CHARACTERIZATION OF GENERALIZED MIXTURES OF EXPONENTIAL DISTRIBUTION BASED ON CONDITIONAL EXPECTATION OF ORDER STATISTICS

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Generalized mixtures of exponential random variables (Everitt and Hand (1981) and Johnson *et al.* (1994)) X_1 and X_2 are identified in terms of their relations between the best predictor of $X_{2:2}$ given $X_{1:2}$ and the functions of the failure rate (or hazard function) of the distribution. Here $X_{1:2}$ and $X_{2:2}$ denote the corresponding order statistics. In addition, we also mention some related theorems in order to characterize the generalized mixtures of exponential distribution. Moreover, when the sample size is n , the above results are also valid.

Key Words and Phrases: characterization, order statistics, conditional expectation, failure rate, generalized mixtures of exponential distribution.

1. Introduction

Let X_1 and X_2 be independent and identically distributed (i.i.d.) non-degenerate random variables with a probability density function (p.d.f.) f , and let $X_{1:2} < X_{2:2}$ denote the corresponding order statistics. Consider the problem of predicting the value of $X_{2:2}$ when the observed value of $X_{1:2}$ is known. This problem is of some importance in life testing and replacement policy situations where one desires to predict the time of failure from the times of early failures in the same sample. As is well known, the almost surely (a.s.) unique best mean squared error predictor of $X_{2:2}$ given $X_{1:2} = x$ is $E[X_{2:2} | X_{1:2} = x]$. Moreover, "failure" here can be interpreted in a perfectly general way as an event - the probability density function for the times of scores in a game may be exponentially distributed.

Hence, Dallas (1973) proved that if the support of X_1 and X_2 is the set of all positive real numbers then $E[X_{2:2} | X_{1:2} = x] = x + a$, $x > 0$ and $a > 0$ if and only if X_1 and X_2 have an exponential distribution (also see Nagaraja (1988)). In addition, when we consider that failures may arise for a number of different reasons it hardly seems surprising that a superposition of exponential densities - a mixture - might provide a better description of the failure properties. For example, the causes of electronic valve failures have been grouped into three categories: gaseous defects, mechanical defects, and normal deterioration of the cathode (also see Everitt and Hand (1981)). El-Arishy and Ahmed (1996) have provided an extension of this result when X_1 and X_2 are taken from a mixture of two exponential distributions.

In this paper, we also consider the prediction when X_1 and X_2 are taken

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from a generalized mixture of an exponential distribution (see Everitt and Hand (1981) and Johnson *et al.* (1994)).

We prove that if $\phi_s(x) = f^{(s)}(x)/p(X_1 \geq x)$, $s=0, 1, 2, \dots, k-2$ and $f^{(0)}(x) \equiv f(x)$, then

$$E[X_{2:2} | X_{1:2} = x] = x + \sum_{i=1}^k a_i - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x),$$

if and only if X_1 has the generalized mixtures p.d.f.

$$f(x) = \sum_{i=1}^k \frac{\theta_i}{a_i} \exp\left(-\frac{x}{a_i}\right), \quad x > 0, \quad a_i > 0, \quad \theta_i \geq 0, \quad \text{and} \quad \sum_{i=1}^k \theta_i = 1.$$

Thus, we provide an extension of the results obtained by Dallas (1973) and El-Arshy and Ahmed (1996) for the exponential or mixtures exponential distributions. In addition, we also mention some related theorems in order to characterize the generalized mixtures of exponential distributions. References on various characterizations of the exponential distribution include Azlarov and Voldin (1986), Beg and Kirmani (1974), Galambos and Kotz (1978), Nagaraja (1988), Nassar and Mahmoud (1985), Ouyang (1982, 1983), and Tanis (1964).

