrow{a.s.} 1$ and $s_N \xrightarrow{a.s.} \sigma$ as $d \rightarrow 0$, it follows from (4.20) that for $j=1, 2$

$$\begin{aligned} & |(BU_N + C_N V_N + D^{(j)}) - (BU_N + C V_N)| \\ & = |(C_N - C) V_N + D^{(j)}| \xrightarrow{p} 0 \quad \text{as } d \rightarrow 0, \end{aligned}$$

which, together with (4.20), implies that for $j=1, 2$

$$BU_N + C_N V_N + D^{(j)} \xrightarrow{L} BU + CV \quad \text{as } d \rightarrow 0.$$

Thus, by the fact that $A_N \xrightarrow{a.s.} 1$ as $d \rightarrow 0$ we get that for $j=1, 2$

$$A_N(BU_N + C_N V_N + D^{(j)}) \xrightarrow{L} BU + CV \quad \text{as } d \rightarrow 0,$$

which, together with (4.18) and the fact that $BU + CV$ has a standard normal distribution, yields that for $j=1, 2$

$$P\{\theta \in I_N^{(j)}\} \rightarrow \Phi(z) - \Phi(-z) = 1 - \alpha \quad \text{as } d \rightarrow 0.$$

Therefore the proof of Theorem 2.1 is complete.

PROOF of THEOREM 2.2. This theorem is an immediate consequence of (4.2), Lemma 4.1 (vi) and the fact that $l=1-L$.

Finally, we shall give the outline of the proof of Theorem 3.1. Before that, we state a lemma without proof.

LEMMA 4.6. *The following results hold.*

- (i) $E(\bar{Y}_t - 1)^2 = \frac{2}{n^*} + (-2\nu + 2l + 16)(n^*)^{-2} + o(d^4)$ if $k > 7$.
- (ii) $E(\bar{Y}_t - 1)^3 = -16(n^*)^{-2} + o(d^4)$ if $k > 8$.
- (iii) $E[(\bar{Y}_t - 1)^4 B_t^{-7/2}] = 12(n^*)^{-2} + o(d^4)$ if $k > 16$,
where $B_t \equiv 1 + \xi(\bar{Y}_t - 1)$ and ξ is a random variable lying between 0 and 1.
- (iv) $E(s_N^{-2}) = \sigma^{-2} + o(1)$ if $k > 6$.
- (v) $E(1 - \sigma s_N^{-1}) = -\frac{7}{4n^*} + o(d^2)$ if $k > 10$.

PROOF of THEOREM 3.1. Since $\bar{X}_n$ and $(s_2^2, \dots, s_n^2)$ are independent, it follows from the definition of N that

$$(4.21) \quad E\{\hat{\theta}_N^{(1)} - \theta\}^2 = a^2 \sigma^2 E(N^{-1}) + b^2 E(s_N - \sigma)^2.$$

From Taylor's expansion, (4.2), (4.3), Theorem 2.2 and Lemma 4.6 we have

$$\begin{aligned} (4.22) \quad E(s_N - \sigma)^2 &= (\sigma^2/4)E(\bar{Y}_t - 1)^2 - (\sigma^2/8)E(\bar{Y}_t - 1)^3 + (5\sigma^2/64)E[(\bar{Y}_t - 1)^4 B_t^{-7/2}] \\ &= \frac{\sigma^2}{2n^*} + \frac{\sigma^2}{(n^*)^2} \left(-\frac{\nu}{2} + \frac{l}{2} + \frac{111}{16} \right) + o(d^4) \end{aligned}$$

and

$$\begin{aligned} (4.23) \quad E(N^{-1}) &= (n^*)^{-1} - (n^*)^{-2}(\nu - l - 1 + o(1)) \\ &\quad + (n^*)^{-2}E[(t^* + (n^*)^{-1/2})^2\{1 + \xi(\eta - 1 + (n^*)^{-1})\}^{-3}], \end{aligned}$$

where ξ is a random variable with $\xi \in (0, 1)$. From Lemma 4.1 we have

$$J \equiv (t^* + (n^*)^{-1/2})^2 \{1 + \xi(\eta - 1 + (n^*)^{-1})\}^{-3} \xrightarrow{L} 2\chi_1^2,$$

which, together with the uniform integrability of $\{J, 0 < d \leq d_0\}$ and (4.23), implies

$$(4.24) \quad E(N^{-1}) = (n^*)^{-1} + (-\nu + l + 3)(n^*)^{-2} + o(d^4).$$

From (4.21), (4.22), (4.24) and the fact that $l = 1 - L$ we obtain (3.1). It follows from (2.8) and (2.12) that

$$E\{\hat{\theta}_N^{(2)} - \theta\}^2 = E\{\hat{\theta}_N^{(1)} - \theta\}^2 + \frac{5b^2\sigma^2}{2n^*}E(1 - \sigma s_N^{-1}) + \frac{25b^2\sigma^4}{16(n^*)^2}E(s_N^{-2}),$$

which, together with Lemma 4.6 and (3.1), yields (3.2). Therefore the proof of Theorem 3.1 is complete.

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