2. The Characterization Theorem

Let X_1 and X_2 be i.i.d. random variables taking positive values with p.d.f. $f(x)$. And let $X_{1:2} < X_{2:2}$ denote the corresponding order statistics. Let $m(x) = p(X \geq x)$, $\phi_0(x) = f(x)/m(x)$ be the failure rate (or hazard function) of X_1 and $\phi_s(x) = f^{(s)}(x)/m(x)$, where $f^{(s)}(x) = d^s f(x)/dx^s$, $s=1, 2, \dots, k-2$. Here, $\phi_s(x)$ is the function for the failure rate of the distribution (i.e., $\phi_1(x) = \phi'_0(x) - \phi_0^2(x)$, and $\phi_2(x) = \phi'_1(x) - \phi_1(x)\phi_0(x) = \phi''_0(x) - 3\phi_0(x)\phi'_0(x) + \phi_0^3(x)$, $\dots$, etc.). Moreover, suppose that $E(X_1) < \infty$. We have the following result.

THEOREM 1. *Let $f(x)$ have continuous derivatives of all orders up to a natural number k on the set of all positive real numbers (i.e. $C^k(R_+)$), then*

$$(2.1) \quad E[X_{2:2} | X_{1:2} = x] = x + \sum_{i=1}^k a_i - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x),$$

if and only if X_1 has the generalized mixtures p.d.f.

$$(2.2) \quad f(x) = \sum_{i=1}^k \frac{\theta_i}{a_i} \exp\left(-\frac{x}{a_i}\right), \quad x > 0, \quad a_i > 0, \quad \theta_i \geq 0, \quad \text{and} \quad \sum_{i=1}^k \theta_i = 1.$$

We shall need the following lemma.

LEMMA 1. *Let X_1 have the generalized mixtures p.d.f. (2.2). Then*

$$\sum_{i=1}^k \theta_i a_i \exp\left(-\frac{x}{a_i}\right) = \left(\sum_{i=1}^k a_i \right) m(x) - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s)}(x).$$

PROOF. It can be shown that if $f(x)$ as (2.2), then

$$(2.3) \quad \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s+2)}(x) + \left(\sum_{i=1}^k a_i \right) f^{(1)}(x) + f(x) = 0.$$

By integrating (2.3) with respect to x and $F(\infty)=1$, we obtain

$$(2.4) \quad \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s+1)}(x) + \left(\sum_{i=1}^k a_i \right) f(x) + F(x) = 1.$$

Using (2.2), we obtain $1 - F(x) = \sum_{i=1}^k \theta_i \exp(-(x/a_i))$. Therefore, (2.4) can be written as

$$(2.5) \quad \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s+1)}(x) + \left(\sum_{i=1}^k a_i \right) f(x) = \sum_{i=1}^k \theta_i \exp\left(-\frac{x}{a_i}\right).$$

By integrating (2.5) with respect to x and $F(\infty)=1$, we obtain

$$\sum_{i=1}^k \theta_i a_i \exp\left(-\frac{x}{a_i}\right) = \left(\sum_{i=1}^k a_i \right) m(x) - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s)}(x).$$

The proof is completed. $\square$

PROOF of THEOREM 1. Suppose that X_1 has the generalized mixtures p.d.f. (2.2). Then, for each $x > 0$,

$$\begin{aligned} E[X_{2,2} | X_{1,2} = x] &= \int_x^\infty y_2 f(y_2) dy_2 / m(x) \\ &= \int_x^\infty y_2 \sum_{i=1}^k \frac{\theta_i}{a_i} \exp\left(-\frac{y_2}{a_i}\right) dy_2 / m(x) \\ &= \sum_{i=1}^k \theta_i \int_x^\infty \frac{y_2}{a_i} \exp\left(-\frac{y_2}{a_i}\right) dy_2 / m(x) \\ &= \left\{ x \sum_{i=1}^k \theta_i \exp\left(-\frac{x}{a_i}\right) + \sum_{i=1}^k \theta_i a_i \exp\left(-\frac{x}{a_i}\right) \right\} / m(x) \\ &\quad \text{(by integration by parts and } E[X_1] < \infty) \\ &= x + \sum_{i=1}^k \theta_i a_i \exp\left(-\frac{x}{a_i}\right) / m(x) \\ &= x + \left(\sum_{i=1}^k a_i \right) - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x). \\ &\quad \text{(by Lemma 1)} \end{aligned}$$

To prove the converse, suppose that (2.1) holds. Then

$$(2.6) \quad \begin{aligned} &\int_x^\infty y_2 f(y_2) dy_2 \\ &= x \cdot m(x) + \left(\sum_{i=1}^k a_i \right) m(x) - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s)}(x). \end{aligned}$$

Since $f(x) \in C^k(R_+)$, so differentiating (2.6) twice with respect to x , we obtain

$$(2.7) \quad \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} f^{(s+2)}(x) + \left(\sum_{i=1}^k a_i \right) f^{(1)}(x) + f(x) = 0,$$

which is a homogeneous k th order differential equation whose characteristic function is

$$\sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} t^{(s+2)} + \left(\sum_{i=1}^k a_i \right) t + 1 = 0,$$

having roots $t = -(1/a_i)$, $i = 1, 2, \dots, k$.

Thus, the solution of (2.7) is

$$f(y) = \sum_{i=1}^k c_i \exp\left(-\frac{y}{a_i}\right),$$

where c_i , $i = 1, 2, \dots, k$ are arbitrary constants and $a_i > 0$.

Now, using the initial condition that $\int_0^\infty f(y) dy = 1$, $f(x) \geq 0$, for all $x > 0$, and $a_i > 0$ we get that $\sum_{i=1}^k c_i a_i = 1$, and $c_i \geq 0$. Hence, let $\theta_i = c_i a_i$, we obtain

$$f(y) = \sum_{i=1}^k \frac{\theta_i}{a_i} \exp\left(-\frac{y}{a_i}\right),$$

where $\sum_{i=1}^k \theta_i = 1$, and $\theta_i \geq 0$. Hence, the result follows. $\square$

REMARK 1. If we take $a_1 = a_2 = \dots = a_k = a$, then Theorem 1 reduces to the result of Dallas (1973) and Nagaraja (1988).

REMARK 2. If we take $k = 2$, then Theorem 1 reduces to the result of El-Arishy and Ahmed (1996).

REMARK 3. If we take $k = 3$, then

$$E[X_{2:2} | X_{1:2} = x] = x + (a_1 + a_2 + a_3) - (a_1 a_2 + a_2 a_3 + a_1 a_3) \phi_0(x) - a_1 a_2 a_3 \phi_1(x),$$

if and only if X_1 has the generalized mixtures p.d.f. as (2.2)

3. Extensions and Possible Applications

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed. And let $X_{1:n} < X_{2:n} < \dots < X_{n:n}$ denote the corresponding order statistics. In practice, $X_{1:n} < X_{2:n} < \dots < X_{n:n}$ may represent the failure times of the components of a system composed of n units. If all of the components are connected in parallel, the system will continue to operate as long as at least one of its components has not yet failed. Thus, an engineer might be wondering as to when the system will fail at an instant of failure for $(n-1)$ th component. On average, this time is given by $E[X_{n:n} | X_{n-1:n} = x]$. Moreover, we can also show that for any integer $n \geq 2$,

$$\begin{aligned} E[X_{n:n} | X_{n-1:n} = x] &= \frac{\int_x^\infty y f(y) dy}{m(x)} \\ &= E[X_{2:2} | X_{1:2} = x]. \end{aligned}$$

Therefore, by Theorem 1 and the equations, we obtain the following result.

THEOREM 2. *Let the condition of $f(x)$ as Theorem 1, then*

$$E[X_{n:n}|X_{n-1:n}=x]=x+\sum_{i=1}^k a_i - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x),$$

if and only if X_1 has the generalized mixtures p.d.f. as (2.2)

We also note some related theorems in order to characterize the generalized mixtures of exponential distribution. We can show that for each fixed $1 \leq j \leq n$,

$$p[X_j < y | X_{1:n} = x] = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{F(y) - F(x)}{1 - F(x)}, \quad \text{if } y > x,$$

while this conditional distribution is zero for $y \leq x$. Moreover, for each fixed $1 \leq j \leq n$,

$$\begin{aligned} E[X_j | X_{1:n} = x] &= \frac{x}{n} + \frac{n-1}{n} \cdot \frac{\int_x^\infty y f(y) dy}{m(x)} \\ &= \frac{x}{n} + \frac{n-1}{n} \cdot E[X_{2:2} | X_{1:2} = x]. \end{aligned}$$

So, we use the relation to obtain the following result.

THEOREM 3. *Let the condition of $f(x)$ as Theorem 1, then for each fixed $1 \leq j \leq n$,*

$$E[X_j | X_{1:n} = x] = x + \frac{n-1}{n} \left[\sum_{i=1}^k a_i - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x) \right],$$

if and only if X_1 has the generalized mixtures p.d.f. as (2.2)

COROLLARY. *Given the condition of $f(x)$ as in Theorem 1, then*

$$E \left[\sum_{j=1}^n \alpha_j X_j | X_{1:n} = x \right] = x + \frac{n-1}{n} \left[\sum_{i=1}^k a_i - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x) \right],$$

for any $\alpha_j \geq 0$ and $\sum_{j=1}^n \alpha_j = 1$ if and only if X_1 has a generalized mixtures p.d.f. as (2.2)

REMARK 4. If we take $k=2$, then Theorem 2 reduces to the result of El-Arshy and Ahmed (1996).

REMARK 5. If we take $a_1 = a_2 = \dots = a_k = a$, then Theorem 3 and Corollary reduce to the result of Beg and Kirmani (1974) and Galambos and Kotz (1978).

Finally, we also use the relation between the mean residual life (MRL) of X_1 and the functions of the failure rate of the distribution in order to characterize the generalized mixtures of exponential distribution. Therefore, we have the following result.

THEOREM 4. Let the condition of $f(x)$ as Theorem 1, then

$$E[X_1 - x | X_1 > x] = \sum_{i=1}^k a_i - \sum_{s=0}^{k-2} \left\{ \sum_{1 \leq i_1 < i_2 < \dots < i_{s+2} \leq k} \left[\prod_{j=1}^{s+2} a_{i_j} \right] \right\} \phi_s(x),$$

if and only if X_1 has a generalized mixture p.d.f. as in (2.2)

REMARK 6. If we take $a_1 = a_2 = \dots = a_k = a$, then Theorem 4 reduces to the result of Shanbhag (1970).

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REFERENCES

- [1] Azlarov, T. and Voldin, N. (1986). *Characterization Problem Associated with the Exponential Distribution*, Springer-Verlag, New York.
- [2] Beg, M. I. and Kirmani S. N. U. A. (1974). On a characterization of exponential and related distributions, *The Austral. J. Statist.*, **16**, 163-166.
- [3] Dallas, A. (1973). A characterization of the exponential distribution, *Bull. Sor. Math. Greece*, **14**, 172-177.
- [4] El-Arishy, S. and Ahmed, A. N. (1996). Identification of mixtures of exponential distributions by means of a predictor and expectations of order statistics, *J. Japan Statist. Soc.*, **26**, 119-125.
- [5] Everitt, B. S. and Hand, D. J. (1981). *Finite Mixture Distribution*, Chapman and Hall, London, New York.
- [6] Galambos, J. and Kotz, S. (1978). *Characterization of Probability Distributions*, Springer-Verlag, Berlin.
- [7] Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions-Volume 1*, John Wiley & Sons, Inc, New York, Second Edition.
- [8] Nagaraja, N. (1988). Some characterizations of continuous distributions based on regressions of adjacent order statistics and record values, *Sankhyā Ser. A.*, **50**, 70-73.
- [9] Nassar, M. and Mahmoud, M. (1985). On characterizations of a mixture of exponential distributions, *IEEE TRANS.*, **R-34**, 484-488.
- [10] Ouyang, L. Y. (1982). On characterizations of the exponential and its related distributions by order statistics, *Tamkang Journal of Management Sciences*, **3**, 17-42.
- [11] Ouyang, L. Y. (1983). A characterizations of the exponential distribution, *Tamkang Journal of Management Sciences*, **4**, 23-28.
- [12] Shanbhag, D. N. (1970). The characterizations for exponential and geometric distributions, *J. Amer. Statist. Assoc.*, **65**, 1256-1259.
- [13] Tanis, E. (1964). Linear forms in the order statistics from an exponential distribution, *Ann. Math. Statist.*, **35**, 270-276.